Problem Set no. 4

Given: January 17, 2018
Due: February 1, 2018

A (nested) blossom B w.r.t. a matching M, or an M-*blossom* for short, is defined recursively as either a single vertex, or an odd "alternating" cycle, w.r.t. M, composed of smaller M-blossoms.

Exercise 4.1 A graph $G = (V, E)$ is said to be *factor-critical* if for every $v \in V$, the subgraph $G \setminus \{v\}$ contains a perfect matching. (A matching is sometimes called a *factor* or a 1-*factor*.) Show that a blossom B, with respect to some matching M, is a factor-critical subgraph of the original graph. **Bonus:** Prove the converse. If G is factor-critical and M matches all vertices of G except one, then there is an M-blossom in G that contains all the vertices of G.

Exercise 4.2 Let $G = (V, E, w)$ be a weighted graph. Let M be a matching in G at the beginning of some iteration of Edmonds' algorithm, and let B be an M-blossom at the end of the iteration. Let P be the M-augmenting path found by this iteration. Prove that B is also an $(M \oplus P)$-blossom.

Exercise 4.3 Prove that the M-augmenting path P found in each iteration of Edmonds' algorithm for finding a maximum weight matching is an augmenting path of maximum weight $\bar{w}_M(P)$, where M is the matching at the beginning of the iteration. Direction: Recall that the *slack* s_e of an edge $(u, v) \in E$ is defined as $s_e = y_u + y_v + \sum_{B \supset e} z_B - w_e$. Every edge e on P is tight, i.e., $s_e = 0$. For every $e \in E$, $s_e \geq 0$. (a) Show that $\bar{w}_M(P) = 2y$, where y is the joint dual value of all unmatched vertices when the augmenting path P is found. (Hint: The y's and the z's in the slacks of the edges on the path "telescope". To show that the z's telescope, show that the intersection of P and any blossom B is an even alternating path.) (b) Show that for every M-augmenting path P' we have $\bar{w}_M(P') \leq 2y$. (Hint: Show that for every blossom B, P' uses at least as many matched edges of B as unmatched edges of B.)

Exercise 4.4 The input to the *Chinese Postman Problem* is a weighted undirected graph $G = (V, E, w)$, where $w : E \to \mathbb{R}^+$. The goal is to find a *tour* of minimal total weight that traverses each edge *at least* once. Describe a polynomial time algorithm for the problem. (A tour here is simply a *closed walk*, i.e., a closed path. It may use an edge more than once, but then the weight of the edge is also taken the appropriate number of times.)

Exercise 4.5 (a) Describe a polynomial time algorithm for finding a minimum weight *perfect* matching in a weighted undirected graph $G = (V, E, w)$.

(b) Let $G = (V, E, w)$, where $w : E \to \mathbb{R}^+$, be a weighted *directed* graph with no negative cycles. Let $s, t \in V$ and assume that there is a path from s to t in G. Define a bipartite graph $\bar{G} = (V_0, V_1, \bar{E}, \bar{w})$ as follows. For every $v \in V \setminus \{t\}$, there is a vertex $v_0 \in V_0$ and for every $v \in V \setminus \{s\}$, there is a vertex $v_1 \in V_1$. For every edge $e = (u, v) \in E$, where $u \neq t$ and $v \neq s$, there is an edge $\bar{e} = (u_0, v_1) \in \bar{E}$ with $\bar{w}(\bar{e}) = w(e)$. Finally, for every $v \in V \setminus \{s, t\}$, there is an edge $(v_0, v_1) \in \bar{E}$ with $\bar{w}(v_0, v_1) = 0$. Show that the length (i.e., weight) of a shortest path in G from s to t equals the minimum weight of a perfect matching in $\bar{G}$.