Problem Set no. 3

Given: December 26, 2017
Due: January 9, 2018

Exercise 1.1 (a) Describe an $O(m)$-time implementation of the process described in slide 13 of the presentation on Goldberg's shortest paths algorithm. (b) Are the processes described on slides 13 and 14 equivalent, i.e., do they produce the same reduced costs of all edges?

Exercise 1.2 Prove the Mendelson-Dulmage Theorem: Let $G = (S, T, E)$ be a bipartite graph. Let M_1, M_2 be two matchings of G. Then, there exists a matching $M \subseteq M_1 \cup M_2$ that matches all the vertices of S matched by M_1 and all the vertices of T matched by M_2.

Exercise 1.3 Prove Hall's Theorem: Let $G = (S, T, E)$ be a bipartite graph. There is a matching that matches all the vertices of S if and only if $|N(X)| \geq |X|$ for every $X \subseteq S$. (Here $N(X)$ is the set of neighbors of the vertices in X.)

Exercise 1.4 Describe a linear time algorithm for finding a maximal set of *disjoint* shortest augmenting paths with respect to a given matching M of a bipartite graph $G = (S, T, E)$.

Exercise 1.5 A *quasi-blossom* with respect to a matching M in a graph $G = (V, E)$ is an odd "alternating" cycle, i.e., a alternating path of odd length that starts and ends at the same vertex. (Note that a *blossom* is a quasi-blossom which is part of a *flower*, i.e., it has a *stem*.) Let B be a quasi-blossom and let G/B be the graph obtained by contracting B. Prove or disprove: G has an M-augmenting path if and only if G/B as an $(M \setminus B)$-augmenting path.