Problem Set no. 2 — Minimum spanning trees

Given: November 22, 2017
Due: December 7, 2017

Exercise 1.1 Let $G = (V, E)$ be an undirected graph on n vertices and let $0 < p < 1$. Let $G' = (V, E')$ be a random subgraph of G to which each edge of G is added, independently, with probability p. Show that the expected number of edges of G that connect different connected components of G' is at most n/p.

Exercise 1.2 In this exercise we obtain an alternative proof, due to Chan, of the following variant of the sampling lemma used to obtain the randomized linear MST algorithm:

> **Lemma:** Let $G = (V, E)$ be a weighted graph with distinct edge weights and let $1 \leq r \leq m$. Let $G' = (V, R)$ be a random subgraph of G containing *exactly* r edges, all choices being equally likely. Let F be a minimum spanning forest of G'. Then, the expected number of edges of G that are F-light is at most nm/r.

(a) Show that $e \in E$ is F-light if and only if $e \in \mathit{MSF}(R \cup \{e\})$, where $\mathit{MSF}(R \cup \{e\})$ is the minimum spanning forest of the subgraph $(V, R \cup \{e\})$.

(b) Show that if e is a random edge of G and R is a random subset of exactly r edges of G then $\Pr[e \in \mathit{MSF}(R \cup \{e\})] < n/r$. (Hint: note that e is a random element of $R \cup \{e\}$, a random set of size r or $r + 1$. How many edges from this set are in $\mathit{MSF}(R \cup \{e\})$? More precisely, show that $\Pr[e \in \mathit{MSF}(R \cup \{e\})] \leq \frac{r}{m}\frac{n-1}{r} + (1 - \frac{r}{m})\frac{n-1}{r+1} < \frac{n}{r}$.)

(c) Finish of the proof of the lemma.

Exercise 1.3 (a) Give an example of a digraph $G = (V, E, w)$ and a vertex $r \in V$ in which the *lightest* edge e in G does not belong to any MDST rooted at r, even though e does belong to some DST rooted at r. (b) Give an example of a digraph $G = (V, E, w)$ and a vertex $r \in V$ in which the *heaviest* edge e belongs to all MDST rooted at r, even though there are some DST rooted at r that avoids e.

Exercise 1.4 Prove or dispove: Let $G = (V, E, w)$ be a weighted digraph, let $r \in V$, and let T be a DST rooted at r. Then, T is a MDST iff for every edge $(u, v) \in T$ and any $(u', v) \in E \setminus T$ such that v is not an ancestor of u' in T, $w(u, v) \leq w(u', v)$. (See slide 7 of the MDST presentation.)

Exercise 1.5 Give linear time reductions between the problems of finding a MDST rooted at a specified vertex r, a MDST rooted at an arbitrary vertex, and the problem of finding a maximal branching. (See slide 23 of the MDST presentation.)