Problem Set no. 1 — Minimum spanning trees

Given: November 11, 2017
Due: November 23, 2017

Exercise 1.1 Prove the correctness of the blue and red rules. (Prove the Theorem in slide 25 of the MST presentation.)

Exercise 1.2 Prove that a spanning tree T is an MST if and only if it is a lexicographically MST. (See slide 18 of the MST presentation.)

Exercise 1.3 Let $G = (V, E, w)$ be a weighted undirected graph and let T be an MST of G. Assume that all the edge weights in G are distinct. Prove that for every $u, v \in V$, the unique path connecting u and v in T is a lexicographically minimal path between u and v in G. (See slide 19 of the presentation.)

Exercise 1.4 Describe a *deterministic* linear time algorithm for finding a spanning tree whose maximal weight is minimal. (Hint: Start by computing the median of the edge weights.)

Exercise 1.5 (a) Describe a simple implementation of Borůvka's algorithm that runs in $O(n^2)$ time. (Hint: Use an adjacency matrix representation of the graph.) (b) Describe a simple implementation of of Borůvka's algorithm that runs in $O(m \log n)$ time. (Hint: Use an adjacency lists representation of the graph, as was essentially done in class.) (c) Combine the two implementations to obtain an implementation that runs in $O(m \log \frac{n^2}{m})$ time. (Hint: Start with the second implementation and switch after a certain number of iterations to the first one. Assume that $n \leq m < n^2$.) (Note that $O(m \log \frac{n^2}{m})$ is better than both $O(n^2)$ and $O(m \log n)$.)

Exercise 1.6 Describe a *deterministic* linear time algorithm algorithm for finding a minimum spanning tree in a *planar* graph. (Hint: a contraction of a planar graph is planar graph. A planar graph on n vertices with no parallel edges has at most $3n - 6$ edges.)