Lecture notes on "Analysis of Algorithms": **Directed Minimum Spanning Trees**

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Solutions to selected exercises

Exercise 1 Give a linear time algorithm that given a directed graph $G = (V, E)$ and a vertex $r \in V$, finds all edges that appear in some DST rooted at r.

Solution to Exercise 1 The solution of this exercise, as it turns out, requires some data structures that were not covered in the course. (The simple solution that I had in mind was flawed.) Here is a sketch of the solution.

An edge (u, v) appears in some DST rooted at r if and only if there is a *simple* path from r to v that ends with the edge (u, v). (If there is such a simple path, then it is easy to extend it to a DST rooted at r.) An equivalent condition is that there is a *simple* path from r to u that does not pass through v, or in other words, if u is *not dominated* by v.

We say that u is *dominated* by v if and only if every path from r to u passes through v. It is not difficult to show that if u is dominated by both v_1 and v_2, then either v_1 dominates v_2, or v_2 dominates v_1. We say that v is the *immediate dominator* of u if and only if every dominator of u is also a dominator of v. It follows from the previous observation that every vertex u has a unique immediate dominator, which might be r. We define the *domination tree* of the graph as a tree in which the parent of each vertex other than r is its immediate dominator. There are linear-time algorithms for constructing the domination tree of a graph with respect to a given root.

Now, v dominates u if and only if v is an ancestor of u in the domination tree. It is possible to preprocess a tree a linear time and then answer every *Lowest Common Ancestor* (LCA) query in constant time. Clearly v is an ancestor of u if and only if $LCA(u, v) = v$.

Open problem: Is there a more elementary solution?

Exercise 8 Let $G = (V, E, w)$ be a weighted directed graph, let $r \in V$ and let T be a MDST rooted at r. Let $r' \in V \setminus \{r\}$ such that G contains a DST rooted at r', and let T' be the subtree of T rooted at r'. Prove or disprove: There is an MDST of G rooted at r' that contains T'.

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Solution to Exercise 8 (Contributed by Yahav Nussbaum who took the course in 2008.) Yes. There is always a MDST of G rooted at r' that contains all the edges of T', the subtree of T rooted at r'. Let $V' = V(T') \setminus \{r'\}$ be the vertices of T', excluding r', and let $A = E(T')$ be the edges of T' entering V'. Let S be a MDST of G rooted at r'. Let B be the edges in S that enter the vertices of V'. If $A = B$, we are done, as S contains all the edges of T'. Assume, therefore, that $A \neq B$. Let $\bar{T} = (T \setminus A) \cup B$ and $\bar{S} = (S \setminus B) \cup A$. We show below that $\bar{T}$ is also a DST of G rooted at r, and that $\bar{S}$ is also a DST of G rooted at r'. As $\bar{S}$ contains T', it is enough to show that $\bar{S}$ is also a MDST rooted at r'. Now $w(\bar{S}) - w(S) = w(A) - w(B) = w(T) - w(\bar{T})$. Thus, if $w(\bar{S}) > w(S)$, then $w(T) > w(\bar{T})$, contradicting the assumption that T is an MDST rooted at r.

We next show that $\bar{T} = (T \setminus A) \cup B$ is a DST rooted at r. As the in-degree of each vertex other than r in $\bar{T}$ is exactly 1, it is enough to show that that every vertex is reachable from r in $\bar{T}$. All vertices of $V \setminus V'$ are reachable from r in $\bar{T}$ using the same paths leading to them in T. Let $v_0 \in V'$. Let $(v_1, v_0) \in \bar{T}$ be the edge entering v_0, let $(v_2, v_1) \in \bar{T}$ be the edge entering v_1, and so on. If $v_i \in V'$, then $(v_{i+1}, v_i) \in B$. As B does not contain cycles, as it is part of S, we must eventually reach a vertex $v_i \notin V'$ and we already know that such vertices are reachable from r.

Finally, we show that $\bar{S} = (S \setminus B) \cup A$ is a DST rooted at r'. The proof here is similar, with the roles of V' and $V \setminus V'$ reversed. All vertices in V' are reachable from r' in $\bar{S}$ as they are reachable from r' in T'. Let $v_0 \notin V'$, and let $(v_1, v_0), (v_2, v_1), \ldots$ be edges obtained by following incoming edges from v_0. As S is acyclic, by following these edges we must eventually reach r' or a vertex of V'.