Problem Set no. 4

Given: January 18, 2021
Due: January 28, 2021

Exercise 4.1 Obtain an upper bound on the number of improving switches performed by the random selection algorithm. (See slide 9 of lecture 6.) More specifically, let $g(m,n)$ be a function that satisfies the following recurrence relation: $g(m,n) = g(m-2, n-1) + \frac{1}{n}\sum_{i=1}^{n}(1+g(m-i,n))$, if $m \geq 2n$, $g(m,n) = g(2(m-n), m-n)$, if $m < 2n$, and $g(m,0) = 0$. Prove by induction that $g(m,n)$ is an upper bound on the number of improving switches performed by the algorithm on any game with n states and m actions. (Hint: For any action $a \in A$, let $\mathit{Val}^+(a)$ be the value of the optimal strategy that uses a. Let $a_1, a_2, \ldots, a_n$ be the actions in the current strategy such that $\mathit{Val}^+(a_1) \leq \mathit{Val}^+(a_2) \leq \cdots \leq \mathit{Val}^+(a_n)$. Continue along the lines of the analysis of the random removal algorithm.)

Exercise 4.2 Let $f(m,n) = f(m-1,n) + \frac{1}{m-n}\sum_{i=1}^{n}(1+f(m-2i, n-i))$, if $m \geq 2n$, $f(2n-1,n) = f(2n-2, n-1)$, $f(m,0) = 0$ be the recurrence relation obtained by analysing the random removal algorithm. (See slide 10 of lecture 6.) (a) What is $f(m,1)$? (b) Estimate the asymptotic behavior of $f(m,2)$.

Exercise 4.3 Show that the following algorithm is a non-recursive version of the random removal algorithm given in class: Choose a random permutation $a_1, a_2, \ldots, a_{m-n}$ on the actions that are not used in the initial strategy σ. For $i = 1, 2, \ldots, m-n$ check whether a_i is an improving switch with respect to σ. If no improving switch is found, then σ is an optimal strategy. Otherwise, if a_i is the first improving switch, let $\sigma' = \sigma[a_i]$, and let a' be the action in σ that a_i replaced. Let $a'_1, a'_2, \ldots, a'_i$ be a random permutation of $a_1, a_2, \ldots, a_{i-1}, a'$. Continue in the same manner with σ' and the permutation $a'_1, a'_2, \ldots, a'_i, a_{i+1}, \ldots, a_{m-n}$ on the actions not used by σ'.

Exercise 4.4 Prove that the product construction does indeed produce an (A)USO. (See slide 17 of lecture 7.)

Exercise 4.5 State formally the hypersink replacement lemma and prove it. (See slide 18 of lecture 7.)

Exercise 4.6 Let $A : \{0,1\}^n \to \{0,1\}^n$ be an *acyclic* orientation of the n-cube, where $n \geq 2$. Prove that A is an n-AUSO if and only if every 2-dimensional subcube of A has a unique sink.