# Solution of Problem Set no. 3

**Exercise 3.1** Show that any sorting network is equivalent to a sorting network in *standard form* with the same number of comparators.

**Solution 3.1** A general comparator network is an acyclic graph with nodes of three different types: (a) *sources*, which have indegree 0 and outdegree 1; (b) *comparators*, which have indegree 2 and outdegree 2; (c) *sinks*, which have have indegree 0 and outdegree 2. The two outputs of each comparator as marked as min or max. The sources are of course the inputs of the network and the sinks are the outputs of the network. It is easy to see that the number of outputs is equal to the number of inputs.
We now prove by induction on the number of comparators in the network, that each general sorting network is equivalent to a standard comparator network, as defined in class. (Briefly, a standard comparator network with $n$ inputs and outputs is composed of $n$ lines. Each comparator is between lines $i$ and $j$, where $i < j$, where the minimum of the compared items is output on line $i$ and the maximum of line $j$.) We can arbitrarily choose the order of the $n$ inputs. The order of the $n$ outputs is then uniquely dictated.
Here is the inductive argument. If there are $n$ inputs and outputs but no comparators, then the standard network is simply composed of $n$ lines. Consider now a network $N$ with $k$ comparators and suppose that the claim holds for all networks with less than $k$ comparators. Let $N'$ be a network obtained from $N$ by removing a comparator whose two outgoing edges feed output nodes. (Consider, for example, the last comparator in a topological ordering of the nodes of the network.) The two edges that feed this comparator feed two output nodes in $N'$. By the induction hypothesis, there is a standard sorting network which is equivalent to $N'$. Suppose that the comparator we removed from $N'$ compares outputs $i$ and $j$, where $i < j$, of this standard network. To get a standard comparator network equivalent to $N$, simply add a comparator between lines $i$ and $j$. The output node fed by the outgoing edge of the comparator marked with min corresponds to line $i$, while the output fed by max corresponds to line $j$.
Consider now a general sorting network. The inputs of the networks are numbered 1 to $n$ is some arbitrary manner. The outputs are also numbered 1 to $n$ in a non-arbitrary manner. Output $i$ is the output in which the $i$-th largest item is guaranteed to end up. By the argument above, we know that the comparator network is equivalent to a standard network. It remains to show, that line $i$ does indeed correspond to the $i$-th output of the original network. To see this, feed the sorting network with the identity permutation. No swaps are performed, so the $i$-th largest item ends up on the $i$-th line. Thus, the $i$-th line does indeed correspond to the $i$-th output of the sorting network.

**Exercise 3.2** (a) Prove, using the 0-1 principle, that Batcher's Bitonic Sorter, for $n = 2^k$, sorts any *cyclic* shift of a bitonic sequence. Justify the use of the 0-1 principle. (b) Prove that Batcher's Bitonic Sorter, for $n = 2^k$, sorts any *cyclic* shift of a bitonic sequence *without* using the 0-1 principle.

**Solution 3.2** See the chapter 27 of the 2nd edition of "Introduction to Algorithms", by Cormen et al. (This chapter was removed from the 3rd edition...)

**Exercise 3.3** (a) Show that if items fed into $k$ input wires may end up in a given output wire, then the depth of the network is at least $\lg k$. (b) Show that the depth of an *halver* of $n$ items must be at least $\lg(\frac{n}{2}+1)$. (c) Show that the depth of network that merges two sorted sequences of $n$ items each is at least $\lg(2n)$.

**Solution 3.3** (a) Define the depth of a comparator in a network to be the length of the longest path leading to it from an input. The depth of a network is then the maximum death of a comparator in the network. We can easily prove by induction that a comparator at depth $d$ can be reached from at most $2^d$ of the inputs. This clearly holds for $d=0$. A comparator of depth $d$ is fed by outputs of two comparators at depths at most $d-1$. Thus, the number of inputs that can reach it is at most $2 \cdot 2^{d-1} = 2^d$, as claimed. It follows that if items fed into $k$ different input wires may end up in a given output wire, than the depth of the comparator which outputs this wire is at least $\lg k$.
(b) In a halving network, each output wire must be reachable from at least $n/2+1$ different input wires. Suppose, for the sake of contradiction, that this not the case. Suppose, for concreteness, that the wire is one of the output wires on which one of the smallest $n/2$ items should end up. If at most $n/2$ input wires may reach this output wire, then if all these wires contain items which are among the $n/2$ largest, then our output wire will also contain one the $n/2$ largest items, showing that the network in not a halver.
(c) Consider the $n$-th (or $n+1$-st) output of a network that merges two sorted sequences $a_1 < a_2 < \ldots < a_n$ and $b_1 < b_2 < \ldots < b_n$. For every $1 \le i \le n$, $a_i$ may be the $n$-th largest item in the merged list, e.g., if $b_1 < b_2 < \ldots < b_{n-i} < a_1 < a_2 < \ldots < a_i < a_{i+1} < \ldots a_n < b_{n-i+1} < \ldots < b_n$. Similarly, each $b_i$ can be the $n$-th largest of the merged sequence. Hence items fed to all $2n$ input wires may end up on the $n$-th output wire and the depth of that wire is at least $\lg(2n)$.

**Exercise 3.4** Prove the following claims needed in the analysis of the AKS network: (a) Show that the "ideal" distribution considered in slide 72 does indeed exist. (More precisely, let $C$ be any node in the tree. Prove that if we sum up the current sizes of all the descendants of $C$ and the specified fractions of the current sizes of the ancestors of $C$, we get exactly the number of items native to $C$.) (b) Show that at the leaves we always have $m \le 2\lfloor \lambda b \rfloor + 1$, so all items at the leaves are sent up. (c) Show that if at a node $B$ we have $b < A$, then all nodes above the level of $B$ are empty and $m$, the number of items in $B$, is even.

**Solution 3.4** (a) Let $m_j$ be the number of items at nodes of depth $j$. Suppose that $C$ is of depth $i$. We need to show that $\sum_{j \not\equiv i \bmod 2} 2^{j-i} m_j = n/2^i$. This follows as $\sum_{j \not\equiv i \bmod 2} 2^{j-i} m_j = 2^{-i} \sum_{j \not\equiv i \bmod 2} 2^j m_j = n/2^i$.
(b) Until the final time unit, the capacity $b$ of all leaves satisfies $b > 1/\lambda$ and thus $1 \le \lfloor \lambda b \rfloor$. As the leaves are at depth $\lg n - 1$, and each leaf contains the same number of items, a leaf may contain at most 2 items. Thus, $m \le 2 \le 2\lfloor \lambda b \rfloor$, as required.
(c) The capacity of all nodes above the level of $B$ is at most $b/A < 1$. By the third invariant, the number of items in each node is at most the capacity of the node. Hence, all these nodes must be empty. Suppose now that $B$ is at depth $i$. The total number of items in the subtree of $B$ is exactly $n/2^i$ which is even. (Recall that $n = 2^\ell$ is a power of 2.) The number of items in descendants of $B$ is even, as there is an even number of descendants at each level. Thus, the number of items in $B$ must also be even.

**Exercise 3.5** In the analysis of the AKS network we assumed that each node uses a $(\lambda', \epsilon, \epsilon)$-separator, where $\frac{\lambda' m}{2} = \lfloor \lambda b \rfloor$. Suppose that we use now a $(\lambda', \epsilon, \epsilon_0)$-separator, where $\epsilon_0 < \epsilon$. Which of the required inequalities on slide 75 change and to what?

**Solution 3.5** The only inequality that changes is the last inequality. The third term changes from $\frac{\varepsilon}{2A}$ to $\frac{\varepsilon_0}{2A}$.