Problem Set no. 1

Given: December 1, 2016
Due: December 15, 2016

Exercise 1.1 Show that matrix *multiplication* and matrix *squaring* have the same asymptotic complexity.

Exercise 1.2 Give an $O(n^2)$-time randomized algorithm for checking whether $C = AB$, where A, B, C are three $n \times n$ matrices with, say, integer entries. The algorithm should always say 'yes' if $C = AB$, and should say 'no' with a probability of at least 1/2 if $C \neq AB$.

Exercise 1.3 (a) Give an $O(n^\omega)$-time algorithm for finding a triangle (a 3-cycle), if there exists one, in a directed graph on n vertices. (b) Give an $O(n^\omega)$-time algorithm for finding a simple 4-cycle. (c) (Bonus) Give an $O(n^\omega)$-time algorithm for finding a simple 5-cycle. (A cycle is *simple* if all vertices on it are distinct.)

Exercise 1.4 Let $G = (V, E)$ be a graph and let $w : E \to \mathbb{N}$ be an integer weight function. Suppose that G has a *unique* perfect matching M of minimum weight. Let A be the Tutte matrix of G in which $x_{i,j}$ is replaced by $2^{w_{i,j}}$, where $w_{i,j}$ is the weight of the edge $\{i, j\} \in E$. (a) Show that $2^{2W} | \det(A)$, but that $2^{2W+1} \not| \det(A)$. (b) Show that $\{i, j\} \in M$ iff $2^{w_{i,j}} \det(A^{ij})/2^{2W}$ is odd.

Exercise 1.5 Let $G = ([n], E)$ be a directed graph. The *symbolic adjacency matrix* of G is a matrix $A = (a_{i,j})$, where $a_{i,j} = x_{i,j}$, if $\{i, j\} \in E$, and $a_{i,j} = 0$, otherwise, where the $x_{i,j}$'s are indeterminants. Also, $a_{i,i} = 1$, for every $i \in [n]$. Show that there is a directed path from i to j in G iff $(\operatorname{adj}(A))_{i,j} \not\equiv 0$.

Exercise 1.6 (Bonus) Give a randomized polynomial time algorithm for the *exact* matching problem.