Problem Set no. 5

Given: May 27, 2018
Due: June 15, 2018

Exercise 5.1 Suppose we run the CDCL SAT solver defined in class on the formula

$$(a \vee b)(\bar{b} \vee c \vee d)(\bar{b} \vee e)(\bar{d} \vee \bar{e} \vee f)(\bar{a} \vee g)(b \vee \bar{g})(\bar{h} \vee j)(\bar{i} \vee k) \ .$$

Also assume that the order of the variables is fixed to be $c, f, h, i, a, b, d, e, g, h, j, k$ and that we always assign 0 when we take a decision. Describe the computation performed, which clauses are learnt and in which order.

Exercise 5.2 Prove by induction on the number of variables that if a CNF formula is unsatisfiable then we can derive the empty clause by a sequence of resolution steps. (hint: consider $F[x = 0]$ and $F[x = 1]$ for some arbitrary variable x.)

Exercise 5.3 Give a polynomial time algorithm for 2-SAT based on resolution. Describe its implementation precisely and prove a polynomial upper bound on its running time. It is easy to get an $O(n^3)$ time algorithm (and suffices to get the maximum grade for this question). Note that it is also possible to do this in $O(m)$ time, where m is the number of clauses.

Exercise 5.4 Prove that the success probably of Schöning algorithm (slide 3) for an instance of k-SAT is $1/(2 - 2/k)^n$. Repeat the steps of the proof given in class for 3-SAT and indicate what are the required modifications.

Exercise 5.5 Consider the following algorithm for 2-SAT. Start with an arbitrary assignment. Repeat the following step $2n^2$ times: Pick an arbitrary unsatisfied clause, pick uniformly at random one of the literals of the clause and flip the value of the corresponding variable, terminate if the formula is satisfiable. If the loop has not terminated before running through it $2n^2$ times then assert that the formula is not satisfiable.
a) Prove that if the formula is satisfiable then the expected number of steps until we find a satisfying assignment is at most n^2. (Hint: Consider a random walk on the integers $0, 1, \ldots, n$. From n we go to $n - 1$ with probability 1. When at 0 we stay at 0. When we are at j we go to $j + 1$ with probability $1/2$ and to $j - 1$ with probability $1/2$. Define h_j to be the expected number of steps to reach 0 when we start at j. Then $h_0 = 0$, $h_n = h_{n-1} + 1$, and $h_j = (h_{j-1} + 1)/2 + (h_{j+1} + 1)/2$. Find h_j, formulate the connection between this walk and the algorithm.)
b) Use Markov inequality to show that if the formula is satisfiable we find a satisfying assignment with probability at least $1/2$.
c) How would you change the algorithm so that the probability that we mistakenly report that a satisfiable formula is unsatisfiable with probability at most $1/2^k$ for some fixed $k > 1$.