Problem Set no. 4

Given: May 26, 2018
Due: June 8, 2018

Exercise 4.1 a) Let P be an irreducible and aperiodic finite Markov chain and let $A \subseteq S$ be a subset of its states. Prove that if we pick a state according to the stationary distribution and make a step according to P then the probability that we leave a state of A is the same as the probability that we enter a state of A.
b) Consider a bounded queue Q containing at most n elements (it has $n+1$ states according to the number of elements it has). If Q has less than n elements then with probability λ a new element is inserted into Q. If Q is not empty then with probability μ an element leaves Q. Otherwise, Q does not change. Prove that Q has a unique stationary distribution and find it.

Exercise 4.2 Let P be an irreducible Markov chain with n states.
a) For two states x and y, let τ_{xy} be number of steps that we do starting from x until the first time we get to y. Prove that $E(\tau_{xy})$ is finite for any two states x and y.
b) Prove that the stationary distribution of P is unique. (You do not need to prove that P has a stationary distribution.)
(Hint: One way to do this is by proving that the rank of $P - I$ is $n - 1$.)

Exercise 4.3 Let P be a Markov chain obtained from an undirected, non-bipartite, d-regular (all vertices are of the same degree d) and connected graph. (i.e. P picks a neighbor uniformly at random from the d neighbors of v)
a) Prove that P is irreducible and aperiodic.
b) Prove that for any probability distribution x^0, $||x^0 P^t - \pi||_2 \le |\lambda_2|^t$, where π is the stationary distribution of P and λ_2 is the second largest eigenvalue of P in absolute value. ($|| \cdot ||_2$ is the Euclidean L_2 norm).
c) Prove that the mixing time of P is at most $\lceil \log(4\sqrt{n}) / \log(1/|\lambda_2|) \rceil$.

Exercise 4.4 In the Traveling Salesman Problem (TSP) we are given a set $\{1, \dots, n\}$ of n cities and the distances $d(i, j)$ between any pair i, j of cities. Our goal is to find a permutation $\pi_1, \dots . \pi_n$ of the cities that minimized $\sum_{i=1}^{n} d(\pi_i, \pi_{i+1})$ (where we define $\pi_{n+1} = \pi_1$). A popular local search algorithm for TSP, called 2OPT, defines two permutations π^1 and π^2 as neighbors if π^2 can be obtained from π^1 by reversing an interval. I.e. if there exist two indices k and ℓ, $1 \le k < \ell \le n$, such that $\pi^2_j = \pi^1_{k+\ell-j}$ for $k \le j \le \ell$ and $\pi^2_j = \pi^1_j$ for $j < k$ and $j > \ell$. Describe a simulated annealing algorithm for TSP which is based on a random walk on the graph which is defined by this local search scheme. Write down the transition matrix of the underlying chain at a fixed temperature T. Prove that this chain is irreducible.

Exercise 4.5 Let $S = \{s_1, s_2, s_3, s_4\}$ and let $f : S \to R$ be given by $f(s_1) = 1$, $f(s_2) = 2$, $f(s_3) = 0$, $f(s_4) = 2$. Suppose we want to find the minimum of $f(s_i)$ using simulated annealing.
a) Construct the Metropolis chain for the Boltzman distribution with respect to f with parameter T using an underlying chain which is a random walk on the cycle $(s_1, s_2), (s_2, s_3), (s_3, s_4), (s_4, s_1)$ (when at s_i you choose each of your two neighbors with the same probability). Write the transition probabilities for this Metropolis chain.

b) Suppose we set the temperature at step k to be T_k, what is the probability, P_n, that if we start at state s_1, we never leave s_1 during the first n steps.
c) Suppose that $T_k = 1/(2\ln(k+1))$ for $k = 1, 2, \ldots$, what is $\lim_{n\to\infty} P_n$? is it good or bad ?