Problem Set no. 3

Given: May 7, 2016
Due: May 25, 2016, Box 288

Exercise 3.1 Recall that a *group* is a set G with an associative operation $\times$ and an *unit* element $\mathrm{id} \in G$ such that $\mathrm{id} \times g = g \times \mathrm{id} = g$ for all $g \in G$. Furthermore, for each element $g \in G$ there exists an *inverse* $g^{-1} \in G$ such that $g^{-1} \times g = g \times g^{-1} = \mathrm{id}$. Assume G is finite.
Given a distribution μ on G we define a random walk P_μ on G as follows. It is a Markov chain with state space G which moves by multiplying the current state on the left by a random element of G, selected according to μ. That is $P_\mu(g, h \times g) = \mu(h)$ for all $g, h \in G$.
a) Prove that the uniform distribution on G is a stationary distribution of P_μ.
A subset $H \subseteq G$ *generates* G if every element of G can be written as a product of elements of H and their inverses.
b) Prove that P_μ is irreducible iff $S = \{g \in G \mid \mu(g) > 0\}$ generates G.
c) Suppose that $G = S_n$ is the set of permutations of $1, 2, \ldots, n$, and $\times$ is the composition of permutations. Assume μ is a uniform distribution on the transpositions. (A transposition is a permutation that swaps 2 different elements and maps all other elements to themselves.) Is P_μ irreducible ? Is P_μ aperiodic ?

Exercise 3.2 a) Given a chain P which is reversible with respect to a distribution π. Prove that

$$P[X_0 = s_0, X_1 = s_1, \ldots, X_j = s_j] = P[X_0 = s_j, X_1 = s_{j-1}, \ldots, X_j = s_0]$$

if we draw the initial state X_0 according to π.
b) Consider a random walk on an undirected unweighted connected simple graph as defined in class (we go from a vertex to one of its neighbors picked uniformly at random). How many self loops do we need to add to each vertex so that the stationary distribution is uniform.

Exercise 3.3 a) You are given a random walk P on a given graph G as in the previous question. Apply the Metropolis transformation to obtain a Markov chain P' whose stationary distribution on the vertices of G is uniform.
b) Suppose you are given a Gibbs chain P for some distribution π over the functions $f : V \to B$. Now we use P as the underlying chain to obtain a Metropolis chain P' for π. What is the relation between P and P'? Prove your answer.

Exercise 3.4 In the Traveling Salesman Problem (TSP) we are given a set $\{1, \ldots, n\}$ of n cities and the distances $d(i, j)$ between any pair i, j of cities. Our goal is to find a permutation $\pi_1, \ldots . \pi_n$ of the cities that minimized $\sum_{i=1}^{n} d(\pi_i, \pi_{i+1})$ (where we define $\pi_{n+1} = \pi_1$). A popular local search algorithm for TSP, called 2OPT, defines two permutations π^1 and π^2 as neighbors if π^2 can be obtained from π^1 by reversing an interval. I.e. if there exist two indices k and ℓ, $1 \le k < \ell \le n$, such that $\pi^2_j = \pi^1_{k+\ell-j}$ for $k \le j \le \ell$ and $\pi^2_j = \pi^1_j$ for $j < k$ and $j > \ell$. Describe a simulated annealing algorithm for TSP which is based on a random walk on the graph which is defined by this local search scheme. Write down the transition matrix of the underlying chain at a fixed temperature T. Prove that this chain is irreducible.

Exercise 3.5 Let $S = \{s_1, s_2, s_3, s_4\}$ and let $f : S \to R$ be given by $f(s_1) = 1$, $f(s_2) = 2$, $f(s_3) = 0$, $f(s_4) = 2$. Suppose we want to find the minimum of $f(s_i)$ using simulated annealing.

a) Construct the Metropolis chain for the Boltzman distribution with respect to f with parameter T using an underlying chain which is a random walk on the cycle $(s_1, s_2), (s_2, s_3), (s_3, s_4), (s_4, s_1)$ (when at s_i you choose each of your two neighbors with the same probability). Write the transition probabilities for this Metropolis chain.
b) Suppose we set the temperature at step k to be T_k, what is the probability, P_n, that if we start at state s_1, we never leave s_1 during the first n steps.
c) Suppose that $T_k = 1/(2\ln(k+1))$ for $k = 1, 2, \ldots$, what is $\lim_{n\to\infty} P_n$? is it good or bad ?