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AN OPERATIONAL CALCULUS CONNECTED WITH PERIODIC SPLINES

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Periodic splines of defect 1 with equally spaced nodes have a dual convolution nature. On the one hand, they form a semigroup with respect to continuous convolution. On the other hand, if such a spline is written as a linear combination of b-splines, then it can be regarded as a discrete convolution. This consideration led to the construction, on the basis of such splines, of an operational calculus that is an adequate mathematical apparatus for solving many problems in which various forms of convolution appear, and in which the information about the objects under study is in discrete form.

We introduce some notation. Suppose that $w = e^{-2\pi i/N}$, is an even number, $v_n = 2\sin(\pi n/N)$, and $V_n = \sin(\pi n/N)/(\pi n/N)$. The discrete Fourier transform (DFT) of a vector $\mathbf{a} = \{a_k\}_0^{N-1}$ is $T_n(a) = N^{-1}\sum_k w^{-nk}a_k$. Here and below, $\sum_k$ stands for $\sum_{k=0}^{N-1}$. The norm of a vector $\mathbf{a}$ is

$$\|\mathbf{a}\| = \left(N^{-1}\textstyle\sum_k a_k^2\right)^{1/2}. \tag{1}$$

We mention some known properties of the DFT:

$$a_k = \textstyle\sum_n w^{nk} T_n(\mathbf{a}); \tag{2}$$

$$N^{-1}\textstyle\sum_k a_k b_k = \sum_n T_n(\mathbf{a})\overline{T_n(\mathbf{b})} \Rightarrow \|\mathbf{a}\|^2 = \sum_n |T_n(\mathbf{a})|^2. \tag{3}$$

The discrete convolution of a vector $\mathbf{a}$ with a vector $\mathbf{b}$ and its DFT are:

$$\mathbf{a} * \mathbf{b} = \left\{N^{-1}\textstyle\sum_k a_{l-k} b_k\right\}, \qquad T_n(\mathbf{a} * \mathbf{b}) = T_n(\mathbf{a})T_b(\mathbf{b}). \tag{4}$$

We now introduce two nets on the x-axis:

$$\Xi = \{x_k = k/N\} \quad \text{and} \quad \Xi^p = \{x_k^p = (k + p/2)/N\}.$$

We denote by $\mathfrak{S}^p$ the space of 1-periodic splines of degree $p-1$ and defect 1 with nodes at the points x_k^p; and by $M^p(x)$ the central 1-periodic b-spline of degree $p-1$ [1]:

$$M^p(x) = \sum_{n=-\infty}^{\infty} V_n^p e^{2\pi i n x}. \tag{5}$$

REMARK 1. The definition of the b-spline $M^p(x)$ given by (5) can be extended to arbitrary integer values of p. Here if $p < 0$, then $M^p(x)$ is a generalized function [2]. In particular, $M^0(x)$ is the periodically extended Dirac δ-function. Let

$$m^p = \{M^p(x_k)\}_0^{N-1}, \qquad u_n^p = T_n(m^p) = N^{-1}\textstyle\sum_k w^{-nk} M^p(x_k). \tag{6}$$

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The functions u_n^p were studied in [1], [3], and [4]. Recursion formulas and explicit representations for the initial values of p are known. For us it is important that

(7) $$0 < \kappa_{p-1} = u_{N/2}^p \leq u_n^p \leq u_0^p = 1\,, \qquad \kappa_p = \mathrm{K}_p(2/\pi)^p\,,$$

where K_p is the Favard constant.

Every spline in $\mathfrak{S}^p$ can be represented in the form

(8) $$S^p(x) = N^{-1}\textstyle\sum_k q_k M^p(x - x_k)\,.$$

REMARK 2. Formula (8) can be extended also to values $p \leq 0$ according to Remark 1. Furthermore, it can be understood as the definition of a spline of nonpositive order.

It is clear from (8) that the spline $S^p(x)$ is completely determined by its order p and by the vector $\mathbf{q} = \{q_n\}_0^{N-1}$ of its coefficients. Let $\{T_n(q)\}_0^{N-1} = \mathbf{Q}(S^p)$, and call this vector the *image* of the spline $S^p(x)$.

The correspondence $S^p(x) \Leftrightarrow \mathbf{Q}(S^p)$ is one-to-one: given the vector $\mathbf{Q}$, it is easy to recover the spline $\mathfrak{S}^p$ with the help of (2) and (8). The passage from the space $\mathfrak{S}^p$ to $\mathbf{Q}(\mathfrak{S}^p)$ leads to a distinctive operational calculus. We present the basic formulas for this operational calculus. Let $s^p = \{S^p(x_k)\}_0^{N-1}$. We have

(9) $$T_n(s^p) = T_n(\mathbf{q})u_n^p\,.$$

THEOREM 1. *If $S^p \in \mathfrak{S}^p$, then its derivative satisfies $(S^p)^{(s)} \in \mathfrak{S}^{p-s}$:*

(10) $$S^p(x)^{(s)} = N^{-1}\textstyle\sum_k q_k^s M^{p-s}(x - x_k)\,.$$

Further, with the notation $s^{s,p} = \{(S^p(x_k))^{(s)}\}_0^{N-1}$, the following relations hold:

(11) $$T_n(\mathbf{q}^s) = T_n(\mathbf{q})(iNv_n)^s\,, \qquad T_n(s^{s,p}) = T_n(\mathbf{q})(iNv_n)^s u_n^{p-s}\,.$$

Suppose that $S^l \in \mathfrak{S}^l$:

(12) $$S^l(x) = N^{-1}\textstyle\sum_k r_k M^l(x - x_k)\,.$$

The integral

$$f * g(x) = \int_0^1 f(x-y)g(y)\,dy$$

will be called the *convolution* of two 1-periodic functions.

THEOREM 2. *The convolution of two splines $S^l \in \mathfrak{S}^l$ and $S^p \in \mathfrak{S}^p$ is a spline $S^{p+l} \in \mathfrak{S}^{p+l}$:*

(13) $$S^{l+p}(x) = N^{-1}\textstyle\sum_k j_k M^{l+p}(x - x_k)\,.$$

Further, with the notation $s^{l+p} = \{S^{l+p}(x_k)\}_0^{N-1}$ the following relations hold:

(14) $$T_n(\mathbf{j}) = T_n(\mathbf{q})T_n(\mathbf{r})\,, \qquad T_n(\mathbf{s}^{l+p}) = T_n(\mathbf{q})T_n(\mathbf{r})u_n^{p+l}\,.$$

REMARK 3. Comparing (11) and (14), we can see that the operation of s-fold differentiation can be regarded as the convolution of the spline S^p with the spline

(15) $$D^s(x) = N^{-1}\textstyle\sum_k d_k^s M^{-s}(x - x_k)\,, \qquad T_n(\mathbf{d}^s) = (iNv_n)^s\,.$$

We have formulas of the Parseval equality type.

THEOREM 3. *Suppose that* $S^l \in \mathfrak{S}^l$ *and* $S^p \in \mathfrak{S}^p$ *are splines defined by* (8) *and* (12). *Then*

$$\int_0^1 S^p(x)S^l(x)\,dx = \sum_n T_n(\mathbf{q})\overline{T_n(\mathbf{r})}u_n^{p+l}. \tag{16}$$

For the convolution,

$$\int_0^1 S^p(x) * S^l(x))^2\,dx = \sum_n |T_n(\mathbf{q})T_n(\mathbf{r})|^2 u_n^{2(p+l)}. \tag{17}$$

Hence, in particular

$$\int_0^1 (S^p(x)^{(s)})^2\,dx = \sum_n |T_n(\mathbf{q})| u_n^{2(p-s)}(Nv_n)^{2s}. \tag{18}$$

Discrete "Parseval equalities" can be obtained from (11) and (14).

THEOREM 4. *Suppose that* $S^l \in \mathfrak{S}^l$ *and* $S^p \in \mathfrak{S}^p$ *are splines defined by* (8) *and* (12). *Then*

$$N^{-1}\sum_k S^p(x_k)S^l(x_k)\,dx = \sum_n T_n(\mathbf{q})\overline{T_n(\mathbf{r})}u_n^p u_n^l. \tag{19}$$

Hence, in particular,

$$N^{-1}\sum_k (S^p(x_k)^{(s)})^2 = \sum_n |T(\mathbf{q})|(u^{p-s})^2(Nv)^{2s}. \tag{20}$$

We consider some typical problems whose solutions use the formulas above.

PROBLEM 1 (the basic one). Splines $S^m(x) \in \mathfrak{S}^m$ are given,

$$S^m(x) = N^{-1}\sum_k t_k M^m(x - x_k), \tag{21}$$

along with a spline $S^l(x) \in \mathfrak{S}^l$, given by (12), and a vector $\mathbf{z} = \{z_k\}_0^{N-1}$.

It is required: a) to find a spline $S^p(x) \in \mathfrak{S}^p$ minimizing the functional

$$I(S^p) = \int_0^1 (S^m(x) * S^p(x))^2\,dx$$

under the condition

$$E(S^p) = N^{-1}\sum_k (S^l * S^p(x_k) - z_k)^2 \le \varepsilon^2;$$

b) to find a spline $S^p(x) \in \mathfrak{S}^p$ minimizing the functional

$$J_\rho(S^p) = E(S^p) + \rho I(S^p);$$

c) to find a spline $S^p(x) \in \mathfrak{S}^p$ satisfying the conditions

$$S^l * S^p(x_k) = z_k, \qquad k = 0, \dots, N-1.$$

SOLUTION OF PROBLEM 1b). Let $\mathbf{r} = \{r_k\}_0^{N-1}$ and $\mathbf{t} = \{t_k\}_0^{N-1}$. Then the solution of Problem 1b) is the spline

$$S_\rho^p(\mathbf{z}, x) = N^{-1}\sum_k q_k(\rho)M^p(x - x_k), \qquad \mathbf{q}(\rho) = \{q_k(\rho)\}_0^{N-1}, \tag{22}$$

$$T_n(\mathbf{q}(\rho)) = \frac{\overline{T_n(\mathbf{r})}T_n(\mathbf{z})u_n^{p+l}}{A_n(\rho)},$$

$$A_n(\rho) = \rho|T_n(\mathbf{t})|^2 u_n^{2(p+m)} + |T_n(\mathbf{r})|^2(u_n^{p+l})^2. \tag{23}$$

SOLUTION OF PROBLEM 1a). Let $e(\rho) = E(S_\rho^p)$, $O_N =$ the set of indices n such that $T_n(\mathbf{r}) = 0$ and $m_N(\mathbf{z}) = \sum_{n\in O_N} |T_n(\mathbf{z})|^2$.

LEMMA 1. *The function*

$$e(\rho) = \sum_n \frac{\rho^2 |T_n(\mathbf{t}) T_n(\mathbf{z}) u_n^{2(p+m)}|^2}{A_n(\rho)^2}$$

is strictly monotonically increasing, and $e(0) = m_N(\mathbf{z})$ *and* $\lim_{\rho\to\infty} e(\rho) = \|\mathbf{z}\|^2$.

THEOREM 5. *Problem* 1a) *has a solution for any value of* ε *with* $m_N(\mathbf{z}) \le \varepsilon^2 \le \|\mathbf{z}\|^2$. *This solution is given by the spline* $S_\rho^p(\mathbf{z}, x)$ *constructed according to* (22) *and* (23), *and the value of the parameter* ρ *is found from the equation* $e(\rho) = \varepsilon^2$.

A solution of Problem 1c) exists for any vector $\mathbf{z}$ if $O_N = \varnothing$. This solution is given by the spline $S_0^p(\mathbf{z}, x)$, and for it

$$T(\mathbf{q}(0)) = T(\mathbf{z})(u^{p+1l} T(\mathbf{r}))^{-1}.$$

PROBLEM 2. a) Find a spline $S^p(x) \in \mathfrak{S}^p$ minimizing the functional

$$I(S^p) = \int_0^1 (S^p(x)^{(m)})^2\, dx$$

under the condition

$$E(S^p) = N^{-1} \sum_k (S^p(x_k - z_k) \le \varepsilon^2.$$

b) Find a spline $S^p(x) \in \mathfrak{S}^p$ minimizing the functional

$$J_\rho(S^p) = E(S^p) + \rho I(S^p).$$

c) Find a spline $S^p(x) \in \mathfrak{S}^p$ satisfying the conditions

$$S^p(x_k) = z_k, \qquad k = 0, \dots, N-1.$$

SOLUTION OF PROBLEM 2b). This solution is given by the spline $S_\rho^p(\mathbf{z}, x)$ constructed according to (22), with $l = 0$, $T_n(\mathbf{r}) = 1$, and

$$A_n(\rho) = \rho |N v_n|^{2m} u_n^{2(p-m)} + (u_n^p)^2.$$

SOLUTION OF PROBLEM 2a). Since $T_n(\mathbf{r}) = 1$, it follows that $m_N(\mathbf{z}) = 0$, and the problem has a unique solution $S_\rho^p(\mathbf{z}, x)$ that can be found in a way analogous to that for the solution of Problem 1a) for all $\varepsilon \le \|\mathbf{z}\|$.

A solution of Problem 2c) also exists for any vector $\mathbf{z}$. This solution is given by the spline $S_0^p(\mathbf{z}, x)$, and for it $T_n(\mathbf{q}(0)) = T_n(\mathbf{z})/u_n^p$.

REMARK 4. The spline $S_\rho^p(\mathbf{z}, x)$ is an interpolation spline for the vector $\mathbf{z}$. Splines of type $S_\rho^p(\mathbf{z}, x)$ are said to be *smoothing*. This concept first appeared in [5]. In the same place it was established that for $p = 2m$ the spline $S_\rho^{2m}(\mathbf{z}, x)$ is a solution of the following problem (see also [6] and [7]).

PROBLEM 3. a) Find a function $f \in \widehat{W}_2^m$ that minimizes the functional

$$I(f) = \int_0^1 (f(x)^{(m)})^2\, dx$$

under the condition

$$E(f) = N^{-1} \sum_k (f(x_k) - z_k)^2 \le \varepsilon^2.$$

b) Find a function $f \in \widehat{W}_2^m$ that minimizes the functional $J_\rho(f)$. Here $\widehat{W}_2^m$ is the intersection of the space $W_2^m(0, 1)$ with the space of 1-periodic functions. Note that for $p = 2m$

$$T_n(\mathbf{q}(\rho)) = (\rho (N v_n)^{2m} + u_n^{2m})^{-1} T_n(\mathbf{z}).$$

REMARK 5. The explicit formulas established above for smoothing and interpolation splines allow the construction of these splines without resorting to the solution of systems of equations. Here it is natural to use fast Fourier transform algorithms. The volume of computations hardly increases at all as the degree of the spline increases. We note also that the explicit representation permitted a complete investigation of the approximation and smoothing properties of splines. Here essential use was made of results in [8].

We now consider an approximate solution of a convolution integral equation. This is a classical example of an ill-posed problem. We present an algorithm for solving this problem that is stable and very efficient for computations.

PROBLEM 4. Suppose that $h \in \widehat{W}_2^l$, $f \in \widehat{W}_2^m$, and $g(x) = f * h(x)$. We have at our disposal the following reference vectors on the interval $[0, 1]$: $\mathbf{z} = \{z_k = g(x_k) + e_k\}_0^{N-1}$, $x_k = k/N$; $\mathbf{h} = \{h(x_k)\}_0^{N-1}$; and $\mathbf{e} = \{e_k\}_0^{N-1}$, a vector of random errors about which it is known that $\|\mathbf{e}\| \leq \varepsilon$. It is required to construct a family of functions $f_\varepsilon(N, x) \in W_2^m$ such that $f_\varepsilon(N, x) \to f(x)$ in the norm of $\widehat{W}_2^{m-1}$ as $\varepsilon \to 0$ and $N \to \infty$.

SOLUTION OF PROBLEM 4. Under the conditions of Problem 1 let $p = 2m$ and $S^l(x) = S_0^l(\mathbf{h}, x)$, where $S_0^l(\mathbf{h}, x)$ is an interpolation spline constructed according to the scheme of Problem 2c) from the data $\mathbf{h}$. Let

$$I(S^p) = \sum_{i=0}^{m} \int_0^1 (S^p(x)^{(i)})^2 \, dx.$$

Then a solution of Problem 1b) is given by a spline $S_\rho^{2m}(N, \mathbf{z}, x) \in \mathfrak{S}^{2m}$ such that in (22)

$$A_n(\rho) = \rho \sum_{i=0}^{m} u_n^{2(p-i)} (N v_n)^{2i} + |T_n(\mathbf{r})|^2 (u_n^{p+l})^2, \qquad T_n(\mathbf{r}) = T_n(\mathbf{h})/u_n^l.$$

PROPOSITION 1. *There is a constructive algorithm for choosing a parameter $\rho = \rho(\varepsilon, N)$ such that the family of splines $S_\rho^{2m}(N, \mathbf{z}, x)$ is a solution of Problem* 4.

REMARK 6. In practical applications, problems of the type of Problem 4 are most often encountered on the "intervals" $(-\infty, \infty)$ and $(0, \infty)$. The algorithm can be modified in the standard way for these cases.

Problem 1 is a basis for solving a whole series of direct and inverse problems connected with the solution of differential equations and difference equations with constant coefficients and integro-differential equations, and of systems of such equations. We consider one such problem.

PROBLEM 5. Suppose that the function $u(x, t)$ is a solution of the initial-value boundary-value problem for the heat equation $u''_{xx} = u'_t$ for $x \in [0, 1]$ under the conditions $u(0, t) = u(1, t)$, $u(0, t)'_t = u(1, t)'_t$, and $u(x, \tau) = g(x)$. Notation: $f(x) = u(x, T)$, $f \in \widehat{W}_2^r$, $r \geq 3$, where $T \geq 0$ and $\tau \geq 0$ are certain given numbers. We have at our disposal the following reference vector on the interval $[0, 1]$: $\mathbf{z} = \{z_k = g(x_k) + e_k\}_0^{N-1}$, $x_k = k/N$; and $\mathbf{e} = \{e_k\}_0^{N-1}$, a vector of random errors about which it is known that $\|\mathbf{e}\| \leq \varepsilon$. It is required to construct a family of continuous functions $f_\varepsilon(N, x)$ such that $f_\varepsilon(N, x) \to f(x)$ uniformly with respect to $x \in [0, 1]$ as $\varepsilon \to 0$ and $N \to \infty$.

We remark that Problem 5 is stable for $T > \tau$, while for $T < \tau$ this problem becomes ill-posed and requires regularization. This is an example of an inverse problem.

A problem of similar type was treated in [9]. We formulate an auxiliary problem.

PROBLEM 6. Find a spline $S^p_\rho(\mathbf{z}, x, t) \in \mathfrak{S}^p(x)$,

$$S^p_\rho(z, x, t) = N^{-1}\textstyle\sum_k q_k(t) M^p(x - x_k), \qquad \mathbf{q}(t) = \{q_k(t)\}_0^{N-1} \tag{24}$$

satisfying the conditions

$$S^p_\rho(\mathbf{z}, x_k, t)'_t = S^p_\rho(\mathbf{z}, x_k, t)''_{xx}, \qquad k = 0, \dots, N-1,$$

and minimizing the functional $J_\rho(S^p_\rho) = E(S^p_\rho) + \rho I(S^p_\rho)$, where

$$I(S^p_\rho) = \int_0^1 (S^p_\rho(\mathbf{z}, x, T)^{(m)})^2\,dx, \qquad E(S^p_\rho) = N^{-1}\textstyle\sum_k (S^p_\rho(\mathbf{z}, x_k, \tau) - z_k),$$

$$m < p, \qquad \tau, T \ge 0.$$

A solution of Problem 6 is given by the spline $S^p_\rho(\mathbf{z}, x, t)$ from (24), and $T_n(\mathbf{q}(t)) = B_n \exp(-G_n t)$, $G_n = (Nv_n)^2 u_n^{p-2}/u_n^p$, and

$$B_n = \frac{T_n(z) u_n^p \exp(-\tau G_n)}{A_n(\rho)}$$

$$A_n(\rho) = \rho |Nv_n|^{2m} u_n^{2(p-m)} \exp(-2G_n T) + (u_n^p)^2 \exp(-2G_n \tau).$$

PROPOSITION 2. *There is a constructive algorithm for choosing the parameter* $\rho = \rho(\varepsilon, N)$ *such that the family of splines* $S^p_\rho(\mathbf{z}, x, T) \in \mathfrak{S}^p$, $p = 2r$, *is a solution of Problem* 5.

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STATISTICAL ANALYSIS OF THE STAR CATALOGUE "ALMAGEST"

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1. Goal of the investigation. The numerical data resulting from astronomical observations and collected in Ptolemy's *Almagest* has been of great interest to investigators (see [1] and [2]). The contradictory character of this data has even given a basis for doubting the authenticity of the measurements carried out for it [1]. In our paper we consider the material contained only in the *Almagest*. We propose a method of analysis based on the detection of various types of errors committed by the observer. The method not only removes many of the contradictions, but also enables us to determine the time interval in which the star catalogue was compiled.

2. A preliminary analysis of the catalogue. The catalogue contains 1025 stars, whose coordinates (the ecliptic longitude and latitude) are represented with an accuracy of $10'$ (the accuracy claimed by its compiler). All the stars are grouped according to constellations, arranged in a certain sufficiently natural order. The version of the catalogue presented in the classical work of Peters and Knobel [3] was analyzed; [3] also contains results on identification of stars in the *Almagest* with stars in the modern sky, along with the real errors in determining the coordinates of the stars. Far from all the stars in the catalogue have been identified with certainty. The errors in the coordinates of some stars are too large (1° or more). Such stars are not very informative. At the preliminary analysis stage the catalogue was "cleansed" of such "doubtful" stars. This cleansing was realized both on the basis of results of the analysis presented in [3] and on the basis of computations newly carried out by us. The "cleansed" catalogue contained 864 stars in all, and was subjected to further statistical investigation. It should only be mentioned that 2 of the 12 known stars of the catalogue fell among the stars eliminated: Canopus and Provindemiatrix.

NOTATION. l_i and b_i are the ecliptic longitude and latitude of the ith star in the "cleansed" catalogue. Let $L_i(t)$ and $B_i(t)$ be the true values of the longitude and latitude of this star in the year t, which are easy to find (using a computer) by starting out from the contemporary coordinates of the stars and using the very precise formulas of Newcomb and Kinoshita [4].

The thorough statistical analysis carried out in [1] showed that the longitudes l_i may have been not direct results of measurements, but of certain special recalculations (for example, due to a desire to "reduce" the catalogue to this or that date). Therefore, the subsequent analysis related the accuracy properties of the latitude coordinates alone. It turned out that these properties enable us to distinguish the groups of "well" and "poorly" measured stars, and to obtain an interval of possible dating of the *Almagest*.

We remark that the "original" mean-square accuracy

$$\sigma = \left[\frac{1}{N}\sum_{i=1}^{N}(b_i - B_i(t))^2\right]^{1/2}$$

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