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ASYMPTOTIC FORMULAS FOR LOCAL SPLINE APPROXIMATION ON A UNIFORM MESH

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There have recently appeared local schemes for spline approximation of functions which, unlike interpolation schemes, are based on the construction of a spline S_f approximating a function f as a linear combination of B-splines with coefficients $l_k(f)$ given explicitly as functionals of f. The simplest local scheme is that of Schoenberg [1], in which a spline is constructed on a uniform mesh with step α, and $l_k(f) = f(t_k)$. This scheme ensures an approximation of order $O(\alpha^2)$ for $f \in C_2$. To get a higher order of approximation it is necessary to choose the functionals in a more complicated way (see [2]–[5]). In these references the $l_k(f)$ are chosen so that the spline S_f is exact on polynomials of degree $n \leqslant r$ (r is the degree of the spline). Furthermore, if $f \in C_s$, $s \leqslant n$, then

$$\left\|f^{(p)} - S_f^{(p)}\right\|_C \leqslant M\alpha_{\max}^{s-p}\omega\left(f^{(s)}, \alpha_{\max}\right). \tag{1}$$

We note, however, that the general formulas obtained in [2]–[5] for $l_k(f)$ are so complicated that they can hardly be used in practice except for the case $r = 3$, which is what is done there. Moreover, this approach does not permit us to get satisfactory values for the constant M in (1), nor to observe how the constructed splines approximate smoother functions $f \in C_s$, $s > n$. In the book [5] it was possible to get good estimates only on a uniform mesh for a cubic spline that is exact on third degree polynomials by means of techniques employing a computer.

Another approach is used in the present note. New properties established by the author for B-splines constructed on a uniform mesh lead to asymptotic formulas in powers of α for the elementary splines of Schoenberg, and to fairly sharp estimates of the remainder terms. These formulas permit us in turn to explicitly construct splines ensuring an approximation of $f^{(l)}$ of order $o(\alpha^r)$ for $f \in C_{p+l}$, where $r \leqslant p$. Asymptotic formulas are obtained for these splines in powers of α, along with fairly sharp estimates of the remainder terms. The degree of the constructed splines is any number $m \geqslant p + 1$.

The function

$$B_\alpha^m(t) = \frac{\alpha^{-m}}{(m-1)!}\sum_{k=0}^{[t/\alpha]}(-1)^k\binom{m}{k}(t-\alpha k)_+^{m-1}$$

is a B-spline of degree $m - 1$ on a uniform mesh with step α. B-splines were first considered by Schoenberg in [6]. There it was observed that $B_\alpha^m(t)$ is the probability density for a sum of m independent random variables distributed uniformly on $[0, \alpha]$. The

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following properties of B-splines noted in [6] and [7] are consequences of this fact:

1) $B_\alpha^m(t) \geqslant 0$, and $\operatorname{supp} B_\alpha^m(t) = (0, \alpha m)$.

2) If

$$\mu_s^m = \frac{1}{\alpha^s} \int_0^{\alpha m} (t - \alpha m/2)^s B_\alpha^m(t)\, dt,$$

then

$$\mu_1^m = 0, \qquad \mu_2^m = m/12. \tag{2}$$

Pushing this further, we can find also the subsequent moments

$$\mu_4^m = \frac{m(5m-2)}{240}, \qquad \mu_6^m = \frac{m(35m^2 - 42m + 16)}{4032}, \tag{3}$$
$$\mu_8^m = \frac{m(175m^3 - 420m^2 + 404m - 144)}{34560};$$

all the odd moments are zero.

The formulas

$$\alpha \sum_{r=0}^{m-1} B_\alpha^m(\alpha(\tau + r)) = 1, \qquad \alpha \sum_{r=0}^{m-1} \alpha(\tau + r) B_\alpha^m(\alpha(\tau + r)) = \frac{\alpha m}{2}, \qquad \tau \in [0, 1],$$

are known (see [8]). From them it is clear that we can define a random variable $Z_\alpha^m(\tau)$ "inscribed" in the distribution B_α^m, namely:

$$P\{Z_\alpha^m(\tau) = \alpha(\tau + r)\} = \alpha B_\alpha^m(\alpha(\tau + r)), \qquad \tau \in [0, 1].$$

The mathematical expectation $E(Z_\alpha^m(\tau)) = \alpha m/2$ is independent of τ.

THEOREM 1. *All the moments of the random variable $Z_\alpha^m(\tau)$ through order $m - 1$ are independent of $\tau \in [0, 1]$ and coincide with the corresponding moments of the continuous distribution with density $B_\alpha^m(t)$*:

$$\alpha \sum_{r=0}^{m-1} (\alpha(\tau + r))^s B_\alpha^m(\alpha(\tau + r)) = \int_0^{\alpha m} t^s B_\alpha^m(t)\, dt, \qquad m > s.$$

REMARK. Obviously, the central moments of $Z_\alpha^m(\tau)$ coincide with those of the distribution $B_\alpha^m(t)$. This gives us the identity

$$\alpha \sum_{r=0}^{m-1} (\tau + r - m/2)^s B_\alpha^m(\alpha(\tau + r)) = \mu_s^m, \qquad s < m. \tag{4}$$

Let f be a continuous function.

Let $N = [t/\alpha]$ and $f_l^m = f(\alpha(l + m/2))$. We consider splines of degree $m - 1$ and defect 1 of the form

$$S_\varphi^m(t) = \alpha \sum_{k=N-m+1}^{N} \varphi_k B_\alpha^m(t - \alpha k),$$

where the φ_k are linear combinations of the values f_k^m. The derivative satisfies

$$S_\varphi^m(t)^{(s)} = \alpha \sum_{k=N-m+s+1}^{N} \Delta_\alpha^s \varphi_k B_\alpha^{m-s}(t - \alpha k).$$

Consider first an elementary spline of the kind studied by Schoenberg:

$$S_f^m(t) = \alpha \sum_{k=N-m+1}^{N} f_k^m B_\alpha^m(t - \alpha k).$$

It is known (see [1]) that $S_f^m(t) = f(t) + o(\alpha)$ for $f \in C_1$, while for $f \in C_2$

$$S_f^m(t) = f(t) + (\alpha^2 m f''/24)(t) + o(\alpha^2).$$

The formulas (4) enable us to establish the next result.

THEOREM 2. *Let $f \in C_{s+p}$. If $m > s + p$, then*

$$S_f^m(t)^{(s)} = f(t)^{(s)} + \sum_{n=1}^{[p/2]} \frac{\alpha^{2n} f(t)^{(s+2n)} \mu_{2n}^m}{(2n)!} + F_p^{sm}(t). \tag{5}$$

With the notation

$$\nu_l^r = \alpha \sum_{k=N-r+1}^{N} |\tau + k - r/2|^l B_\alpha^r(\alpha(\tau + k))$$

the following estimate holds:

$$|F_p^{sm}(t)| < \frac{\alpha^p \omega(f^{(s+p)}, \alpha m/2)}{p!} \sum_{l=0}^{p} \binom{p}{l} \left(\frac{s}{2}\right)^{p-l} \nu_l^{m-s} = o(\alpha^p). \tag{6}$$

REMARK. If l is even, then $\nu_l^r = \mu_l^r$, and if it is odd, then $\nu_l^r \leqslant \sqrt{\mu_{2s}^r \mu_{2q}^r}$, where $s + q = l$.

Using (2)–(4), we can write out explicitly an asymptotic expression for $S_f^m(t)^{(s)}$ up to $p = 9$.

Theorem 2 enables us to construct splines which approximate functions and their derivatives with great accuracy. Let

$$S_{fr}^m(t) = \alpha \sum_{k=N-m+1}^{N} f_k^{rm} B_\alpha^m(t - \alpha k),$$

where

$$f_k^{rm} = f_k^m + \sum_{n=1}^{[r/2]} \alpha^{2n} \beta_n \Delta_\alpha^{2n} f_{k+n}^m.$$

If we find β_n from the system of equations

$$\sum_{l=1}^{n} \frac{\beta_l}{(2(n-l))!} \mu_{2(n-l)}^{m+2l} = -\frac{\mu_{2n}^m}{(2n)!},$$

then the following assertion holds.

THEOREM 3. *Let $f \in C_{s+p}$, $m > s + p$, and $r \leqslant p$.*(1) *Then*

$$S_{fr}^m(t)^{(s)} = f(t)^{(s)} + \sum_{n=[r/2]+1}^{[p/2]} \alpha^{2n} \gamma_n^r f(t)^{(s+2n)} + F_{pr}^{sm}(t),$$

where

$$\gamma_n^r = \frac{\mu_m^{2n}}{(2n)!} + \sum_{l=1}^{[r/2]} \frac{\beta_l}{(2(n-l))!} \mu_{2(n-l)}^{m+2l},$$

$$F_{rp}^{sm}(t) = F_p^{sm}(t) + \sum_{n=1}^{[r/2]} \alpha^{2n} \beta_n F_{p-2n}^{m+2n,s+2n} = o(\alpha^p).$$

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(1) $S_{fr}^m(t)^{(s)} = f(t)^{(s)} + F_{pr}^{sm}(t)$ when $[r/2] = [p/2]$.

We write out the first few values of β_n:

$$\beta_1 = -m/24, \qquad \beta_2 = \frac{m(5m+22)}{5760}, \qquad \beta_3 = -\frac{m(35m^2+462m+1528)}{2903040},$$
$$\beta_4 = \frac{m(175m^3+4620m^2+40724m+121154)}{1393459200}.$$

It is now possible to find γ_n^r, $r = 2, 3, \ldots, 9$, $n_0 \leqslant n \leqslant 4$, explicitly, where $n_0 = [r/2] + 1$. We mention that $\gamma_{n_0}^r = -\beta_{n_0}$. Let $\varepsilon_k = 0$ for $p < k$ and $\varepsilon_k = 1$ for $p \geqslant k$. If $f \in C_{s+p}$, $m > s + p$, and $p \leqslant 9$, then

a) $f_k^{rm} = f_k^m + \beta_1\alpha^2\Delta_\alpha^2 f_{k+1}^m$ and $S_{fr}^m(t)^{(s)} = f(t)^{(s)} - \varepsilon_4\alpha^4\beta_2 f(t)^{(s+4)} + \varepsilon_6\alpha^6\gamma_3^r f(t)^{(s+6)} + \varepsilon_8\alpha^8\gamma_4^r f(t)^{(s+8)} + F_{rp}^{sm}(t)$ for $r = 2$ or 3;

b) $f_k^{rm} = f_k^m + \beta_1\alpha^2\Delta_\alpha^2 f_{k+1}^m + \beta_2\alpha^4\Delta_\alpha^4 f_{k+2}^m$ and $S_{fr}^m(t)^{(s)} = f(t)^{(s)} - \varepsilon_6\alpha^6\beta_3 f(t)^{(s+6)} + \varepsilon_8\alpha^8\gamma_4^r f(t)^{(s+8)} + F_{pr}^{sm}(t)$ for $r = 4$ or 5;

c) $f_k^{rm} = f_k^m + \sum_{n=1}^3 \alpha^{2n}\beta_n\Delta_\alpha^{2n} f_{k+n}^m$ and $S_{fr}^m(t)^{(s)} = f(t)^{(s)} - \varepsilon_8\alpha^8\beta_4 f(t)^{(s+8)} + F_{pr}^{sm}(t)$ for $r = 6$ or 7; and

d) $f_k^{rm} = f_k^m + \sum_{n=1}^4 \alpha^{2n}\beta_n\Delta_\alpha^{2n} f_{k+n}^m$ and $S_{fr}^m(t)^{(s)} = f(t)^{(s)} + F_{pr}^{sm}(t)$ for $r = 8$ or 9, where

$$F_{pr}^{sm}(t) = F_p^{sm}(t) + \sum_{n=1}^{4} \alpha^{2n}\beta_n F_{p-2}^{m+2n,s+2n}(t) = o(\alpha^p).$$

To obtain estimates of F_{pr}^{sm} in explicit form it is necessary to use (6). We mention especially some particular cases.

a) A cubic spline which is exact on 3rd-degree polynomials ($m = 4$),

$$S_{f3}^4(t) = \alpha \sum_{k=N-3}^{N} f_k^{34} B_\alpha^4(t - \alpha k),$$

where

$$f_k^{34} = f_k^4 - \tfrac{\alpha^2}{6}\Delta_\alpha^2 f_{k+1}^4 = \tfrac{4}{3} f_k^4 - \tfrac{1}{6}\left(f_{k-1}^4 + f_{k+1}^4\right).$$

If $f \in C_3$, then $|S_{f3}^4(t) - f(t)| = |F_{33}^{04}(t)| < 0.6\alpha^3\omega(f^{(3)}, 3\alpha)$.

b) A cubic spline approximating f',

$$S_f^5(t)' = \alpha \sum_{k=N-3}^{N} \Delta_\alpha f_k^{35} B_\alpha^4(t - \alpha k)$$
$$= \alpha \sum_{k=N-3}^{N} \left[\frac{39}{24}\Delta_\alpha f_k^5 - \frac{5}{8}\Delta_{3\alpha} f_{k+1}^5\right] B_\alpha^4(t - \alpha k).$$

If $f \in C_4$, then $|S_f^5(t)' - f(t)'| = |F_{33}^{15}(t)| \leqslant 1.4\alpha^3\omega(f^{(4)}, 7/2\alpha)$.

c) A cubic spline approximating f'',

$$S_{f3}^6(t)'' = \alpha \sum_{k=N-3}^{N} \Delta_\alpha^2 f_k^{36} B_\alpha^4(t - \alpha k)$$
$$= \alpha \sum_{k=N-3}^{N} \left(2\Delta_\alpha^2 f_k^6 - \Delta_{2\alpha}^2 f_{k+1}^6\right) B_\alpha^4(t - \alpha k).$$

If $f \in C_5$, then $|S_{f3}^6(t)'' - f(t)''| = |F_{33}^{26}(t)| \leqslant 1.4\alpha^3\omega(f^{(5)}, 3\alpha)$.

Similarly, we can write out cubic splines approximating $f^{(k)}$ for $k > 2$.

d) A fifth-degree spline which is exact on polynomials of fifth degree,

$$S_{f5}^6(t) = \alpha \sum_{k=N-3}^{N} f_k^{56} B_\alpha^6(t - \alpha k),$$

where

$$f_k^{56} = \tfrac{73}{40} f_k^6 - \tfrac{7}{15}\left(f_{k-1}^6 + f_{k+1}^6\right) + \tfrac{13}{240}\left(f_{k-2}^6 + f_{k+2}^6\right).$$

If $f \in C_5$, then $|S_{f5}^6(t) - f(t)| = |F_{55}^{06}(t)| \leqslant 0.8\alpha^5\omega(f^{(5)}, 5\alpha)$.

e) A fifth-degree spline approximating f',

$$S_{f5}^7(t)' = \alpha \sum_{k=N-5}^{N} \Delta_\alpha f_k^{57} B_\alpha^6(t - \alpha k)$$
$$= \alpha \sum_{k=N-5}^{N} \left(\frac{55}{24}\Delta_\alpha f_k^7 - \frac{25}{16}\Delta_{3\alpha} f_{k+1}^7 + \frac{13}{48}\Delta_{5\alpha} f_{k+2}^7\right) B_\alpha^6(t - \alpha k).$$

If $f \in C_6$, then $|S_{f5}^7(t)' - f(t)'| = |F_{55}^{17}(t)| < \alpha^5\omega(f^{(6)}, 5.5\alpha)$.

f) A fifth-degree spline approximating f'',

$$S_{f5}^8(t)'' = \alpha \sum_{k=N-5}^{N} \Delta_\alpha^2 f_k^{58} B_\alpha^6(t - \alpha k)$$
$$= \alpha \sum_{k=N-5}^{N} \left(\frac{29}{8}\Delta_\alpha^2 f_k^8 - \frac{51}{15}\Delta_{2\alpha}^2 f_{k+1}^8 + \frac{31}{40}\Delta_{3\alpha}^2 f_{k+2}^8\right) B_\alpha^6(t - \alpha k).$$

If $f \in C_7$, then $|S_{f5}^8(t)'' - f(t)''| = |F_{55}^{28}(t)| < 2\alpha^5\omega(f^{(7)}, 6\alpha)$.

In an analogous manner it is possible to write out fifth-degree splines approximating higher-order derivatives, as well as seventh- and ninth-degree splines which are exact on polynomials of the same degree. We underscore that the splines obtained that approximate derivatives enable us to construct difference schemes of high accuracy for solving differential equations.

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ON THE SPECTRUM OF AUTOMORPHIC LAPLACIANS IN SPACES OF PARABOLIC FUNCTIONS

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In his well-known report [8], A. Selberg devoted a great deal of attention to questions connected with the presence of small eigenvalues of the Laplace operator in spaces of automorphic forms. His interest in such questions is primarily due to the fact that eigenvalues of automorphic Laplacians which are small (i.e., in the interval $(0, 1/4)$) to a large extent determine the asymptotic behavior of the general Kloosterman sums which play an important role in the methods that exist for estimating the Fourier coefficients of automorphic forms. As Selberg showed, the eigenvalues of the automorphic Laplacians may be arbitrarily close to zero, and as a result the well-known methods for estimating the Fourier coefficients, which rely upon estimates for Kloosterman sums, do not, in general, enable one to obtain the desired results (see [8], p. 13). In particular, Selberg gave a method for constructing subgroups of the modular group for which the Laplace operator in the space of automorphic functions (automorphic forms of weight 0) will have an arbitrarily small first eigenvalue.

In Selberg's example, the small eigenvalues of automorphic Laplacians had as eigenfunctions the residues at the poles of meromorphic continuations of Eisenstein series. In the present note we construct analogous examples, except with the small eigenvalues corresponding to eigenfunctions which are parabolic (exponentially decreasing at infinity) (Theorem 2). The basic technical device to do this is Theorem 1, which generalizes to the case of Fuchsian groups of the first kind the Yang-Yau inequality [9], in which the first eigenvalue of the Laplace operator on a compact Riemann surface is bounded from above by an expression which depends only upon its genus and area. As far as the author is aware, the question of small eigenvalues of automorphic Laplacians in spaces of parabolic functions has not been examined before in the literature, although small eigenvalues (lying outside the continuous spectrum) are very important from various points of view, including number theory and representation theory (see [1]–[3]). We note that in scattering theory, for example, for the Schrödinger operator with rapidly decreasing potential, the poles of the resolvent which lie outside the continuous spectrum are also poles of the scattering matrix. It follows from our results that the situation is different in the spectral theory of automorphic functions: the automorphic scattering matrix may actually not have poles at the poles of the resolvent.

We proceed to the precise statements. Let Γ be a Fuchsian group of the first kind, regarded as a discrete group of motions of the upper half-plane H with the Poincaré metric. It is well known that Γ can be given by a system of generators $A_1, \ldots, A_g$,

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