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Tight and sibling frames originated from discrete splines ☆

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Abstract

We present a new family of frames, which are generated by perfect reconstruction filter banks. The filter banks are based on the discrete interpolatory splines and are related to Butterworth filters. Each filter bank comprises one interpolatory symmetric low-pass filter, one band-pass and one high-pass filters. In the sibling frames case, all the filters are linear phase and generate symmetric scaling functions with analysis and synthesis pairs of framelets. In the tight frame case, all the analysis waveforms coincide with their synthesis counterparts. In the sibling frame, we can vary the framelets making them different for synthesis and analysis cases. This enables us to swap vanishing moments between the synthesis and the analysis framelets or to add smoothness to the synthesis framelets. We construct dual pairs of frames, where all the waveforms are symmetric and the framelets may have any number of vanishing moments. Although most of the designed filters are IIR, they allow fast implementation via recursive procedures. The waveforms are well localized in time domain despite their infinite support.

Keywords: Frame; Filter bank; Discrete spline

1. Introduction

The theory of wavelet frames or framelets is an extension of wavelet analysis. Currently it is a subject of extensive investigation by researchers working in signal processing and applied mathematics. A wavelet frame is generated by several mother wavelets and provides a redundant expansion of a function or a signal. Due to this redundancy, there is more freedom in the design and implementation of the frame transforms. The frame expansions of signals demonstrate resilience to quantization noise and to coefficients losses [1–3]. Thus, frames may serve as a tool for error correction to signals transmitted through lossy channels. Additional adaptation capabilities of the overcomplete representation of signals has a potential to succeed in feature extraction and identification of signals. Promising results on image reconstruction and error correction are recently reported in [4–6]. Although many types of wavelet frames have been designed by now, there is a demand for framelet transforms, which have properties useful for signal processing, such as symmetry, flat spectra, vanishing moments, interpolation and fast implementation. This is the motivation for the work presented in this paper.

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A common approach to construction of a framelet system in the function space L^2 starts from the introduction of a pair of refinable functions (or one function), which generate(s) the multiresolution analysis (MRA) in L^2. Then, the wavelets are derived by one or another method as linear combinations of refinable functions. Many construction schemes are based on unitary extension principle (UEP) [7] for tight frames and mixed extension principle (MEP) [8] for bi-frames. These principles reduce the construction of a framelet system to the design of a perfect reconstruction filter bank. The masks of the given refinable functions serve as low-pass filters of the filter bank.

On the other hand, the oversampled perfect reconstruction filter banks by themselves generate wavelet-type frames in the signal space [9,10]. In this paper we use filter banks as an engine to construct a new family of frames in the signal space. Under some relaxed conditions infinite iteration of the frame filter banks results in limit functions, the so-called framelets, which generate the wavelet frames in L^2. The framelets are symmetric, interpolatory and have flat spectra combined with fine time-domain localization and efficient implementation of the transforms. The framelets are smooth and may have any number of vanishing moments. The redundancy rate is two.

Recently a new oblique extension principle (OEP) was proposed [11], which essentially extends the tools for the design of wavelet frames in L^2. New wavelet frames with advanced properties were constructed using OEP [11–14]. However, application of the OEP scheme to the construction of frames in the signal space is somewhat problematic. The reason is that the filter banks corresponding to the *combined MRA masks* (see [11]) of the compactly supported L^2-framelets lack the perfect reconstruction property. This property is necessary for a filter bank to generate a frame in the signal space. In the OEP scheme the perfect reconstruction is achieved by additional filtering with infinite impulse response (IIR) filters. Moreover, in order to construct a tight frame in the signal space, one has to use IIR filters with irrational transfer functions. Filters with irrational transfer functions (unlike filters with rational transfer functions employed in our construction) can be implemented exactly only in the periodic setting via the discrete Fourier transform. In a non-periodic case, the filters must be truncated i.e., they are approximated by long FIR filters.

In this paper we continue the investigation of wavelet-type frames in signal space that are generated by 3-channel analysis and synthesis filter banks comprising one low-pass, one band-pass and one high-pass filters. The downsampling factor $N = 2$ and the transfer functions of all filters are rational functions. The low-pass filters in each filter bank are interpolatory. Our approach to the design of interpolatory perfect reconstruction filter banks is, to some extent, similar to the approach, which we used for the construction of biorthogonal wavelet - transforms [15]. For example, the output of the low-pass component of the analysis filter bank is the sum of the even polyphase component of the input signal and the approximation of the even component by the values of the discrete spline of order $2r$, which interpolates the odd samples of the signal. Such a procedure is equivalent to the application of a filter to the signal, whose transfer function is the squared magnitude of the transfer function of the half-band low-pass Butterworth filter of order r, followed by downsampling. By using this approach we construct a family of tight and bi-frames for the signal space. First results of this investigation are reported in [16]. In the present paper we use the same general approach to the problem. The key point to the construction is the matrix factorization scheme in Section 3. Here it is different from the scheme used in [16]. This results in a new family of tight and sibling wavelet frames. In addition, starting from a symmetric interpolatory low-pass analysis filter, whose transfer function is rational and has zero of arbitrary order m, we construct an analysis filter bank such that the framelet generated by the high-pass filter is symmetric and has m vanishing moments. The framelet generated by the band-pass filter is (anti)symmetric and may have arbitrary number of vanishing moments. The synthesis filter bank is dual to the analysis filter bank and has exactly the same properties.

Note that the Butterworth filters were used in [17] for the construction of orthogonal wavelets. The regularity of the refinable functions generated by the Butterworth filters was analyzed in [18]. In our previous papers [15,19] we presented a family of biorthogonal symmetric wavelets related to the Butterworth filters and their application to image compression.

Unlike the majority of wavelet frame schemes, we use IIR filters. Consequently, the devised framelets have infinite support. However, due to rational structure of their transfer functions, filtering is implemented in a fast recursive mode. The computational cost of the transforms implementation is not higher (sometimes even lower) than the cost of processing using finite impulse response (FIR) filters (see [15,19]). Non-compactness of the framelets support is compensated by the fast exponential decay as time goes to infinity.

The paper is organized as follows. In the introductory Section 2 we recall some facts concerning filter banks and frames, which are needed for the rest of the presentation. In Section 3 we describe how to construct a tight frame and a bundle of sibling frames starting from arbitrary low-pass filter. Having a pair of interpolatory low-pass filters, we construct a set of bi-frames. In Section 4 we present the derivation of the interpolatory filters from discrete splines and explain the relation between the designed filters and the Butterworth filters. In addition, we establish some properties of these filters and their corresponding waveforms. Section 5 is devoted to the construction of tight, semi-tight and bi-frames using the designed filters. We provide a number of examples.

2. Filter banks and frames: preliminaries

In this section we introduce notation and briefly outline the necessary facts about filter banks and their relation to signal space frames. More detailed presentation is given in [16].

2.1. Filter banks

We call the sequences $\mathbf{x} \triangleq \{x_k\}$, $k \in \mathbb{Z}$, which belong to the space l_1, (and, consequently, to l_2) discrete-time signals. The z-transform of a signal $\mathbf{x}$ is defined as $X(z) \triangleq \sum_{k\in\mathbb{Z}} z^{-k} x_k$. Throughout the paper we assume that $z = e^{j\omega}$.

We designate a filter by its transfer function $F(z) \triangleq \sum_{n\in\mathbb{Z}} z^{-n} f_n$, where the sequence $\{f_n\}$ is called the impulse response of the filter. The function $\widehat{F}(\omega) \triangleq F(e^{-j\omega n})$ is called the frequency response of the filter.

In this paper we consider only 3-channel filter banks that contain one low-pass, one band-pass and one high-pass filters, whose transfer functions are rational functions and the downsampling factor is two. The analysis and synthesis low-pass filters are denoted by $\tilde{H}(z)$ and $H(z)$, respectively, the band-pass filters are denoted by $\tilde{G}^1(z)$ and $G^1(z)$ and the high-pass filters are denoted by $\tilde{G}^2(z)$ and $G^2(z)$. We denote the output signals from the analysis filter bank by $\mathbf{s}^1$, $\mathbf{d}^{r,1}$, $r = 1, 2$. These signals are used as the input for the synthesis filter bank. Then, the analysis and synthesis formulas are

$$s_l^1 = 2\sum_{n\in\mathbb{Z}} \tilde{h}_{n-2l} x_n \Leftrightarrow S^1(z^2) = \tilde{H}(1/z)X(z) + \tilde{H}(-1/z)X(-z), \tag{2.1}$$

$$d_l^{r,1} = 2\sum_{n\in\mathbb{Z}} \tilde{g}^r_{n-2l} x_n \Leftrightarrow D^{r,1}(z^2) = \tilde{G}^r(1/z)X(z) + \tilde{G}^r(-1/z)X(-z), \quad r = 1, 2, \tag{2.2}$$

$$\hat{x}_l = \sum_{n\in\mathbb{Z}} h_{l-2n} s_n^1 + \sum_{r=1}^{2} \sum_{n\in\mathbb{Z}} g^r_{l-2n} d_n^{r,1} \Leftrightarrow \hat{X}(z) = H(z)S^1(z^2) + \sum_{r=1}^{2} G^r(z)D^{r,1}(z^2). \tag{2.3}$$

Polyphase representation of filtering: The functions

$$F_e(z) \triangleq \sum_{k\in\mathbb{Z}} z^{-k} f_{2k}, \quad F_o(z) \triangleq \sum_{k\in\mathbb{Z}} z^{-k} f_{2k+1},$$

$$E(z) \triangleq \sum_{k\in\mathbb{Z}} z^{-k} x_{2k}, \quad O(z) \triangleq \sum_{k\in\mathbb{Z}} z^{-k} x_{2k+1}$$

are called the polyphase components of $F(z)$ and $X(z)$, respectively. Then, the polyphase components of $Y(z) \triangleq F(z)X(z)$ are

$$Y_e(z) = F_e(z)E(z) + z^{-1}F_o(z)O(z),$$
$$Y_o(z) = F_o(z)E(z) + F_e(z)O(z). \tag{2.4}$$

We introduce the analysis $\tilde{\mathbf{P}}(z)$ and the synthesis $\mathbf{P}(z)$ *polyphase matrices*, respectively:

$$\tilde{\mathbf{P}}(z) \triangleq \begin{pmatrix} \tilde{H}_e(z) & \tilde{H}_o(z) \\ \tilde{G}^1_e(z) & \tilde{G}^1_o(z) \\ \tilde{G}^2_e(z) & \tilde{G}^2_o(z) \end{pmatrix},$$

$$\mathbf{P}(z) \triangleq \begin{pmatrix} H_e(z) & G^1_e(z) & G^2_e(z) \\ H_o(z) & G^1_o(z) & G^2_o(z) \end{pmatrix}.$$

Then the analysis and synthesis formulas can be represented as

$$\begin{pmatrix} S^1(z) \\ D^{1,1}(z) \\ D^{2,1}(z) \end{pmatrix} = 2\tilde{\mathbf{P}}(1/z)\begin{pmatrix} E(z) \\ O(z) \end{pmatrix},$$

$$\begin{pmatrix} \hat{E}(z) \\ \hat{O}(z) \end{pmatrix} = \mathbf{P}(\mathbf{z})\begin{pmatrix} S^1(z) \\ D^{1,1}(z) \\ D^2(z) \end{pmatrix}.$$

Here, $\hat{E}(z)$ and $\hat{O}(z)$ are the z-transforms of the even and odd components of the output signal $\hat{\mathbf{x}}$, respectively. If the signal $\hat{\mathbf{x}} = \mathbf{x}$ then the analysis and synthesis filter banks form a perfect reconstruction filter bank. Analytically, this property is expressed via the polyphase matrices as

$$\mathbf{P}(z)\tilde{\mathbf{P}}(1/z) = \tfrac{1}{2}\mathbf{I}, \tag{2.5}$$

where $\mathbf{I}$ denotes the 2×2 identity matrix. Thus, the synthesis polyphase matrix must be a left inverse of the analysis matrix (up to factor $\frac{1}{2}$). Obviously, if such a matrix exists, it is not unique.

2.2. Frames

In this section we provide a definition of frames in signal space and describe the relation between filter bank processing and frame expansion of signals.

Definition 2.1. The system $\tilde{\Phi} \triangleq \{\tilde{\phi}_j\}_{j\in\mathbb{Z}}$ of signals forms a frame in the signal space if there exist positive constants A and B such that for any signal $\mathbf{x} = \{x_l\}_{l\in\mathbb{Z}}$

$$A\|\mathbf{x}\|^2 \leqslant \sum_{j\in\mathbb{Z}} |\langle \mathbf{x}, \tilde{\phi}_j\rangle|^2 \leqslant B\|\mathbf{x}\|^2.$$

If the frame bounds A and B are equal to each other then the frame is said to be tight.

If the system $\tilde{\Phi}$ is a frame then there exists another frame $\Phi \triangleq \{\phi_i\}_{i\in\mathbb{Z}}$ of the signals space such that any signal $\mathbf{x}$ can be expanded into the sum $\mathbf{x} = \sum_{i\in\mathbb{Z}} \langle \mathbf{x}, \tilde{\phi}_i\rangle \phi_i$. The frames $\tilde{\Phi}$ and Φ can be interchanged. Together they form the so-called bi-frame. If a frame is tight then $\tilde{\Phi} = c\Phi$.

Let the analysis $\tilde{H}(z)$, $\tilde{G}^1(z)$, $\tilde{G}^2(z)$ and the synthesis $H(z)$, $G^1(z)$, $G^2(z)$ filter banks form a perfect reconstruction filter bank. We denote for $r = 1, 2$ and $n \in \mathbb{Z}$

$$\tilde{\varphi}^1 \triangleq \{\tilde{\varphi}^1(n) \triangleq 2\tilde{h}(n)\}, \quad \tilde{\psi}^{r,1} \triangleq \{\tilde{\psi}^{r,1}(n) \triangleq 2\tilde{g}^r(n)\},$$
$$\varphi^1 \triangleq \{\varphi^1(n) \triangleq 2h(n)\}, \quad \psi^{r,1} \triangleq \{\psi^{r,1}(n) \triangleq 2g^r(n)\}.$$

Then, the analysis and synthesis formulas ((2.1) and (2.2), respectively) can be presented in the following way:

$$s_l^1 = \langle \mathbf{x}, \tilde{\varphi}^1(\cdot - 2l)\rangle, \quad d_l^{r,1} = \langle \mathbf{x}, \tilde{\psi}^{r,1}(\cdot - 2l)\rangle,$$
$$r = 1, 2, \ \ l \in \mathbb{Z},$$

$$\mathbf{x} = \frac{1}{2}\sum_{l\in\mathbb{Z}} \langle \mathbf{x}, \tilde{\varphi}^1(\cdot - 2l)\rangle \varphi^1(\cdot - 2l) + \frac{1}{2}\sum_{r=1}^{2}\sum_{l\in\mathbb{Z}} \langle \mathbf{x}, \tilde{\psi}^{r,1}(\cdot - 2l)\rangle \psi^{r,1}(\cdot - 2l).$$

The results in [9,10] imply the following condition for a filter bank to yield a frame expansion of the signal $\mathbf{x}$.

Proposition 2.1. *Assume the impulse responses of the perfect reconstruction filter bank* $\tilde{H}(z)$, $\tilde{G}^1(z), \tilde{G}^2(z)$ *and* $H(z)$, $G^1(z), G^2(z)$ *belong to* l_1. *Then, the filter bank provides a frame expansion of signals* $\mathbf{x} \in l_2$ *and the set of two-sample shifts of the signals* $\tilde{\varphi}^1$, $\tilde{\psi}^{r,1}$, φ^1, $\psi^{r,1}$, $r = 1, 2$, *form a bi-frame of the signal space.*

One solution to (2.5) is the parapseudoinverse of $\tilde{\mathbf{P}}$:

$$\mathbf{P}(z) = \tilde{\mathbf{P}}^{+}(z) \triangleq \tfrac{1}{2}(\tilde{\mathbf{P}}^{\mathrm{T}}(z) \cdot \tilde{\mathbf{P}}(1/z))^{-1} \cdot \tilde{\mathbf{P}}^{\mathrm{T}}(z). \tag{2.6}$$

The synthesis frame that corresponds to the polyphase matrix $\tilde{\mathbf{P}}^{+}(z)$ is dual to the analysis frame. If $\mathbf{P}(z) = \tilde{\mathbf{P}}^{\mathrm{T}}(z)$ then the signals $\tilde{\varphi}^1$ and $\tilde{\psi}^{r,1}$, $r = 1, 2$, generate a tight frame.

2.3. Multiscale frame transforms

The N times iterated application of the analysis filter bank to the output from the low-pass component of the analysis filter bank lead to the following frame expansion of the signal $\mathbf{x}$:

$$\mathbf{x} = 2^{-N}\sum_{l\in\mathbb{Z}} \langle \mathbf{x}, \tilde{\varphi}^N(\cdot - 2^N l)\rangle \varphi^N(\cdot - 2^N l) + \sum_{r=1}^{2}\sum_{v=1}^{N} 2^{-v}\sum_{l\in\mathbb{Z}} \langle \mathbf{x}, \tilde{\psi}^{r,v}(\cdot - 2^v l)\rangle \psi^{r,v}(\cdot - 2^v l),$$

where

$$\tilde{\varphi}^N(l) \triangleq 2 \sum_{n\in\mathbb{Z}} h_n \tilde{\varphi}^{N-1}(n-2l),$$
$$\tilde{\psi}^{r,v}(l) \triangleq 2 \sum_{n\in\mathbb{Z}} g_n^r \tilde{\varphi}^{v-1}(n-2l), \quad r=1,2$$

and

$$\varphi^N(l) \triangleq 2 \sum_{n\in\mathbb{Z}} h_n \varphi^{N-1}(n-2l),$$
$$\psi^{r,v}(l) \triangleq 2 \sum_{n\in\mathbb{Z}} g_n^r \varphi^{v-1}(n-2l), \quad r=1,2.$$

The new bi-frame consists of shifts of the signals $\tilde{\varphi}^N$, $\{\tilde{\psi}^{r,v}\}$ and φ^N, $\{\psi^{r,v}\}$, $r=1,2$, $v=1,\ldots,N$.

2.4. Scaling functions and framelets

It is well known [20] that under certain conditions the filter bank $H(z)$, $G^1(z)$, $G^2(z)$ generates the continuous scaling function $\varphi(t)$ and two framelets $\psi^1(t)$ and $\psi^2(t)$. Suppose that $H(1)=1$. If the infinite product

$$\lim_{N\to\infty} \prod_{v=1}^{N} H(e^{j2^{-v}\omega}), \tag{2.7}$$

converges to $\Phi(\omega) \in L^2(\mathbb{R})$, then, the inverse Fourier transform of this function $\Phi(\omega)$ is the scaling function $\varphi(t) \in L^2(\mathbb{R})$, which is a solution of the refinement equation $\varphi(t) = 2\sum_{k\in\mathbb{Z}} h_k \varphi(2t-k)$.

A simple sufficient condition for the existence of a smooth scaling function was established in [20].

Proposition 2.2 (*Daubechies [20]*). *Let the transfer function $H(z)$ be factorized as $H(z) = ((1+z^{-1})/2)^p K(z)$, where $K(z)$ is a rational function such that $K(1)=1$. If the condition $\kappa \triangleq \sup_{|z|=1} |K(z)| < 2^{p-1-m}$ is satisfied then there exists a scaling function $\varphi(t) \in L^2(\mathbb{R})$, which is continuous together with its derivatives up to order m.*

It was proved in [21] that under the conditions of Proposition 2.2 there exist positive numbers A and g such that $|\varphi(t)| \leqslant A e^{-g|t|}$.

Definition 2.2. The set of functions $\{\psi^k(t)\}_{k=1}^n$, such that $\{\{2^{v/2}\psi^k(2^j t - l)\}_{v,l\in\mathbb{Z}}\}_{k=1}^n$ forms a frame for $L^2(\mathbb{R})$, is called a wavelet frame. The functions $\{\psi^k(t)\}$ are called framelets.

The mixed extension principle [8] implies the following statement.

Proposition 2.3. *Let $\tilde{H}$, $\tilde{G}^1, \tilde{G}^2$ and H, G^1, G^2 be a perfect reconstruction filter bank and the impulse response $\{\tilde{h}(n)\}, \{\tilde{g}^r(n)\}$ and $\{h(n)\}, \{g^r(n)\}$, $r=1,2$ decay exponentially. If the low-pass filters $\tilde{H}$ and H generate square integrable scaling functions $\tilde{\varphi}(t)$ and $\varphi(t)$, respectively, then, the functions*

$$\tilde{\psi}^r(t) \triangleq 2 \sum_{k\in\mathbb{Z}} \tilde{g}_k^r \,\tilde{\varphi}(2t-k),$$
$$\psi^r(t) \triangleq 2 \sum_{k\in\mathbb{Z}} g_k^r \varphi(2t-k), \quad r=1,2, \tag{2.8}$$

generate the dual wavelet frames of $L^2(\mathbb{R})$, i.e., they are the dual framelets.

If the scaling functions $\tilde{\varphi}(t)$ and $\varphi(t)$ decay exponentially and the rational functions $\tilde{G}^r(z)$, $G^r(z)$, $r=1,2$, have no poles on the unit circle $|z|=1$, then their impulse response g_i^r, $r=1,2$, decay exponentially. Thus, the framelets $\tilde{\psi}^r(t)$ and $\psi^r(t)$, defined in (2.8), also decay exponentially.

A framelet $\psi^r(t)$ has p vanishing moments if $\int_{-\infty}^{\infty} t^s \psi^r(t)\,dt = 0$, $s=0,\ldots,p-1$. The number of vanishing moments of the framelet $\psi^r(t)$ is equal to the multiplicity of zero of the filter $G^r(z)$ at $z=1$ [22].

3. Interpolatory frames

In this section, which is central in the paper, we describe how to construct frames in signal space starting from one low-pass interpolatory filter. The problem reduces to the design of a perfect reconstruction filter bank with the desired properties. The key point to this design is the factorization scheme of the polyphase matrix (3.1).

3.1. Bi-frames

If the even polyphase component of a filter $F(z)$ is $F_e(z) = \frac{1}{2}$, then, the filter is called interpolatory. If an interpolatory low-pass filter generates the scaling function $\varphi(t)$ then this scaling function is interpolatory, that is $\varphi(n) = \delta_n$, $n \in \mathbb{Z}$. In the rest of the paper we deal exclusively with filter banks, whose low-pass filters are interpolatory:

$$H(z) = \frac{1 + z^{-1} U(z^2)}{2}, \quad \tilde{H}(z) = \frac{1 + z^{-1} \tilde{U}(z^2)}{2}.$$

Denote

$$T(z) \triangleq 1/(1 + U(z)\tilde{U}(z^{-1})), \quad W(z) \triangleq 1 - U(z)\tilde{U}(z^{-1}).$$

We assume that

A1. $U(z)$ and $\tilde{U}(z)$ are rational functions that have no poles on the unit circle $|z|=1$.
A2. $T(z)$ has no poles on the unit circle.
A3. $U(1)=\tilde{U}(1)=1$.
A4. Symmetry:
$z^{-1}U(z^2)=zU(z^{-2}),\ z^{-1}\tilde{U}(z^2)=z\tilde{U}(z^{-2})$.

Thus, the filters $\tilde{H}(z)$ and $H(z)$ have linear phase and the corresponding scaling functions are symmetric about zero.

The polyphase matrices for a filter bank comprising the interpolatory low-pass filters $H(z)$ and $\tilde{H}(z)$ are

$$\tilde{\mathbf{P}}(z) \triangleq \begin{pmatrix} 1/2 & \tilde{U}(z)/2 \\ \tilde{G}_e^1(z) & \tilde{G}_o^1(z) \\ \tilde{G}_e^2(z) & \tilde{G}_o^2(z) \end{pmatrix},$$

$$\mathbf{P}(z) \triangleq \begin{pmatrix} 1/2 & G_e^1(z) & G_e^2(z) \\ U(z)/2 & G_o^1(z) & G_o^2(z) \end{pmatrix}.$$

Then, the perfect reconstruction condition (2.5) leads to equation:

$$\mathbf{P}_g(z)\cdot\tilde{\mathbf{P}}_g(1/z)=\tfrac{1}{4}\mathbf{Q}(z), \tag{3.1}$$

where

$$\tilde{\mathbf{P}}_g(z) \triangleq \begin{pmatrix} \tilde{G}_e^1(z) & \tilde{G}_o^1(z) \\ \tilde{G}_e^2(z) & \tilde{G}_o^2(z) \end{pmatrix}, \quad \mathbf{P}_g(z)\begin{pmatrix} G_e^1(z) & G_e^2(z) \\ G_o^1(z) & G_o^2(z) \end{pmatrix},$$

$$\mathbf{Q}(z) \triangleq \begin{pmatrix} 1 & -\tilde{U}(z^{-1}) \\ -U(z) & 2-U(z)\tilde{U}(z^{-1}) \end{pmatrix}.$$

Therefore, the construction of a frame with the interpolatory low-pass filters $\tilde{H}(z)$ and $H(z)$ reduces to factorization of the matrix $\mathbf{Q}(z)$ as in (3.1). One option is the triangular factorization of $\mathbf{Q}(z)$:

$$\tilde{\mathbf{P}}_g(z)=\begin{pmatrix} 0 & \tilde{w}(z) \\ 1/2 & -\tilde{U}(z)/2 \end{pmatrix},$$

$$\mathbf{P}_g(z)=\begin{pmatrix} 0 & 1/2 \\ w(z) & -U(z)/2 \end{pmatrix},$$

$$\text{where } w(z)\tilde{w}(z^{-1})=\frac{W(z)}{2}. \tag{3.2}$$

Note that in this case the high-pass filters

$$G^2(z)=\frac{1-z^{-1}U(z^2)}{2}=H(-z),$$
$$\tilde{G}^2=\frac{1-z^{-1}\tilde{U}(z^2)}{2}=\tilde{H}(-z)$$

are interpolatory.

The implications of the factorization (3.2) are discussed in [16]. In this paper we consider another factorization scheme that is somewhat related to the scheme by Petukhov [23].

The eigenvalues of the matrix $\mathbf{Q}(z)$ are $\lambda_1(z)=W(z)$, $\lambda_2=2$ and the eigenvectors are

$$v_1(z)=\begin{pmatrix} 1 \\ U(z) \end{pmatrix}, \quad v_1(z)=\begin{pmatrix} -\tilde{U}(z^{-1}) \\ 1 \end{pmatrix}.$$

Define the matrices

$$\mathbf{\Lambda}(z) \triangleq \begin{pmatrix} \lambda_1(z) & 0 \\ 0 & \lambda_2(z) \end{pmatrix},$$

$$\mathbf{V}(z) \triangleq (v_1(z)\, v_2(z)) = \begin{pmatrix} 1 & -\tilde{U}(z^{-1}) \\ U(z) & 1 \end{pmatrix}.$$

Then,

$$\mathbf{V}^{-1}(z)=T(z)\begin{pmatrix} 1 & \tilde{U}(z^{-1}) \\ -U(z) & 1 \end{pmatrix}$$

and the matrix $\mathbf{Q}(z)$ can be represented as follows:

$$\begin{aligned}\mathbf{Q}(z)&=\mathbf{V}(z)\mathbf{\Lambda}(z)\mathbf{V}^{-1}(z)\\ &=\begin{pmatrix} 1 & -\tilde{U}(z^{-1}) \\ U(z) & 1 \end{pmatrix}\begin{pmatrix} W(z)T(z) & 0 \\ 0 & 2T(z) \end{pmatrix}\\ &\quad\times\begin{pmatrix} 1 & \tilde{U}(z^{-1}) \\ -U(z) & 1 \end{pmatrix}.\end{aligned}$$

Let

$$\begin{aligned}\mathbf{P}_g(z)&\triangleq\frac{1}{2}\begin{pmatrix} 1 & -\tilde{U}(z^{-1}) \\ U(z) & 1 \end{pmatrix}\begin{pmatrix} \mu_1(z) & 0 \\ 0 & \mu_2(z) \end{pmatrix}\\ &=\frac{1}{2}\begin{pmatrix} \mu_1(z) & -\mu_2(z)\tilde{U}(z^{-1}) \\ \mu_1(z)U(z) & \mu_2(z) \end{pmatrix},\end{aligned}$$

$$\begin{aligned}\tilde{\mathbf{P}}_g(z)&\triangleq\frac{1}{2}\begin{pmatrix} \tilde{\mu}_1(z) & 0 \\ 0 & \tilde{\mu}_2(z) \end{pmatrix}\begin{pmatrix} 1 & \tilde{U}(z), \\ -U(z^{-1}) & 1 \end{pmatrix}\\ &=\frac{1}{2}\begin{pmatrix} \tilde{\mu}_1(z) & \tilde{\mu}_1(z)\tilde{U}(z) \\ -\tilde{\mu}_2(z)U(z^{-1}) & \tilde{\mu}_2(z) \end{pmatrix},\end{aligned}$$

$$\mu_1(z)\tilde{\mu}_1(z^{-1}) = W(z)T(z), \quad \mu_2(z)\tilde{\mu}_2(z^{-1}) = 2T(z). \tag{3.3}$$

Then, we get a perfect reconstruction filter bank:

$$H(z) = \frac{1+z^{-1}U(z^2)}{2}, \quad \tilde{H}(z) = \frac{1+z^{-1}\tilde{U}(z^2)}{2},$$

$$G^1(z) = \frac{\mu_1(z^2)}{2}(1+z^{-1}U(z^2)),$$

$$\tilde{G}^1(z) = \frac{\tilde{\mu}_1(z^2)}{2}(1+z^{-1}\tilde{U}(z^2)),$$

$$G^2(z) = \frac{\mu_2(z^2)}{2z}(1-z^{-1}\tilde{U}(z^2)),$$

$$\tilde{G}^2(z) = \frac{\tilde{\mu}_2(z^2)}{2z}(1-z^{-1}U(z^2)). \tag{3.4}$$

Proposition 3.1. *Let*

$$1-z^{-1}U(z^2) = (1-z)^p u(z),$$
$$1-z^{-1}\tilde{U}(z^2) = (1-z)^{\tilde{p}}\tilde{u}(z),$$
$$W(z) = (1-z)^q \omega(z).$$

If $p, \tilde{p} \geqslant 1$, $q \geqslant 2$ *and* $\omega(z)$, $u(z)$ *and* $\tilde{u}(z)$ *have no poles on the unit circle then there exist rational functions* $\mu_r(z)$ *and* $\tilde{\mu}_r(z)$, $r = 1, 2$, *satisfying* (3.3), *such that the filters* $G^2(z)$ *and* $\tilde{G}^2(z)$ *are high-pass and the filters* $G^1(z)$ *and* $\tilde{G}^1(z)$ *are band-pass.*

Proof. In order for the filters $G^2(z)$ and $\tilde{G}^2(z)$ to be high-pass, the rational functions $\mu_2(z)$ and $\tilde{\mu}_2(z)$ must be regular or have poles of order $p_2 < p$ $\tilde{p}_2 < \tilde{p}$ at $z = 1$. Due to Assumption A2, the factorization $2T(z) = \mu_2(z)\tilde{\mu}_2(z^{-1})$ exists. Assumption A3 implies that the filters$G^1(z)$ and $\tilde{G}^1(z)$ have zero at $z = -1$. Thus, these filters suppress higher frequencies. Therefore, as it is seen from (3.4), in order for them to be band-pass, the functions $\mu_1(z)$ and $\tilde{\mu}_1(z)$ must vanish at $z = 1$. If $q \geqslant 2$ then we can factorize $W(z)T(z) = \mu_1(z)\tilde{\mu}_1(z^{-1})$ in such a way that $\mu_1(1) = \tilde{\mu}_1(1) = 0$. □

3.2. Tight and sibling frames

Assume $U(z) = \tilde{U}(z)$. Then,

$$H(z) = \tilde{H}(z), \quad T(z) = 1/(1+|U(z)|^2),$$
$$W(z) = 1-|U(z)|^2.$$

Note that if $1-z^{-1}U(z^2) = (1-z)^p u(z)$ then $W(z) = (1-z)^p \omega(z)$.

If there exist rational functions $\mu_1(z)$ and $\mu_2(z)$ such that

$$|\mu_1(z)|^2 = W(z)T(z) = \frac{1-|U(z)|^2}{1+|U(z)|^2},$$
$$|\mu_2(z)|^2 = 2T(z)$$

then the synthesis filter bank coincides with the analysis filter bank and generates a tight frame. This factorization is possible if

$$|U(e^{i\omega})| \leqslant 1 \quad \text{and} \quad p = 2r. \tag{3.5}$$

In this case, the filters $G^1(z)$ and $G^2(z)$ have zero of multiplicity $p/2$ and p at $z = 1$, respectively. Consequently, the framelets ψ^1 and ψ^2 have $p/2$ and p vanishing moments, respectively. In the case when the condition (3.5) is not satisfied, we can construct frames if we allow the function $\mu_1(z)$ to differ from $\tilde{\mu}_1(z)$ and $\mu_2(z)$ to differ from $\tilde{\mu}_2(z)$. Here the analysis and synthesis filter banks generate a pair of *sibling* frames $\{\varphi, \tilde{\psi}^1, \tilde{\psi}^2\}$, $\{\varphi, \psi^1, \psi^2\}$ using the terminology in [24]. Note that even when a tight frame is possible, sometimes it is preferable to have a bi-frame. For example, although both functions $W(z)$ and $T(z)$ are symmetric about inversion $z \to z^{-1}$, the functions $\mu_1(z)$ and $\mu_2(z)$ may lack this property. The (anti)symmetry can be achieved by construction of sibling frames. In addition, using different factorizations in (3.3), we can swap vanishing moments between the synthesis and the analysis framelets and vice versa.

3.3. Dual frame

Let

$$\tilde{\mathbf{P}}(z) \triangleq \frac{1}{2}\begin{pmatrix} 1 & \tilde{U}(z) \\ \tilde{\mu}_1(z) & \tilde{\mu}_1(z)\tilde{U}(z) \\ -\tilde{\mu}_2(z)\tilde{U}(z^{-1}) & \tilde{\mu}_2(z) \end{pmatrix}$$

be the polyphase matrix of the analysis interpolatory filter bank

$$\tilde{H}(z) = \frac{1+z^{-1}\tilde{U}(z^2)}{2},$$
$$\tilde{G}^1(z) = \frac{\tilde{\mu}_1(z^2)}{2}(1+z^{-1}\tilde{U}(z^2)),$$
$$\tilde{G}^2(z) = \frac{\tilde{\mu}_2(z^2)}{2z}(1-z^{-1}U(z^2)).$$

The determinant

$$\mathrm{Det}(\tilde{\mathbf{P}}^{\mathrm{T}}(z)\tilde{\mathbf{P}}(z^{-1})) = \frac{|\tilde{\mu}_2(z)|^2(1+|\tilde{\mu}_1(z)|^2)}{16T(z)}.$$

If $\tilde{\mu}_2(z)$ has no zeros on the unit circle then the matrix $\tilde{\mathbf{P}}(z)$ is of full rank. The filter bank generates the analysis frame. The dual synthesis frame is generated by the filter bank, whose polyphase matrix is the parapseudoinverse $\tilde{\mathbf{P}}^+(z)$ of $\tilde{\mathbf{P}}(z)$ (see (2.6)). It is readily verified that

$$\tilde{\mathbf{P}}^+(z) = T(z)\begin{pmatrix} 1/(1+|\tilde{\mu}_1(z)|^2) & \tilde{\mu}_1(z^{-1})/(1+|\tilde{\mu}_1(z)|^2) & -\tilde{U}(z^{-1})/\tilde{\mu}_2(z) \\ \tilde{U}(z)/(1+|\tilde{\mu}_1(z)|^2) & \tilde{\mu}_1(z^{-1})\tilde{U}(z)/(1+|\tilde{\mu}_1(z)|^2) & 1/\tilde{\mu}_2(z) \end{pmatrix}.$$

Proposition 3.2. *Assume that the interpolatory low-pass symmetric filter* $\tilde{H}(z) = \frac{1}{2} + z^{-1}\tilde{U}(z^2)/2$ *has zero of multiplicity m at* $z = -1$ *and no poles on the unit circle* $|z| = 1$. *Then, it generates a family of invertible analysis filter bank* $\tilde{H}(z)$, $\tilde{G}^1(z)$, $\tilde{G}^2(z)$ *such that the high-pass filter* $\tilde{G}^2(z)$ *is symmetric and has a zero of multiplicity m at* $z = 1$. *The band-pass filters* $\tilde{G}^1(z)$ *are variable. They are (anti)symmetric and have a zero of multiplicity m at* $z = -1$ *and a zero of arbitrary multiplicity n at* $z = 1$. *The dual synthesis filter bank* $H(z)$, $G^1(z)$, $G^2(z)$ *has the same properties.*

Proof. Define $\tilde{\mu}_2(z) \equiv 1$ and $\tilde{\mu}_1(z) \triangleq (z - 1/z)^n$. Then the symmetric filters $\tilde{G}^2(z) = z^{-1}(1/2 - z^{-1}\tilde{U}(z^2)/2)$ and $G^2(z) = T(z^2)(1 - z^{-1}\tilde{U}(z^2))$ have a zero of multiplicity m at $z = 1$. The (anti)symmetric filter $\tilde{G}^1(z) = (z^2 - z^{-2})^n(1/2 + z^{-1}\tilde{U}(z^2)/2)$ has a zero of multiplicity m at $z = -1$ and a zero of multiplicity n at $z = 1$. The same property is inherent to the synthesis filter

$$G^1(z) = \frac{T(z^2)(z^2 - z^{-2})^n}{1 + (z^2 - z^{-2})^{2n}}(1 + z^{-1}\tilde{U}(z^2)).$$

If $n = 2p$ is an even integer then we can take $\tilde{\mu}_1(z) \triangleq (z - 2 + 1/z)^p$ and, consequently,

$$\tilde{G}^1(z) = \frac{(z^2 - 2 + z^{-2})^s}{2}(1 + z^{-1}\tilde{U}(z^2)),$$
$$G^1(z) = \frac{T(z^2)(z^2 - 2 + z^{-2})^s}{1 + (z^2 - 2 + z^{-2})^{2s}}(1 + z^{-1}\tilde{U}(z^2)). \qquad \square$$

Corollary 3.1. *The framelets* $\tilde{\psi}^2(t)$ *and* $\psi^2(t)$ *have m vanishing moments. The framelets* $\tilde{\psi}^1(t)$ *and* $\psi^1(t)$ *may have arbitrary number n of vanishing moments.*

4. Design of interpolatory filters

Our scheme that constructs an interpolatory biframe consists of two steps: 1. Choice of feasible rational functions $\tilde{U}(z)$ and $U(z)$ and 2. Factorization of the functions $T(z)$ and $W(z)$. In this section we discuss the first step.

We assumed above that $\tilde{U}(1) = 1$ and $U(1) = 1$. Thus, $\tilde{H}(z) = (1 + z^{-1}\tilde{U}(z^2))/2$ and $H(z) = (1 + z^{-1}U(z^2))/2$ have zeros of multiplicities $m > 0$ and $\tilde{m} > 0$ at $z = -1$, respectively. They are the low-pass interpolatory filters, which restore sampled polynomials of degree $m-1$ and $\tilde{m}-1$, respectively. In Proposition 2.2 the multiplicities of zero at $z = -1$ are linked to the smoothness of the corresponding scaling functions and framelets. The multiplicities of zero at $z = 1$ of the filters $G^r(z)$ and $\tilde{G}^r(z)$, $r = 1, 2$ are equal to the number of vanishing moments of the corresponding framelets. Eq. (2.4) implies that if $Y(z) = H(z)X(x)$ then

$$Y_e(z) = \frac{E(z) + z^{-1}U(z)O(z)}{2},$$
$$Y_o(z) = \frac{U(z)E(z) + O(z)}{2}.$$

Hence, we see that in order for the filter H to restore a polynomial $\{p_k\}$ of degree $m-1$ sampled on the grid $\{k\}$, the filter $z^{-1}U(z)$, being applied to the array $\{p_{2k+1}\}$ of odd samples of the polynomial, must produce exactly the even array $\{p_{2k}\}$. Vice versa, the filter $U(z)$, being applied to the array of even samples, must produce exactly the array of odd samples. In this paper we operate with the family of IIR filters with rational transfer functions that are derived from the discrete spline insertion rule:

Construct the discrete spline of degree $2r-1$, which interpolates the even samples $\{x_{2k}\}$ of a signal $\mathbf{x}$ and predict the odd samples $\{x_{2k+1}\}$ as the value of the spline at $2k+1$.

The devised filters are related to the Butterworth filters, which are commonly used in signal processing [25].

4.1. Discrete splines

We outline briefly the properties of discrete splines, which will be needed later. For a detailed description of the subject, see [26–28]. The discrete splines are defined on the grid $\{k\}$ and constitute a counterpart to the continuous polynomial splines.

The signal

$$\mathbf{b}^{1,n} = \{b_k^{1,n}\} \triangleq \begin{cases} 1, & \text{as } k = 0, \dots, 2n-1 \\ 0, & \text{otherwise} \end{cases}$$
$$\Longleftrightarrow B^{1,n}(z) = \frac{1 - z^{2n}}{1 - z},$$

is called the discrete B-spline of first order.

We define by recurrence the higher order B-splines via discrete convolutions:

$$\mathbf{b}^{p,n} = \mathbf{b}^{1,n} * \mathbf{b}^{p-1,n} \Longleftrightarrow B^{p,n}(z) = \left(\frac{1 - z^{2n}}{1 - z}\right)^p.$$

In this paper we are interested only in the case when $p = 2r$, $r \in \mathbb{N}$ and $n = 1$. In this case, we have $B^{2r,1}(z) = (1 + z^{-1})^{2r}$. The B-spline $\mathbf{b}^{2r,1}$ is symmetric about the point $k = r$ where it attains its maximal value. We define the centered B-spline $\mathbf{q}^{2r}$ of order $2r$ as a shift of the B-spline: $\mathbf{q}^{2r} \triangleq \{q_k^{2r} = b_{k+r}^{2r,1}\}$, $Q^{2r}(z) = z^r B^{2r,1}(z) = z^r(1 + z^{-1})^{2r}$. The discrete spline $\mathbf{a}^{2r} = \{a_k^{2r}\}_{k \in \mathbb{Z}}$ of order $2r$ on the grid $\{2k\}$is defined as a linear combination with real-valued coefficients of shifts of the centered B-spline:

$$a_k^{2r} \triangleq \sum_{l=-\infty}^{\infty} c_l\, q_{k-2l}^{2r} \Longleftrightarrow A^{2r}(z)$$
$$= C(z^2) Q^{2r}(z) = C(z^2)(\upsilon^{2r}(z^2) + z^{-1}\theta^{2r}(z^2)),$$

$$\upsilon^{2r}(z^2) \triangleq Q_e^{2r}(z^2) = \frac{\rho^r(z) + \rho^r(-z)}{2},$$

$$\theta^{2r}(z^2) \triangleq Q_o^{2r}(z^2) = \frac{\rho^r(z) - \rho^r(-z)}{2},$$

$$\rho(z) \triangleq z + 2 + z^{-1}. \tag{4.1}$$

Our scheme that designs prediction filters using discrete splines consists of the following. We construct the discrete spline $\mathbf{a}^{2r}$, which interpolates even samples $\{e_k = x_{2k+1}\}$ of the signal $\mathbf{x} \triangleq \{x_k\}_{k \in \mathbb{Z}}$: $a_{2k}^{2r} = e_k$, $k \in \mathbb{Z}$. Then, we use the values a_{2k+1}^{2r} for the prediction of the odd samples $\{o_k = x_{2k+1}\}$.

The z-transform of the even component of the spline $\mathbf{a}^{2r}$ is

$$A_e^{2r}(z) = C(z)\upsilon^{2r}(z) = E(z) \Longrightarrow C(z) = E(z)/\upsilon^{2r}(z).$$

Then, the z-transform of the odd component of the spline $\mathbf{a}^{2r}$ is

$$A_o^{2r}(z) = C(z)\theta^{2r}(z) = U^{2r}(z)E(z),$$
$$\text{where } U^{2r}(z) \triangleq \frac{\theta^{2r}(z)}{\upsilon^{2r}(z)}. \tag{4.2}$$

Thus, in order to predict the odd samples of the signal $\mathbf{x}$, we apply the filter $U^{2r}(z)$ to the even subarray of $\mathbf{x}$.

4.2. Properties of filters derived from discrete splines

In this section we prove that the designed filters $U^{2r}(z)$ can serve as a source for the construction of frames. Denote

$$\chi^{2r}(z) \triangleq \tfrac{1}{2}(1 + z^{-1} U^{2r}(z^2)),$$
$$\gamma^{2r}(z) \triangleq \tfrac{1}{2}(1 - z^{-1} U^{2r}(z^2)).$$

Proposition 4.1. *The rational functions $U^{2r}(z)$, defined in* (4.2), *have the following properties*:

P1. *No poles on the unit circle* $|z| = 1$.
P2. $U^{2r}(1) = 1$.
P3. *Symmetry*: $z^{-1} U^{2r}(z^2) = z U^{2r}(z^{-2})$.
P4. $|U^{2r}(z)| \leqslant 1$.
P5. *The function $\chi^{2r}(z)$ has a root of multiplicity $2r$ at $z = -1$ and the function $\gamma^{2r}(z)$ has a root of multiplicity $2r$ at $z = 1$.*

Proof. We substitute $z = e^{j\omega}$ into $z^{-1} U^{2r}(z^2)$. We have

$$z^{-1} U^{2r}(z^2) = \frac{e^{ir\omega}(1 - e^{-j\omega})^{2r} + (-1)^r e^{ir\omega}(1 - e^{-j\omega})^{2r}}{e^{ir\omega}(1 + e^{-j\omega})^{2r} + (-1)^r e^{ir\omega}(1 - e^{-j\omega})^{2r}}$$
$$= \frac{\left(\cos\frac{\omega}{2}\right)^{2r} - \left(\sin\frac{\omega}{2}\right)^{2r}}{\left(\cos\frac{\omega}{2}\right)^{2r} + \left(\sin\frac{\omega}{2}\right)^{2r}}.$$

Hence P1–P4 follow. The function

$$\chi^{2r}(z) = \frac{\left(\cos\frac{\omega}{2}\right)^{2r}}{\left(\cos\frac{\omega}{2}\right)^{2r} + \left(\sin\frac{\omega}{2}\right)^{2r}}$$
$$= \frac{\rho^r(z)}{\rho^r(z) + \rho^r(-z)}, \tag{4.3}$$

$$\gamma^{2r}(z) = \frac{\left(\sin\frac{\omega}{2}\right)^{2r}}{\left(\cos\frac{\omega}{2}\right)^{2r} + \left(\sin\frac{\omega}{2}\right)^{2r}} = \frac{\rho^r(-z)}{\rho^r(z)+\rho^r(-z)}. \tag{4.4}$$

Eqs. (4.3) and (4.4) imply P5. □

Remark. It is well known [25] that the squared magnitude of the frequency response of the half-band low-pass digital Butterworth filter of order r is

$$\beta^r(z) = \frac{\rho^r(z)}{\rho^r(z)+\rho^r(-z)}.$$

Obviously, it coincides with $\chi^{2r}(z)$. Similarly, $\gamma^{2r}(z)$ coincides with the squared magnitude of the frequency response of the half-band high-pass Butterworth filter. The relation between the presented filters and the Butterworth filters is described in more details in [15].

Proposition 4.2 (*Averbuch [16]*). *The filter $\chi^{2r}(z)$ generates the scaling function $\Phi^{2r}(t) \in L^2(\mathbb{R})$ such that*

$$\hat{\Phi}^{2r}(\omega) = \lim_{N\to\infty} \prod_{v=1}^{N} \chi^{2r}(\mathrm{e}^{\mathrm{j}2^{-v}\omega}),$$

$$\Phi^{2r}(t) = 2\sum_{k\in\mathbb{Z}} \chi_k^{2r} \Phi^{2r}(2t-k).$$

The scaling function $\Phi^{2r}(t)$ is continuous together with its derivatives up to order $r-1$ (belongs to C^{r-1}).

There exist improved estimates of smoothness for a few low-orders.

Proposition 4.3 (*Zheludev [21]*). *The filters $\chi^{2r}(z)$, $r = 2,3,4$ generate the scaling functions $\Phi^{2r}(t)$, which decay exponentially as $t \to \infty$. In addition $\Phi^4(t) \in C^2$, $\Phi^6(t) \in C^4$, $\Phi^8(t) \in C^5$.*

5. Frames derived from discrete splines

The considerations in Section 4.2 suggest that the filters $U^{2r}(z)$, $\chi^{2r}(z)$, $\gamma^{2r}(z)$, which originate from the discrete splines, are useful for the construction of frames in signal space. To be specific, we choose $U(z) = U^{2r}(z)$, $H(z) = \chi^{2r}(z)$, $\tilde{U}(z) = U^{2p}(z)$, $\tilde{H}(z) = \chi^{2p}(z)$, where r and p are some integers, which, in particular, may coincide with each other. In the rest of the paper we focus mainly on the case when $\tilde{U}(z) = U(z)$.

5.1. Tight frames

Using the function $\rho(z)$, defined in (4.1), we denote $D^r(z^2) \triangleq \rho^r(z) + \rho^r(-z)$. Then,

$$H(z) = \tilde{H}(z) \triangleq \chi^{2r}(z) = \frac{1+z^{-1}U^{2r}(z^2)}{2} = \frac{\rho^r(z)}{D^r(z^2)}.$$

Due to P4, we get a tight frame as soon as we implement the following factorization:

$$\mu_1(z)\mu_1(1/z) = W(z)T(z) = \frac{1-|U^{2r}(z)|^2}{1+|U^{2r}(z)|^2},$$

$$\mu_2(z)\mu_2(1/z) = 2T(z) = \frac{2}{1+|U^{2r}(z)|^2}.$$

Then,

$$G^1(z) = \frac{\mu_1(z^2)}{2}(1+z^{-1}U^{2r}(z^2)) = \frac{\mu_1(z^2)\rho^r(z)}{D^r(z^2)} = \mu_1(z^2)H(z),$$

$$G^2(z) = \frac{\mu_2(z^2)}{2z}(1-z^{-1}U^{2r}(z^2)) = \frac{\mu_2(z^2)\rho^r(-z)}{zD^r(z^2)} = z^{-1}\mu_2(z^2)H(-z). \tag{5.1}$$

The functions W and T are

$$W(z^2) = \frac{4\rho^r(z)\rho^r(-z)}{(D^r(z^2))^2} = \frac{4(-1)^r z^{-2r}(1-z^2)^{2r}}{(D^r(z^2))^2},$$

$$T(z^2) = \frac{(D^r(z^2))^2}{2D^{2r}(z^2)}.$$

Hence,

$$\mu_1(z)\mu_1(1/z) = \frac{2(-1)^r z^{-r}(1-z)^{2r}}{D^{2r}(z)},$$

$$\mu_2(z)\mu_2(1/z) = \frac{(D^r(z))^2}{D^{2r}(z)}.$$

Proposition 5.1. *The function $D^{2r}(z)$ can be represented by the following product*:

$$D^{2r}(z) = 2d^{2r}(z)d^{2r}(1/z),$$

$$\text{where } d^{2r}(z) \triangleq \prod_{k=0}^{r-1} \frac{1+(\gamma_k^r)^2 z}{\gamma_k^r} \tag{5.2}$$

and

$$\gamma_k^r \triangleq \tan\frac{\pi(2k+1)}{8r}, \quad 0<\gamma_k^r<1, \quad k=0,\ldots,r-1.$$

Proof. We have $D^{2r}(z^2) = z^{-2r}((z+1)^{4r} + (z-1)^{4r})$. Thus, the roots of $D^{2r}(z^2)$ are found from the equation

$$(z+1)^{4r} + (z-1)^{4r} = 0 \Leftrightarrow z + 1 = e^{\frac{\pi j(2k+1)}{4r}}(1-z), \quad k = 0, \ldots, 4r-1.$$

Then, the roots are

$$z_k = j\gamma_k^r, \quad z_{2r-1-k} = 1/\gamma_k^r,$$
$$z_{4r-1-k} = -z_k, \quad k = 0, \ldots, r-1$$

and we get the representation

$$D^{2r}(z^2) = 2z^{-2r} \prod_{k=0}^{r-1} (1 - j\gamma_k^r z)(1 - j\gamma_k^r z)\left(1 + \frac{z}{j\gamma_k^r}\right) \times \left(1 - \frac{z}{j\gamma_k^r}\right).$$

Hence, (5.2) follows. □

Corollary 5.1. *The functions* $\mu_1(z)$ *and* $\mu_2(z)$ *are*

$$\mu_1(z) = \frac{(1-z)^r}{d^{2r}(z)}, \quad \mu_2(z) = \frac{D^r(z)}{\sqrt{2}d^{2r}(z)}. \tag{5.3}$$

Consequently, *the filters*

$$G^1(z) = \frac{(1-z^2)^r \rho^r(z)}{d^{2r}(z^2)D^r(z^2)}, \quad G^2(z) = \frac{z^{-1}\rho^r(-z)}{\sqrt{2}d^{2r}(z^2)}.$$

Note that if $r = 2n$ is an even number then we can define the function μ_1 in a slightly different way:

$$\mu_1(z^2) \triangleq \frac{(z - z^{-1})^{2n}}{d^{2r}(z^2)}.$$

The transfer function $G^2(z)$ has a zero of multiplicity $2r$ as $z = 1$. Therefore, the framelet $\psi^2(t)$ has $2r$ vanishing moments. The transfer function $G^1(z)$ has zero of multiplicity r as $z = 1$ and the framelet $\psi^1(t)$ has br vanishing moments.

Remark. The functions $\mu_1(z)$ and $\mu_2(z)$, defined in (5.3), are neither symmetric nor antisymmetric, therefore, the filters $G^1(z)$ and $G^2(z)$, given in (5.1), are not linear phase. Consequently, unlike the scaling function φ, the framelets ψ^1 and ψ^2 lack symmetry.

Examples. The functions D^r and their factorization: Denote

$$\alpha^2 \triangleq 3 - 2\sqrt{2} \simeq 0.1716,$$
$$\alpha_1^4 \triangleq 7 - 4\sqrt{2} - 2\sqrt{20 - 14\sqrt{2}} \simeq 0.4465,$$
$$\alpha_2^4 \triangleq 7 + 4\sqrt{2} - 2\sqrt{20 - 14\sqrt{2}} \simeq 0.0396.$$

Then

$$D^1(z) = 4, \quad D^2(z) = 2(z + 6 + 1/z) = 2d^2(z)d^2(1/z),$$

$$d^2(z) = \frac{1 + \alpha^2 z}{\sqrt{\alpha^2}},$$

$$D^4(z) = 2(z^{-2} + 28z^{-1} + 70 + 28z + z^2) = 2d^4(z)d^4(1/z),$$

$$d^4(z) = \frac{(1 + \alpha_1^4 z)(1 + \alpha_2^4 z)}{\sqrt{\alpha_1^4 \alpha_2^4}}.$$

The simplest case, $r = 1$: We have

$$U^2(z) = \frac{1+z}{2}, \quad H(z) = \frac{z^{-1} + 2 + z}{4},$$
$$\mu_1(z) = \frac{\sqrt{\alpha^2}(1-z)}{1 + \alpha^2 z}, \quad \mu_2(z) = \frac{\sqrt{8\alpha^2}}{1 + \alpha^2 z},$$

$$G^1(z) = \frac{\sqrt{\alpha^2}(z^{-1} + 2 + z)(1 - z^2)}{4(1 + \alpha^2 z^2)},$$
$$G^2(z) = \sqrt{\frac{\alpha^2}{2}} \frac{z^{-1}(-z^{-1} + 2 - z)}{1 + \alpha^2 z^2}.$$

The filter $U^2(z)$ is FIR and, therefore, the scaling function $\varphi(t)$ is compactly supported unlike the framelets $\psi^1(t)$ and $\psi^2(t)$. The transfer function

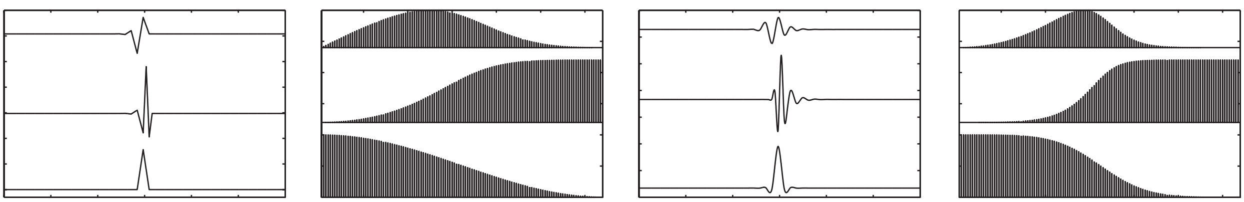

Fig. 1. Filters and framelets for tight frames resulting from the discrete splines of second order (the two leftmost columns) and fourth order (the two rightmost columns). The four rows on the bottom depict the scaling functions $\varphi(t)$ and the frequency response of the low-pass filters $H(z)$, the central four rows display the high-pass filters $G^2(z)$ and the framelets $\psi^2(t)$ and the upper four rows depict the band-pass filters $G^1(z)$ and the corresponding framelets $\psi^1(t)$.

$G^2(z)$ has a zero of multiplicity two at $z=1$. Therefore, the framelet $\psi^2(t)$ has two vanishing moments. The transfer function $G^1(z)$ has a zero of multiplicity one at $z=1$ and the framelet $\psi^1(t)$ has one vanishing moment.

Cubic discrete spline, $r=2$:

$$U^4(z)=4\frac{1+z}{z+6+z^{-1}},\quad H(z)=\frac{(z+2+z^{-1})^2}{2(z^{-2}+6+z^2)},$$

$$G^1(z)=\frac{\sqrt{\alpha_1^4\alpha_2^4}(z-z^{-1})^2(z+2+z^{-1})^2}{2(z^{-2}+6+z^2)(1+\alpha_1^4z^2)(1+\alpha_2^4z^2)},$$

$$G^2(z)=\frac{\sqrt{\alpha_1^4\alpha_2^4}(z-z^{-1})^2}{z(1+\alpha_1^4z^2)(1+\alpha_2^4z^2)}. \quad (5.4)$$

The framelet $\psi^2(t)$ has four vanishing moments. The framelet $\psi^1(t)$ has two vanishing moments. We display in Fig. 1 the filter banks and framelets that are described in the above examples. Although the scaling functions are symmetric, the framelets lack this property. Note that the frequency response of low-pass and high-pass filters are flat, especially for filters derived from cubic discrete splines.

5.2. Sibling frames

Let $\mu_1(z)$, $\tilde{\mu}_1(z)$ and $\mu_2(z)$, $\tilde{\mu}_2(z)$ be pairs of functions such that

$$\mu_1(z)\tilde{\mu}_1(1/z)=W(z)T(z)=\frac{2(-1)^rz^{-r}(1-z)^{2r}}{D^{2r}(z)}=\frac{2\rho^r(-z)}{D^{2r}(z)},$$

$$\tilde{\mu}_2(z)\mu_2(1/z)=2T(z)=\frac{(D^r(z))^2}{D^{2r}(z)}.$$

Then, the analysis and synthesis filters are

$$H(z)=\tilde{H}(z)=\frac{\rho^r(z)}{D^r(z^2)},\quad \tilde{G}^1(z)=\tilde{\mu}_1(z^2)H(z),$$

$$G^1(z)=\mu_1(z^2)H(z),$$

$$\tilde{G}^2(z)=z^{-1}\tilde{\mu}_2(z^2)H(-z),\quad G^2(z)=z^{-1}\mu_2(z^2)H(-z).$$

As we saw in Section 5.1, if we require that $\mu_s(z)=\tilde{\mu}_s(z)$, $s=1,2$ then we have to factorize the symmetric positive function $D^{2r}(z)$ into non-symmetric factors $d^{2r}(z)d^{2r}(1/z)$. Now we can avoid such a factorization and obtain the symmetric functions μ. Consequently, the filters $G^1(z)$, $\tilde{G}^1(z)$ and $G^2(z)$, $\tilde{G}^2(z)$ will be linear phase. In addition, as it was mentioned above, we may swap vanishing moments between analysis and synthesis framelets.

We have the following symmetric factorizations:

$$\tilde{\mu}_1(z)=\frac{\sqrt{2}\rho^{r-p}(-z)}{D^{2r}(z)},$$

$$\mu_1(z)=\sqrt{2}\rho^p(-z),\quad 0\leqslant p\leqslant r,$$

$$\tilde{\mu}_2(z)=\frac{D^r(z)}{D^{2r}(z)},\quad \mu_2(z)=D^r(z).$$

Also the antisymmetric $\tilde{\mu}_1$ and μ_1 are possible:

$$\tilde{\mu}_1(z)=\frac{\sqrt{2}(-1)^rz^{-r}(1-z)^{2r-p}}{D^{2r}(z)},$$

$$\mu_1(z)=\sqrt{2}(-1)^pz^{-p}(1-z)^p. \quad (5.5)$$

Here $p<2r$ is an odd integer. Note that

$$\rho(-z^2)=-(z-z^{-1})^2=\rho(z)\rho(-z).$$

Then, the corresponding symmetric filters are

$$\tilde{G}^1(z)=\frac{\sqrt{2}(-1)^{r-p}(z-z^{-1})^{2(r-p)}\rho^r(z)}{D^{2r}(z^2)D^r(z^2)},$$

$$G^1(z)=\frac{\sqrt{2}(-1)^p(z-z^{-1})^{2p}\rho^r(z)}{D^r(z^2)},$$

$$\tilde{G}^2(z)=z^{-1}\frac{\rho^r(-z)}{D^{2r}(z^2)},\quad G^2(z)=z^{-1}\rho^r(-z). \quad (5.6)$$

All the four filters given in (5.6) are symmetric about inversion $z\to 1/z$. They are IIR except for G^2, which is FIR. The framelet $\psi^2(t)$ has compact support and $2r$ vanishing moments. The support of the framelet $\tilde{\psi}^2(t)$ is infinite but the framelet decays exponentially as $t\to\infty$ and it has $2r$ vanishing moments. Similarly, the framelets $\tilde{\psi}^1(t)$ and $\psi^1(t)$ decay exponentially as $t\to\infty$. The analysis framelet $\tilde{\psi}^1(t)$ has $2(r-p)$ vanishing moments whereas the synthesis framelet $\psi^1(t)$ has $2p$ vanishing moments, where $0\leqslant p\leqslant r$. In particular, we can assign all $2r$ vanishing moments to the analysis framelet. Then the filter G^1 will be low-pass. Recall that the pair of analysis filters can be interchanged with the synthesis pair.

The antisymmetric filters are

$$\tilde{G}^1(z)=\frac{\sqrt{2}(-1)^rz^{-2r}(1-z^2)^{2r-p}\rho^r(z)}{D^{2r}(z^2)D^r(z^2)},$$

$$G^1(z)=\frac{\sqrt{2}(-1)^pz^{-2p}(1-z^2)^p\rho^r(z)}{D^r(z^2)}. \quad (5.7)$$

In this case, the analysis framelet $\tilde{\psi}^1(t)$ has $2r-p$ vanishing moments whereas the synthesis framelet $\psi^1(t)$ has p vanishing moments where $0<p<2r$.

Examples. The simplest case, $r=1$**:** We have

$$U^2(z)=\frac{1+z}{2},\quad H(z)=\frac{z^{-1}+2+z}{4},$$

$$\tilde{\mu}_2(z)=\frac{8}{z+6+1/z},\quad \mu_2(z)\equiv 1,$$

$$\tilde{G}^2(z)=\frac{z^{-1}(-z^{-1}+2-z)}{z^2+6+1/z^2},$$

$$G^2(z)=z^{-1}(-z^{-1}+2-z). \tag{5.8}$$

The filters $\tilde{G}^1(z)$ and $G^1(z)$ can by antisymmetric or symmetric.

- Define

$$\tilde{\mu}_1(z)=\frac{(1-z)}{z+6+1/z},\quad \mu_1(z)=(1-z),$$

then,

$$\tilde{G}^1(z)=\frac{(z^{-1}+2+z)(1-z^2)}{4(z^2+6+1/z^2)},$$

$$G^1(z)=\frac{(z^{-1}+2+z)(1-z^2)}{4}. \tag{5.9}$$

The filter $U^2(z)$ is FIR and, therefore, the scaling function $\varphi(t)$ and the synthesis framelets $\psi^1(t)$ and $\psi^2(t)$ are compactly supported unlike the analysis framelets $\tilde{\psi}^1(t)$ and $\tilde{\psi}^2(t)$. The framelets $\psi^2(t)$ and $\tilde{\psi}^2(t)$ have two vanishing moments. The framelets $\psi^1(t)$ and $\tilde{\psi}^1(t)$ are antisymmetric and have one vanishing moment. We display in Fig. 2 the analysis and synthesis filter banks described in (5.8) and (5.9) and their corresponding framelets.

- The symmetric filters are available

$$\tilde{\mu}_1(z)=\frac{-z+2-1/z}{z+6+1/z},\quad \mu_1(z)\equiv 1,$$

$$\tilde{G}^1(z)=-\frac{(z-1/z)^2}{4(z^2+6+1/z^2)},$$

$$G^1(z)=\frac{z^{-1}+2+z}{4}. \tag{5.10}$$

The analysis framelet $\tilde{\psi}^1(t)$ has two vanishing moments, whereas the synthesis framelet $\psi^1(t)$ has none. We illustrate this example in Fig. 3.

Cubic discrete spline, $r=2$**:**

$$U^4(z)=4\frac{1+z}{z+6+z^{-1}},$$

$$H(z)=\frac{(z+2+z^{-1})^2}{2(z^{-2}+6+z^2)},$$

$$\tilde{\mu}_2(z)=\frac{z+6+z^{-1}}{z^{-2}+28z^{-1}+70+28z+z^2},$$

$$\mu_2(z)=2(z+6+z^{-1}),$$

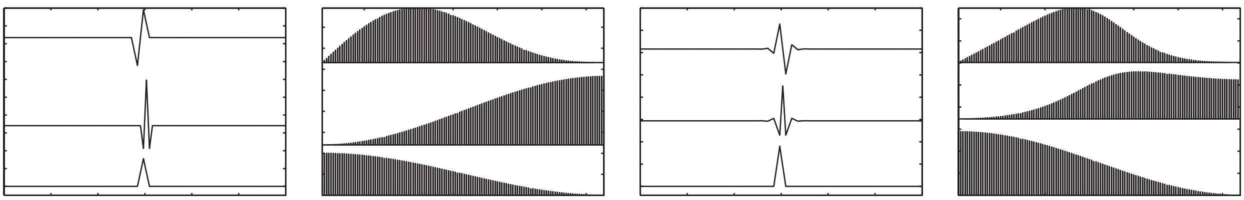

Fig. 2. The analysis and synthesis sibling filter banks that are described in (5.8), (5.9) and their corresponding framelets. From bottom to top of the two rightmost columns: scaling function $\varphi(t)$ and the low-pass filter $H(z)$, symmetric analysis framelet $\tilde{\psi}^2(t)$ and the high-pass analysis filter $\tilde{G}^2(z)$, antisymmetric analysis framelet $\tilde{\psi}^1(t)$ and the band-pass analysis filter $\tilde{G}^1(z)$. From bottom to top of the two leftmost columns: the scaling function $\varphi(t)$, the synthesis framelets $\psi^2(t)$, $\psi^1(t)$ and the synthesis filters $H(z)$, $G^2(z)$, $G^1(z)$.

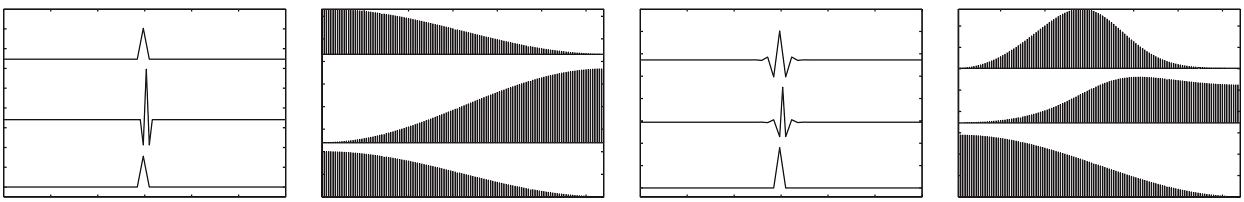

Fig. 3. The analysis and synthesis sibling filter banks that are described in (5.8) and (5.10) and their corresponding framelets. The waveforms and filters are presented in the same order as in Fig. 2. All the displayed waveforms are symmetric.

$$\tilde{G}^2(z) = \frac{(z-2+z^{-1})^2}{2(z^{-4}+28z^{-2}+70+28z^2+z^4)},$$

$$G^2(z) = \frac{(z-2+z^{-1})^2}{z}. \quad (5.11)$$

The framelets $\tilde{\psi}^2(t)$ and $\psi^2(t)$ are symmetric and have four vanishing moments. The synthesis framelet $\psi^2(t)$ is compactly supported. A few options are available for the filters $\tilde{G}^1(z)$ and $G^1(z)$.

- Symmetric factorization:

$$\tilde{\mu}_1(z) = \frac{z-2+z^{-1}}{z^{-2}+28z^{-1}+70+28z+z^2},$$

$$\mu_1(z) = z-2+z^{-1},$$

$$\tilde{G}^1(z) = \frac{(z-z^{-1})^2(z+2+z^{-1})^2}{2(z^{-2}+6+z^2)(z^{-4}+28z^{-2}+70+28z^2+z^4)},$$

$$G^1(z) = \frac{(z-z^{-1})^2(z+2+z^{-1})^2}{2(z^{-2}+6+z^2)}. \quad (5.12)$$

Here, both the analysis $\tilde{\psi}^1(t)$ and the synthesis $\psi^1(t)$ framelets are symmetric and have two vanishing moments. This example is depicted in Fig. 4.

- Another symmetric factorization:

$$\tilde{\mu}_1(z) = \frac{(z-2+z^{-1})^2}{z^{-2}+28z^{-1}+70+28z+z^2}, \quad \mu_1(z) \equiv 1,$$

$$\tilde{G}^1(z) = \frac{(z-z^{-1})^4(z+2+z^{-1})^2}{2(z^{-2}+6+z^2)(z^{-4}+28z^{-2}+70+28z^2+z^4)},$$

$$G^1(z) = \frac{(z+2+z^{-1})^2}{2(z^{-2}+6+z^2)}. \quad (5.13)$$

Here, all the four vanishing moments are assigned to the analysis framelet $\tilde{\psi}^1(t)$. The synthesis framelet $\psi^1(t)$ does not have vanishing moments. We illustrate this example in Fig. 5.

- Antisymmetric factorization:

$$\tilde{\mu}_1(z) = \frac{z^{-2}(1-z)^3}{z^{-2}+28z^{-1}+70+28z+z^2},$$

$$\mu_1(z) = -z^{-1}(1-z),$$

$$\tilde{G}^1(z) = \frac{z^{-4}(1-z^2)^3(z+2+z^{-1})^2}{2(z^{-2}+6+z^2)(z^{-4}+28z^{-2}+70+28z^2+z^4)},$$

$$G^1(z) = \frac{-z^{-2}(1-z^2)(z+2+z^{-1})^2}{2(z^{-2}+6+z^2)}. \quad (5.14)$$

Both the analysis $\tilde{\psi}^1(t)$ and the synthesis $\psi^1(t)$ framelets are antisymmetric. The analysis framelet $\tilde{\psi}^1(t)$ has three vanishing moments, the synthesis framelet $\psi^1(t)$ has one vanishing moment .

The framelet $\psi^2(t)$ has four vanishing moments. The framelet $\psi^1(t)$ has two vanishing moments. It is illustrated in Fig. 6.

Example of the dual pair of frames. We describe the simplest case, $r=1$. In line with Proposition 3.2 we

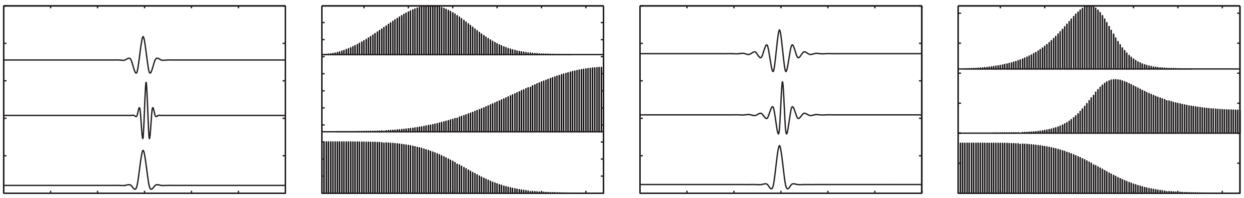

Fig. 4. The analysis and synthesis sibling filter banks that are described in (5.11) and (5.12) and their corresponding framelets. The waveforms and filters are presented in the same order as in Fig. 2. All the displayed waveforms are symmetric.

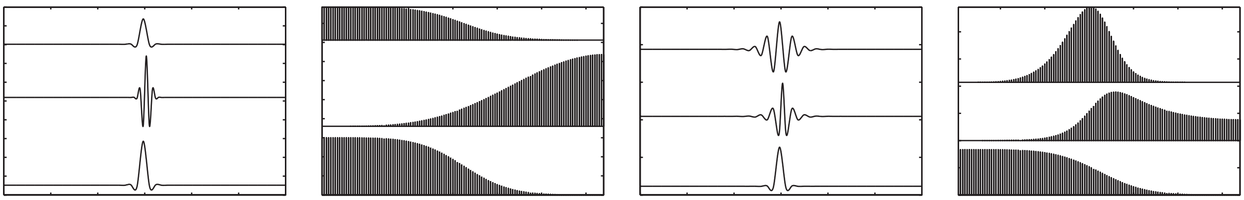

Fig. 5. The analysis and synthesis sibling filter banks that are described in (5.11) and (5.13) and their corresponding framelets. The waveforms and filters are presented in the same order as in Fig. 2. All the displayed waveforms are symmetric.

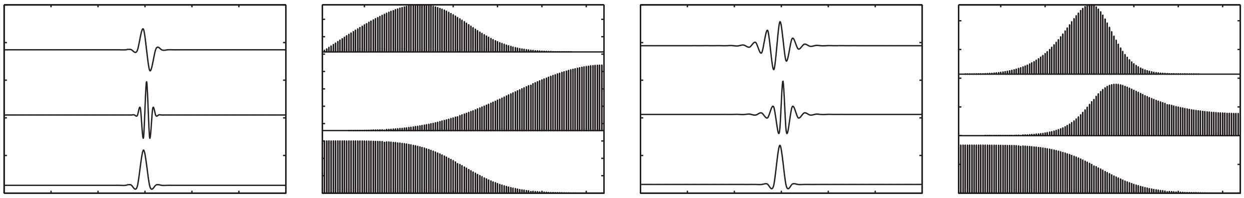

Fig. 6. The analysis and synthesis sibling filter banks, that are described in (5.11) and (5.14) and their corresponding framelets. The waveforms and filters are presented in the same order as in Fig. 2. The scaling functions and the framelets $\tilde{\psi}^2(t)$ and $\psi^1(t)$ are antisymmetric.

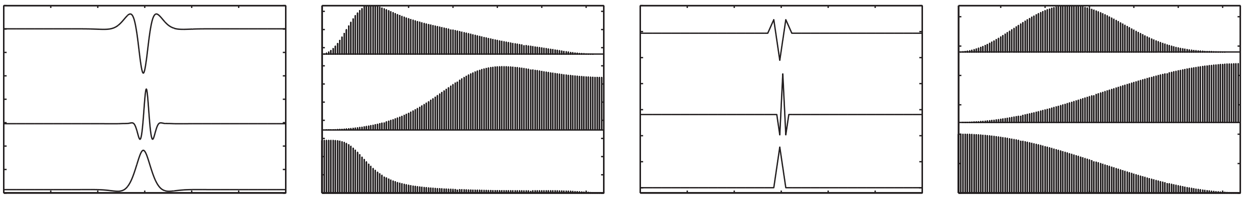

Fig. 7. The pair of dual analysis and synthesis filter banks described in (5.15) and (5.16) and their corresponding framelets. The waveforms and filters are presented in the same order as in Fig. 2. All the displayed waveforms are symmetric.

have

$$U^2(z)=\frac{1+z}{2},\quad \tilde{H}(z)=\frac{z^{-1}+2+z}{4},$$
$$T(z)=\frac{4}{z+6+1/z}$$

and put

$$\tilde{\mu}_2(z)\equiv 1,\quad \tilde{\mu}_1(z)=z-2+z^{-1},$$
$$1+|\tilde{\mu}_1(z)|^2=z^2-4z+7-4z^{-1}+z^2. \tag{5.15}$$

Then,

$$H(z)=\frac{2(z^{-1}+2+z)}{(z^2+6+1/z^2)(z^4-4z^2+7-4z^{-2}+z^4)},$$

$$\tilde{G}^1(z)=\frac{(z^{-1}+2+z)(z^2-2+z^{-2})}{4},$$

$$G^1(z)=\frac{2(z^{-1}+2+z)(z^2-2+z^{-2})}{(z^2+6+1/z^2)(z^4-4z^2+7-4z^{-2}+z^4)},$$

$$\tilde{G}^2(z)=z^{-1}\,\frac{-z^{-1}+2-z}{4},$$

$$G^2(z)=\frac{z^{-1}(-z^{-1}+2-z)}{z^2+6+1/z^2}. \tag{5.16}$$

The analysis filters $\tilde{H}(z)$ and $\tilde{G}^r(z),\ r=1,2$, are FIR and, therefore, the scaling function $\tilde{\varphi}(t)$ and the analysis framelets and $\tilde{\psi}^r(t),\ r=1,2$, are compactly supported unlike the synthesis framelets. All the waveforms are symmetric. All the framelets have two vanishing moments. Recall that the synthesis and analysis filters can be interchanged. The illustrations are given in Fig. 7.

6. Conclusion

We presented a new family of tight and sibling frames, which are generated by perfect reconstruction filter banks. The filter banks are based on discrete interpolatory splines and are related to Butterworth filters. Note that a similar scheme for filter design is possible by using the continuous interpolatory and quasi-interpolatory splines. Each of the designed filter banks comprises one interpolatory symmetric low-pass filter, one band-pass and one high-pass filters. In the tight frames case, the band-pass and high-pass filters lack symmetry, but in sibling frames the high-pass filters are symmetric. The band-pass filters may be symmetric or antisymmetric. These filter banks generate smooth analysis and synthesis scaling functions and pairs of framelets. One step of the framelet transform of a signal of length N produces $1.5N$ coefficients. Thus, the full transform of this signal consists of $\log_2 N$ steps that produces $2N$ coefficients.

While in the tight frame all the analysis waveforms coincide with their synthesis counterparts, in the sibling frames we can vary the framelets making them different for the synthesis and the analysis cases. Therefore, we can, for example, swap the vanishing moments and computational cost between the synthesis and the analysis framelets. We constructed dual pairs of frames starting from a symmetric interpolatory low-pass analysis filter, whose transfer function is rational and has a zero of arbitrary order m. The analysis and synthesis framelets generated by the high-pass filters are symmetric and have m vanishing moments. The framelets generated by the band-pass filters are (anti)symmetric and may have arbitrary number of vanishing moments.

Most of the designed filters are IIR and their transfer functions are rational. Therefore, they allow fast implementation via a cascade of elementary causal and anticausal recursive filters [25]. The waveforms are well localized in time domain despite their infinite support.

We anticipate that this new family of transforms will have a wide range of signal processing applications, in particular in error protection of transmitted signals and denoising of audio signals and images.

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