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A TERNARY INTERPOLATORY SUBDIVISION SCHEMES ORIGINATED FROM SPLINES

AMIR Z. AVERBUCH

School of Computer Science, Tel Aviv University, Tel Aviv 69978, Israel
amir@math.tau.ac.il

VALERY A. ZHELUDEV

School of Computer Science, Tel Aviv University, Tel Aviv 69978, Israel
zhel@post.tau.ac.il

GARY B. FATAKHOV

Department of Applied Mathematics, School of Mathematical Sciences
Tel Aviv University, Tel Aviv 69978, Israel
fat_gar@intermail.co.il

EDUARD H. YAKUBOV

Department of Sciences, Holon Academic Institute of Technology, 52 Golomb Street, Holon 58102, Israel
yakubov@hit.ac.il

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A generic technique for construction of a ternary interpolatory subdivision schemes, which is based on polynomial and discrete splines, is presented. These schemes have rational symbols. The symbols are explicitly presented in the paper. This is accompanied by a detailed description of the design of the refinement masks and by algorithms that verify the convergence these schemes. In addition, the smoothness of the limit functions is investigated. The ternary subdivision schemes, whose construction is based on continuous splines, become tools for fast computation of interpolatory splines of arbitrary order at triadic rational points.

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1. Introduction

Subdivision started as a tool for efficient computation of spline functions. Now, it is an independent subject with many applications. It is being used for the development of new methods for curve and surface design, approximation, generation of wavelets and multiresolution analysis and also for solving some classes of functional equations. Many research paper have been written over the years on various subdivision schemes including binary subdivision schemes.

Ternary subdivision schemes were investigated in [12,13] where it was showed that there exists a family of three and four point ternary subdivision schemes, whose limit functions belong to C^1 and C^2, respectively. They used finite refinement masks and showed that the fundamental functions, which were generated by ternary subdivision schemes, have smaller support than their binary counterparts. For continuity analysis of the ternary schemes, they used the generating function formalism technique that was developed in [1].

A generic technique for construction of different interpolatory binary subdivision schemes, which are based on polynomial and discrete splines, was introduced in [3]. These schemes have rational symbols and infinite masks but they are competitive (regularity, speed of convergence, computational complexity) with schemes that have finite masks. Exponential decay of the basic limit functions of schemes with rational symbols is proved in [3]. This property guarantees the convergence of such schemes on initial data of power growth. A similar approach resulted in construction of a diverse library of wavelet and frame transforms, which proved to be efficient in signal and image processing applications [4,5,6].

In the present paper, we investigate univariate Ternary Interpolatory Subdivision schemes (TISS) that are derived from continuous and discrete interpolatory splines. Our analysis extends the technique that was developed in [1,2,3] for binary schemes. We present a detailed analysis that enables to verify the convergence and the smoothness of TISS. We also show how to derive refinement masks from continuous and discrete splines. A fast algorithm for computation of interpolatory splines of arbitrary order at triadic rational points is described.

The paper is organized as follows. Basic notation and fundamentals of subdivision schemes are given in section 2. Section 3 presents the main results on TISS. Sections 4 and 5 show how to design refinement masks and how to implement the corresponding TISS. Examples of spline-based TISS are given in section 6. The Appendix describes how to evaluate the coefficients of TISS with infinite masks using the discrete Fourier transform. In addition, the scheme for verifying the convergence of TISS is also presented in the Appendix.

2. Notation and fundamentals of interpolatory subdivision schemes

Interpolatory subdivision schemes (ISS) are refinement rules, which iteratively refine the data by inserting values that correspond to intermediate points. This is done by using linear combinations of values in the initial points, while the data in these initial points are retained. Non-interpolatory schemes also update the initial data in addition to the inserted values in intermediate points. Stationary schemes use the same insertion rule at each refinement step. A scheme is called uniform if its insertion rule does not depend on the location in the data. To be more specific, a univariate stationary uniform subdivision scheme with ternary refinement, denoted by S_a, consists of the following: The function $\{f^k\}$, which is defined on the grid $\mathbf{G^k} = \{j/3^k\}_{j\in\mathbb{Z}}$: $f^k(j/3^k) = f^k_j$, is extended onto the grid $\mathbf{G^{k+1}}$ by filtering the array $\{f^k_j\}_{j\in\mathbb{Z}}$ to become:

$$f^{k+1}_j = \sum_{l\in\mathbb{Z}} a_{j-3l} f^k_l. \tag{2.1}$$

This is one refinement step. The next refinement step employs f^{k+1}_j as an initial data. The array $a = \{a_k\}_{k\in\mathbb{Z}}$ is called a refinement mask of a subdivision scheme S. We assume that the series $\sum_{k\in\mathbb{Z}} a_k$ is absolutely convergent. A subdivision scheme S with the refinement mask a is denoted by S_a. The z-transform of a mask a is defined by $a(z) = \sum_{k\in\mathbb{Z}} a_k z^k$. It is also called the symbol of S_a. Throughout the paper, we assume that $z = e^{-i\omega}$. We work with infinite masks and assume that there exist Laurent polynomials $T(z)$ and $P(z)$ such that the symbol is

$$a(z) = \sum_{k \in \mathbb{Z}} a_k z^k = \frac{T(z)}{P(z)}.$$

If S_a is interpolatory then $a_0 = 1$ and $a_{3k} = 0$ for all $k \in \mathbb{Z}, k \neq 0$.
The sequence of the values f_j^k at level k is represented by its z-transform $F_k(z)$ that is formally defined as $F_k(z) = \sum_{j \in \mathbb{Z}} f_j^k z^j$. Equation (2.1) implies that

$$F_{k+1}(z) = a(z) F_k(z^3) \Longrightarrow F_{k+1}(z) = F_1(z^{3^k}) \prod_{i=0}^{k-1} a(z^{3^i}). \tag{2.2}$$

The TISS Eq. (2.1) can be split into:

$$\begin{cases} f_{3i}^{k+1} = f_i^k \\ f_{3i \pm 1}^{k+1} = \sum_{j \in \mathbb{Z}} a_{3j \pm 1} f_{i-j}^k \end{cases}.$$

Definition 2.1. *Given an initial data* $\mathbf{f^0} = \{f_j^0\} \in \ell_1,\ j \in \mathbb{Z}$, *denote by* $f^k(t)$ *the sequence of polygonal lines (second order splines) that interpolate the data generated by* S_a *at the corresponding refinement level* $\{f^k(3^{-k}j) = f_j^k = (S_a^k f^0)_j\},\ j \in \mathbb{Z}$. *If* $\{f^k(t)\}$ *converges uniformly at any finite interval to a continuous function* $f^\infty(t)$, *as* $k \to \infty$, *then we say that the subdivision scheme* S_a *converges on the initial data* $\mathbf{f^0}$ *and* $f^\infty(t)$ *is called its limit function. If* S_a *converges for any* $f^0 \in \ell_1$ *then* S_a *is called the convergent TISS.*

3. Convergence and regularity of TISS with infinite masks

3.1. *Preliminary results*

We restrict the admissible initial data to the sequences $\mathtt{f} = \{f_j^0\} \in \ell_1,\ j \in \mathbb{Z}$. The symbol $a(z) = T(z)/P(z)$ is subject to the following requirements:

P1: The Laurent polynomials $P(z)$ and $T(z)$ are symmetric about inversion: $P(z^{-1}) = P(z),\ T(z^{-1}) = T(z)$. Thus, they are real on the unit circle $|z| = 1$.

P2: The roots of the denominator $P(z)$ are real, simple and do not lie on the unit circle $|z| = 1$.

P3: The symbol $a(z)$ is factorized as follows:

$$a(z) = (1 + z^{-1} + z^{-2})q(z),\ q(1) = 1. \tag{3.1}$$

In the sequel, we say that a subdivision scheme S_a belongs to the class $\mathbf{P}$ if its symbol $a(z)$ possesses the properties **P1**– **P3**.

The above properties imply that the coefficients a_k of the mask of the scheme S_a of the class $\mathbf{P}$ are symmetric about zero. If **P1** and **P2** hold then $P(z)$ can be represented as follows:

$$P(z) = \prod_{n=1}^{r} \frac{1}{\gamma_n}(1 + \gamma_n z)(1 + \gamma_n z^{-1}), \quad 0 < |\gamma_1| < |\gamma_2| < \ldots |\gamma_r| = e^{-g} < 1,\ g > 0. \tag{3.2}$$

Proposition 3.1. *If the symbol of a scheme S_a is $a(z) = T(z)/P(z)$ and Eq. (3.2) holds then the mask satisfies the inequality $|a_j| \le A\,e^{-g|j|}$, where A is a positive constant.*

Proof: Assume that the degree t of $T(z)$ is less than the degree p of $P(z)$. If Eq. (3.2) holds then the symbol can be represented as follows:

$$a(z) = \sum_{n=1}^{r} \left(\frac{A_n^+}{1 + \gamma_n z} + \frac{A_n^- z}{1 + \gamma_n z^{-1}} \right) = \sum_{n=1}^{r} \left(A_n^+ \sum_{j=0}^{\infty} (-\gamma_n)^j z^j + z A_n^- \sum_{j=0}^{\infty} (-\gamma_n)^j z^{-j} \right)$$

$$= \sum_{j=0}^{\infty} \left(a_j^+ z^j + a_j^- z^{1-j} \right),\ a_j^+ = \sum_{n=1}^{r} A_n^+ (-\gamma_n)^j,\ a_j^- = \sum_{n=1}^{r} A_n^- (-\gamma_n)^j,$$

where

$$|a_j^+| \le |\gamma_r|^j \sum_{n=1}^{r} |A_n^+| \le A e^{-gj},\ |a_j^-| \le |\gamma_r|^j \sum_{n=1}^{r} |A_n^-| \le A e^{-gj}. \tag{3.3}$$

If $t \ge p$ then a polynomial of degree $t - p$ is added to the expansion (3.3). Obviously, this addition does not affect the decay of the mask $a(j)$ as j tends to infinity. ■

Lemma 3.1. *Let S_a be the subdivision scheme whose symbol is $a(z) = T(z)/P(z)$ and the Laurent polynomial $P(z)$ satisfies the properties **P1** and **P2**. If Eq. (3.2) holds then for any finite initial data $\mathbf{f^0}$ the following inequalities*

$$|f_j^k| \le A_k\, e^{-gj3^{-k+1}} \tag{3.4}$$

hold, where A_k are positive constants.

Proof: The mask of the scheme S_a decays exponentially, i.e. $|a_j| \leq Ae^{-gj}$. Due to Eq. (2.2), $F_1(z) = a(z)F_0(z^3) = T_1(z)/P_1(z)$, where $T_1(z) \triangleq T(z)F_0(z^3)$ and $P_1(z) = P(z)$. Hence, the roots of $P_1(z)$ are $\rho_n^1 = -\gamma_n,\ \ 1 \leq n \leq r$, and, therefore, $|f_j^1| \leq A_1\, e^{-gj}$. The next refinement step produces the following $z-$ transform:

$$F_2(z) = a(z)F_1(z^3) = \frac{T_2(z)}{P_2(z)}, \quad P_2(z) = P(z)P(z^3).$$

The roots of $P_2(z)$ satisfy the inequality $|\rho_n^2| \leq \sqrt[3]{|\gamma_r|} = e^{-g/3}$. Hence, $|f_j^2| \leq A_2\, e^{-gj/3}$. Then, Eq. (3.4) is derived by induction. ■

Denote $\Delta f_j^k \triangleq f_{j+1}^k - f_j^k$ and the z-transform of the difference sequence $\left\{\Delta f_j^k\right\}$ by $Q_k(z)$. Then, $Q_k(z) = (z^{-1}-1)F_k(z)$.

Proposition 3.2. *If Eq. (3.1) holds then*

$$Q_{k+1}(z) = q(z)Q_k(z^3). \tag{3.5}$$

Proof:

$$\begin{aligned} Q_{k+1}(z) &= (z^{-1}-1)F_{k+1}(z) = (z^{-1}-1)a(z)F_k(z^3) = (z^{-1}-1)(z^{-2}+z^{-1}+1)q(z)F_k(z^3) \\ &= q(z)(z^{-3}-1)F_k(z^3) = q(z)Q_k(z^3). \end{aligned}$$

■

Equation (3.5) implies that there exists a ternary subdivision scheme of differences S_q defined by:

$$S_q:\ \Delta f_j^{k+1} = \sum_{l\in\mathbb{Z}} q_{j-3l}\Delta f_l^k. \tag{3.6}$$

Denote by $F(z;f)$ the z-transform of a sequence $f \triangleq \{f_j\}$. Then, the z-transform of the difference sequence $\Delta f \triangleq \{\Delta f_j\}$ is $F(z;\Delta f)$=$(z^{-1}-1)F(z;f)$ and $F(z;S_af) \triangleq a(z)F(z^3;f)$.

We get

$$\begin{aligned} F(z;\Delta S_af) &= (z^{-1}-1)F(z;S_af) = (z^{-1}-1)a(z)F(z^3;f) \\ &= (z^{-1}-1)(z^{-2}+z^{-1}+1)q(z)F(z^3;f) \\ &= (z^{-3}-1)q(z)F(z^3;f) = q(z)(z^{-3}-1)F(z^3;\Delta f) = F(z;S_q\Delta f). \end{aligned}$$

Thus, $\Delta(S_af) = S_q\Delta f$.

TISS originated from the second order (first degree) interpolatory splines We define the values $f_{3i\pm1}^{k+1}$ as the values at the points $(i \pm 1/3)\,3^{-k}$ of the piece-wise linear spline, which interpolates the data f_i^k on the grid $\{3^{-k}i\}$. Then,

$$\begin{cases} f_{3i}^{k+1} = f_i^k \\ f_{3i-1}^{k+1} = \frac{1}{3}\cdot f_{i-1}^k + \frac{2}{3}\cdot f_i^k \ , \\ f_{3i+1}^{k+1} = \frac{2}{3}\cdot f_i^k + \frac{1}{3}\cdot f_{i+1}^k \end{cases}$$

$$F_{k+1}(z) = a_{lin}(z)F_k(z^3), \quad a_{lin}(z) \triangleq \frac{(z^2+z+1)^2}{3z^2}. \tag{3.7}$$

3.2. *Convergence of subdivision schemes*

Denote $\|f^k\|_\infty \triangleq \max_{i\in\mathbb{Z}} |f_i^k|$. Then, we have

$$\|f^{k+1}\|_\infty \le \|S_a\|\|f^k\|_\infty, \text{ where } \|S_a\| \triangleq \max\left\{\sum_{k\in\mathbb{Z}} |a_{3k}|, \sum_{k\in\mathbb{Z}} |a_{3k+1}|, \sum_{k\in\mathbb{Z}} |a_{3k+2}|\right\}.$$

After L refinement steps, the following inequality holds

$$\|f^{k+L}\|_\infty \le \|S_a^L\|\|f^k\|_\infty, \text{ where } \|S_a^L\| \triangleq \max_n\left\{\sum_{k\in\mathbb{Z}} \left|a_{3^L k+n}^{[L]}\right|,\right\}, \quad n = 0, ..., 3^L-1, \tag{3.8}$$

and $a_k^{[L]}$ is the mask of the operator $S_a^{[L]}$ whose symbol is $a^{[L]}(z) = a(z), \ldots, a(z^{3^{L-1}})$.

Theorem 3.1. *Let S_a be a TISS of Class **P** and S_q be the subdivision scheme of differences defined by Eq. (3.6). The scheme S_a converges if for some $L \in \mathbb{N}$*

$$\|S_q^L\|_\infty = \mu < 1. \tag{3.9}$$

Remark A subdivision scheme whose norm satisfies the inequality (3.9) is called contractive.

Proof: Recall that $f^k(t)$ are the second order splines that interpolate the subsequently refined data $f^k(3^{-k}i) = f_i^k,\ i \in \mathbb{Z}$, where the initial data is $\{f_i^0\}$. We prove that $\{f^k(t)\}_{k\in\mathbb{Z}}$ is a Cauchy sequence that is for a given $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that for all $n, m > N_1$: $\sup_{t\in\mathbb{R}} |f^n(t) - f^m(t)| < \varepsilon$.

We can write $f^n(t) - f^m(t) = \sum_{k=m}^{n-1} D^{k+1}(t)$, where $D^{k+1}(t) \triangleq f^{k+1}(t) - f^k(t)$.

Denote $g_{3i}^{k+1} = f_i^k,\ g_{3i-1}^{k+1} = (f_{i-1}^k + 2f_i^k)/3,\ g_{3i+1}^{k+1} = (2f_i^k + f_{i+1}^k)/3$. From Eq. (3.7) we have $G_{k+1}(z) = a_{lin}(z)F_k(z^3)$, where $a_{lin}(z) = z^2\left(z^{-2} + z^{-1} + 1\right)^2/3$. The maximal absolute value of the piecewise linear function $D^{k+1}(t)$ at the interval $[3^{-k}i, 3^{-k}(i+1)]$ is reached at its breakpoints. Therefore,

$$|D^{k+1}(t)| \le \max\left\{\left\|f_{3i-1}^{k+1} - g_{3i-1}^{k+1}\right\|,\ \left\|f_{3i+1}^{k+1} - g_{3i+1}^{k+1}\right\|\right\},$$
$$\sup_{t\in\mathbb{R}} |D^{k+1}(t)| = \sup_{t\in\mathbb{R}} |f^{k+1}(t) - g^{k+1}(t)| = ||f^{k+1} - g^{k+1}||_\infty. \quad (3.10)$$

From Eq. (3.1), we have $a(z) = (z^{-2} + z^{-1} + 1)q(z),\ q(1) = 1$. Thus, $q(z) - (z^2 + z^1 + 1)/3 = (z^{-1} - 1)r(z),\ r(z) = \sum_{n\in\mathbb{Z}} r_n z^n$ and we have

$$F_{k+1}(z) - G_{k+1}(z) = a(z)F_k(z^3) - a_{lin}(z)F_k(z^3)$$
$$= (z^{-2} + z^{-1} + 1)(q(z) - \frac{(z^2 + z^1 + 1)}{3})F_k(z^3) = (z^{-2} + z^{-1} + 1)(z^{-1} - 1)r(z)F_k(z^3)$$
$$= (z^{-3} - 1)r(z)F_k(z^3) = r(z)Q_k(z^3) \quad (3.11)$$

where $Q_k(z) = (\Delta f^k)(z)$. The rational function $r(z)$ has the same denominator as $a(z)$ and, by Proposition (3.1), the coefficients $|r_n| \le Ce^{-g|n|},\ g > 0$. Therefore, $|r(z)| \le \sum_{n\in\mathbb{Z}} |r_n| = R < \infty$. Equation (3.11) implies that

$$f_i^{k+1} - g_i^{k+1} = \sum_{j\in\mathbb{Z}} r_{i-3j}\Delta f_j^k. \quad (3.12)$$

By combining Eqs. (3.10) and (3.12) we get

$$\sup_{t\in\mathbb{R}} |f^{k+1}(t) - g^{k+1}(t)| = ||f^{k+1} - g^{k+1}||_\infty = ||\sum_{j\in\mathbb{Z}} r_{i-3j}\Delta f_j^k||_\infty$$
$$\le \sum_{n\in\mathbb{Z}} |r_n| \cdot ||\Delta f^k||_\infty = R \cdot ||\Delta f^k||_\infty = R \cdot ||S_q^k \Delta f^0||_\infty.$$

Using Eq. (3.9), we obtain

$$\sup_{t\in\mathbb{R}} |f^n(t) - f^m(t)| = \sup_{t\in\mathbb{R}} |\sum_{k=m}^{n-1} [f^{k+1}(t) - f^k(t)]| = \sup_{t\in\mathbb{R}} |\sum_{k=m}^{n-1} [f^{k+1}(t) - g^{k+1}(t)]|$$
$$\le \sum_{k=m}^{n-1} \sup_{t\in\mathbb{R}} |f^{k+1}(t) - g^{k+1}(t)| \le R \cdot \sum_{k=m}^{n-1} ||S_q^k \Delta f^0||_\infty \le \sum_{k=m}^{n-1} R\mu^{\frac{k}{L}} \cdot \max_{0\le i<L} |\Delta f^i| \le A \cdot \eta^n,$$

where $\eta = \mu^{\frac{1}{L}} < 1,\ A > 0,\ n > L.$ ■

The proof of the next proposition is straightforward.

Proposition 3.3. *If TISS S_a converges on the initial data f^0 and $f^k = S_a f^{k-1}$ then*

$$\lim_{k\to\infty} \Delta d f^k = 0. \tag{3.13}$$

Remark The key practical problem in the application of TISS algorithms with infinite masks is the evaluation of the sums of the coefficients in (3.9). We present in the Appendix a method to evaluate the coefficients via the discrete Fourier transform and an algorithm for verifying the convergence of such subdivision schemes.

Basic limit function

Definition 3.1. *Let S_a be a convergent TISS. Assume that the initial data is the Kronecker delta $\mathbf{f^0} = \{\delta(k)\}_{k\in\mathbb{Z}}$. Then, the limit function $\varphi_a(t) \triangleq S_a^\infty f^0(t)$ is called the basic limit function (BLF).*

If the mask of the subdivision scheme is finite then its BLF exists and has a compact support. This is not the case for the schemes with infinite masks. However, for the class of TISSs that we deal with, the BLFs exist and decay exponentially when their arguments grow.

Theorem 3.2. *Let S_a be a TISS of Class $\boldsymbol{P}$ and S_q be the subdivision scheme of differences defined by Eq. (3.6). If for some $L \in \mathbb{N}$, the inequality (3.9) holds, then there exists a continuous BLF $\varphi_a(t)$ of the scheme S_a, which decays exponentially as $t \to \infty$. Namely, if (3.2) holds then for any $\varepsilon > 0$ a constant $\Phi_\varepsilon > 0$ exists such that the following inequality $|\varphi_a(t)| \le \Phi_\varepsilon e^{-(g-\varepsilon)|t|}$ holds.*

A similar fact about the binary ISS has been established in [3]. The proof of the statement for the TISS almost literally coincides with the proof in the binary case provided in [3]. Therefore, we omit it in the present paper. We also refer to [3] for the proof of the following fact.

Corollary 3.1 ([3]). *Assume that TISS S_a satisfies the conditions of Theorem 3.2. Then, for any initial data $f^0 = \{f_j^0\}_{j\in\mathbb{Z}}$, the limit function $f^\infty(t)$ can be represented* *by the sum*

$$S_a^\infty f^0(t) = \sum_j f_j^0 \varphi_a(t-j), \tag{3.14}$$

where $\varphi_a(t)$ is the BLF of the scheme S_a.

3.3. *Smoothness of the limit functions*

In this section, we establish conditions for a TISS to produce a limit function that possesses some number of derivatives.

Lemma 3.2. *Assume that the support* $\text{supp}(\psi)$ *of a function* $\psi \in C(\mathbb{R})$ *is compact and the identity* $\sum_{i\in\mathbb{Z}} \psi(x-i) = 1, x \in \mathbb{R}$, *holds. If* S_a *is a convergent TISS then*

$$\lim_{k\to\infty} \sum_{j\in\mathbb{Z}} (S^k f^0)_j \ \psi(3^k t - j) = S^\infty f^0(t). \tag{3.15}$$

Proof: We evaluate the difference

$$e_k(x) \triangleq |\sum_{j\in\mathbb{Z}} (S^k f^0)_j \ \psi(3^k x - j) - S^\infty f^0(x)| = |\sum_{j\in\mathbb{Z}} [(S^k f^0)_j - S^\infty f^0(x)]\psi(3^k x - j)|$$
$$\leq \sum_{j\in\mathbb{Z}} |(S^k f^0)_j - S^\infty f^0(3^{-k}j)||\psi(3^k x - j)| + \sum_{j\in\mathbb{Z}} |S^\infty f^0(3^{-k}j) - S^\infty f^0(x)||\psi(3^k x - j)|$$
$$= \sum_{j\in\Gamma_{3^k x}} |S^\infty f^0(3^{-k}j) - S^\infty f^0(x)||\psi(3^k x - j)|,$$

where $\Gamma_{3^k x} = \{j : 3^k x - j \in \text{supp}(\psi) \wedge j \in \mathbb{Z}\}$. Due to compactness of $\text{supp}(\psi)$, $\Gamma_{3^k x}$ is a finite set. For $j \in \Gamma_{3^k x}, x - 3^{-k} j = y \in 3^{-k}\text{supp}(\psi)$. Due to the uniform convergence of S_a at the interval $\Omega_{x,\delta} = \{y : ||x-y||_\infty < \delta\}$ and due to the continuity of $S^\infty f^0(x)$, for any $\epsilon > 0$, there exists $N(\epsilon, \Omega_{x,\delta})$ such that $e_k(x) \leq \epsilon M||\psi||_\infty$, $\forall k > N(\epsilon, \Omega_{x,\delta})$. ■

We introduce the divided differences $(df^k)_j \triangleq 3^k \Delta f^k = 3^k(f_{j+1}^k - f_j^k)$, $d^j f^k \triangleq 3^{kj}\Delta^j f^k$.

It follows from Proposition 3.2 that if S_a is a TISS, whose symbol $a(z)$ is factorized as $a(z) = (1 + z^{-1} + z^{-2})q(z),\ q(1) = 1$, then the subdivision scheme $S_{a^{[1]}}$ such that

$$df^{k+1} = S_{a^{[1]}} df^k \tag{3.16}$$

has the symbol $a^{[1]}(z) = 3q(z) = 3a(z)/(z^{-2} + z^{-1} + 1)$.

Denote

$$a^{[j]}(z) \triangleq 3^j a(z)/\left(z^{-2} + z^{-1} + 1\right)^j, \quad q^{[j]}(z) \triangleq a^{[j]}(z)/\left(z^{-2} + z^{-1} + 1\right) = a^{[j+1]}(z)/3, \; j = 0, 1..... \tag{3.17}$$

The function $q^{[j]}(z)$ is the symbol of the subdivision scheme of differences $S_{q^{[j]}}$: $\Delta d_n^{k+1} = \sum_{l\in\mathbb{Z}} q_{n-3l}^{[j]} \Delta d_l^k$.

Lemma 3.3. *Assume that the subdivision scheme $S_{q^{[j+1]}}$ is contractive. Then, the scheme $S_{q^{[j]}}$ is contractive as well.*

Proof: Consider the case when $L = 1$ in Eq. (3.8). Assume

$$\|S_{q^{[j+1]}}\|_\infty = \max\left\{\sum_{i\in\mathbb{Z}} |q_{3k}^{[j+1]}, \sum_{i\in\mathbb{Z}} |q_{3k+1}^{[j+1]}, \sum_{i\in\mathbb{Z}} |q_{3k+2}^{[j+1]}, \right\} \le \mu < 1.$$

From to Eq. (3.17),

$$a^{[j+1]}(z) = q^{[j+1]}(z)\left(z^{-2} + z^{-1} + 1\right) \Longleftrightarrow a_l^{[j+1]} = q_l^{[j+1]} + q_{l-1}^{[j+1]} + q_{l-2}^{[j+1]}.$$

Hence,

$$\begin{aligned}\sum_{i\in\mathbb{Z}} |a_{3k}^{[j+1]}| &= \sum_{i\in\mathbb{Z}} |q_{3k}^{[j+1]} + q_{3k-1}^{[j+1]} + q_{3k-2}^{[j+1]}| \\ &\le \sum_{i\in\mathbb{Z}} |q_{3k}^{[j+1]} + \sum_{i\in\mathbb{Z}} |q_{3k-1}^{[j+1]} + \sum_{i\in\mathbb{Z}} |q_{3k-2}^{[j+1]} \le 3\mu.\end{aligned}$$

We have similar estimations for the sums $\sum_{i\in\mathbb{Z}} |a_{3k+1}^{[j+1]}|$ and $\sum_{i\in\mathbb{Z}} |a_{3k+2}^{[j+1]}|$. Thus, $\|S_{a^{[j+1]}}\|_\infty\| \le 3\mu$. But, by definition, $q^{[j]}(z) = a^{[j+1]}(z)/3$ and we have $\|S_{q^{[j]}}\|_\infty\| \le \mu < 1$.

The proof for $L > 1$ is similar.■

Corollary 3.2. *Assume that S_a is a TISS, whose symbol $a(z)$ is factorized as*

$$a(z) = \left(\frac{z^{-2} + z^{-1} + 1}{3}\right)^m a^{[m]}(z), \quad a^{[m]}(0) = 3. \tag{3.18}$$

If the subdivision scheme $S_{q^{[m]}}$ is contractive then the schemes $S_{a^{[j]}}$, $j = 1, .., m$, and S_a are convergent.

Proof: The convergence of S_a follows from Theorem 3.2 and Lemma 3.3.■

Theorem 3.3. *Assume that S_a is a TISS of class $\boldsymbol{P}$ whose symbol is factorized as in Eq. (3.18) and the subdivision scheme $S_{q^{[m]}}$ is contractive. Then, S_a converges on any initial data $f^0 \in l_1$ to the limit function $S_a^\infty f^0 \in C^m$ and*

$$\frac{d^m}{dx^m} S_a^\infty f^0 = S_{a^{[m]}}^\infty \Delta^m f^0.$$

Moreover, for $j = 1, ..., m$, the scheme $S_{a^{[j]}}$ with the symbol $a^{[j]}(z) = \Big(3/(z^{-2} + z^{-1} + 1)\Big)^j$, satisfies

$$S_{a^{[j]}} d^j(S_a^k f^0) = d^j(S_a^{k+1} f^0)), \ S_{a^{[j]}}^\infty d^j f^0 = \frac{d^j}{dx^j} S_a^\infty f^0. \tag{3.19}$$

Proof: The convergence of S_a and $S_{a^{[j]}}$, $j = 1, .., m$, follows from Corollary 3.2. We prove the theorem for $m = 1$. Assume that the initial data $\varphi^0 = \delta$ is the Kronecker delta. Then, $\varphi^k = S_a^k \delta$ and $\varphi^\infty = \varphi(x)$, which is the BLF of S_a. From (3.16) $d\varphi^{k+1} = S_{a^{[1]}} d\varphi^k$. We introduce the sequence of the second order splines $\{g^k(x)\}_{k\in\mathbb{Z}}$, $x \in \mathbb{R}$, which interpolate the subsequently refined data $g^k(3^{-k} j) = d\varphi_j^k, \ j \in \mathbb{Z}$, while the initial data is $\{d\varphi_j^0\}$. Since $S_{a^{[1]}}$ is convergent, the sequence of functions $\{g^k(x)\}_{k\in\mathbb{Z}}$ at any finite interval converges uniformly to a limit function $g = S_{a^{[1]}}^\infty \Delta\delta \in C(\mathbb{R})$. Define the piecewise constant function $h^k(x) \stackrel{\Delta}{=} (d\varphi^k)_j, \ 3^{-k} j \le x < 3^{-k}(j+1), \ j \in \mathbb{Z}$.

It is clear that

$$|g^k(x) - h^k(x)| \le ||\Delta d\varphi^k||_\infty,$$

and, from Eq. (3.13) we get that $h^k(x)$ converges uniformly to $g(x)$. All the functions considered here decay exponentially and that $|\int_{-\infty}^x g(t)dt - \int_{-\infty}^x h^k(t)dt| \le \int_{-\infty}^x |g(t) - h^k(t)|dt,$. Thus, the sequence $\{\int_{-\infty}^x h^k(t)dt : k \in \mathbb{Z}_+\}$ converges uniformly to the function $\int_{-\infty}^x g(t)dt$. From the definition of $h^k(x)$ we get

$$\int_{-\infty}^x h^k(t)dt = \sum_{j\in\mathbb{Z}} (S_a^k \delta)_j \psi(3^k x - j),$$

where $\psi(x) = 1 - |x|, x \in [-1, 1]$. Since S_a is convergent, Lemma (3.2) implies that $\int_{-\infty}^x h^k(t)dt$ converges uniformly to $\varphi(x)$, and, therefore, $\varphi(x) =$

$\int_{-\infty}^{x} g(t)dt$, $\frac{d}{dx}\varphi(x) = g(x) = S^{\infty}_{a^{[1]}}\Delta\delta \in C(\mathbb{R})$. We conclude from (3.14) that for all the initial data $f^0 \in \ell_1$, (3.19) holds with $m = 1$. The proof for $m > 1$ is similar.■

4. TISS derived from interpolatory splines

In this section, we present two families of TISS of type **P**, which are derived from the continuous and the discrete interpolatory splines.

The subdivision scheme, which originated from continuous splines, is presented in [14]. In the present paper we briefly outline this scheme. The subdivision scheme, which is based on discrete splines, will be described in more details.

4.1. *TISS based on continuous splines*

We denote by $\mathfrak{S}^p$ the space of polynomial splines $\sigma^p(x)$ of order p (degree $p-1$) defined on the uniform grid $\mathbf{g}^0 \triangleq \{k\}$, $k \in \mathbb{Z}$, such that the arrays $\{S^p(k)\}$, $k \in \mathbb{Z}$, belong to l_1. The refinement mask of the subdivision scheme is derived from the following:

Continuous Spline Triadic Insertion Rule (CSTIR): We construct on the grid $\mathbf{g}^j \triangleq \left\{k3^{-j}\right\}, j = 0, 1, ...,$ $k \in \mathbb{Z}$, a spline S^j, which interpolates the sequence $\mathbf{f}^j \triangleq \left\{f^j_k\right\}$ on the grid $\mathbf{g}^j$. Then,

$$f^{j+1}_k = S^j\left(k3^{-j-1}\right), \quad k \in \mathbb{Z}.$$

In this section, we use a signal processing terminology in which refinement masks designate a filter.

Note that the value of a spline at any point can be expressed as a linear combination of its values at grid points. In other words, any value f^{j+1}_k can be derived by some filtering of the sequence $\mathbf{f}^j$. We present explicit expressions for these filters for splines of arbitrary order. Moreover, it turns out that for any $j \in \mathbb{N}$, $f^j_k = S^0\left(k3^{-j}\right)$, $k \in \mathbb{Z}$. Thus, we obtain a fast algorithm for computing the values of a spline from $\mathfrak{S}^p$, which interpolates the sequence $\mathbf{f}^0$ on the grid $\mathbf{g}^0$ at triadic rational points $\left\{k3^{-j}\right\}$, $k \in \mathbb{Z}$.

4.1.1. *Interpolatory splines*

The centered B-spline of the first order is the characteristic function of the interval $[-1/2, 1/2]$. The centered B-spline of order p is the convolution $M^p(x) = M^{p-1}(x) * M^1(x) \quad , p \geq 2$.

The B-spline of order p is supported on the interval $(-p/2, p/2)$. It is positive within its support and symmetric about zero. The B-spline M^p consists of pieces of polynomials of degree $p-1$ that are linked to each other at the nodes such that $M^p \in C^{p-2}$. Nodes of B-splines of even and odd orders are located at points $\{k\}$ and $\{k+1/2\},\ k \in \mathbb{Z}$, respectively.

The explicit representation of a B-spline is

$$M^p(t) = \frac{1}{(p-1)!} \sum_{k=0}^{p-1} (-1)^k \binom{p}{k} \left(t + \frac{p}{2} - k\right)_+^{p-1}$$

where $t_+ \triangleq (t + |t|)/2$. Shifts of B-splines form a basis in $\mathfrak{S}^p$. Namely, any spline $\sigma^p(x) \in \mathfrak{S}^p$ has the following representation:

$$\sigma^p(x) = \sum_{k \in \mathbb{Z}} q_k \, M^p(x-k).$$

The $z-$transform of the sampled B-spline

$$U^p(z) \triangleq \sum_{k \in \mathbb{Z}} z^k \, M^p(k) \tag{4.1}$$

is a Laurent polynomial, which is symmetric about inversion. Its roots are real, simple and do not lie on the unit circle $|z| = 1$ (see [10,11]). This polynomials $U^p(z)$ can be calculated explicitly for any order p of the splines.

Assume that the values of the spline at the grid points $S^p(k) = f_k^0,\ k \in \mathbb{Z}$. The $z-$transform is $\sum_{k \in \mathbb{Z}} z^k \ S^p(k) = q(z)U^p(z) = f^0$. Thus, the $z-$transform of the coefficients of the spline, which interpolates the data f^0 is

$$q(z) = \frac{f^0(z)}{U^p(z)}.$$

4.1.2. *Subdivision with refinement masks derived from continuous splines*

When the subdivision is implemented according to CSTIR where continuous splines are involved, the symbol of the schemes can be presented explicitly.

Theorem 4.1 ([14])**.** *The symbol $C^p(z)$ of the TISS S^p_C, which is generated by CSTIR with a continuous interpolatory spline of order p, is*

$$C^p(z) = \frac{(z^{-1}+1+z)^p U^p(z)}{3^{p-1} U^p(z^3)}, \tag{4.2}$$

*where the Laurent polynomial $U^p(z)$ is defined in Eq. (4.1). The scheme S^p_C is of type **P**.*

The symbol $C^p(z)$ can be represented in a polyphase form:

$$C^p(z) = \sum_{k\in\mathbb{Z}} z^k\, c^p_k = zC^p_{-1}(z^3) + C^p_0(z^3) + z^{-1}C^p_1(z^3),$$
$$C^p_0(z) \triangleq \sum_{k\in\mathbb{Z}} z^k\, c^p_{3k}, \quad C^p_{\pm1}(z) \triangleq \sum_{k\in\mathbb{Z}} z^k\, c^p_{3k\pm1}.$$

Respectively, the z-domain representation of the subdivision step is

$$f^{k+1}(z) = C^p(z) f^k(z^3) = z f^k_{-1}(z^3) + f^k_0(z^3) + z^{-1} f^k_1(z^3),$$
$$f^k_m(z) = C^p_m(z) f^k(z),\ m = -1, 0, 1.$$

The scheme S^p_C is interpolatory, therefore, $C^p_0(z) \equiv 1$. Due to the symmetry of $U^p(z)$, $C^p_1(z) = C^p_{-1}(z^{-1})$.

Examples of symbols:

Linear spline:

$$C^2_1(z) = C^2_{-1}(z^{-1}) = \frac{z+2}{3} \tag{4.3}$$
$$C^2(z) = z^{-1}C^2_1(z^3) + 1 + zC^2_{-1}(z^3) = \frac{\left(z+1+z^{-1}\right)^2}{3}$$

This is a single symbol in the family, whose mask is finite. All the masks, which were derived from splines of higher orders, are infinite but exponentially decaying.

Quadratic spline:

$$C^3(z) = \frac{\left(z+6+z^{-1}\right)\left(z+1+z^{-1}\right)^3}{9\left(z^3+6+z^{-3}\right)}, \quad C^3_1(z) = C^3_{-1}(z^{-1}) = \frac{25z+46+z^{-1}}{9\left(z+6+z^{-1}\right)}. \tag{4.4}$$

Cubic spline:

$$C^4(z) = \frac{\left(z+4+z^{-1}\right)\left(z+1+z^{-1}\right)^4}{27\left(z^3+4+z^{-3}\right)}, \quad C_1^4(z) = C_{-1}^4(z^{-1}) = \frac{z^2+60z+93+8z^{-1}}{27\left(z+4+z^{-1}\right)}.$$

Spline of fourth degree:

$$C^5(z) = \frac{\left(z^2+76z+230+76z^{-1}+z^{-2}\right)\left(z+1+z^{-1}\right)^5}{81\left(z^6+76z^3+230+76z^{-3}+z^{-6}\right)},$$
$$C_1^5(z) = C_{-1}^5(z^{-1}) = \frac{625z^2+11516z+16566+2396z^{-1}+z^{-2}}{81\left(z^2+76z+230+76z^{-1}+z^{-2}\right)}.$$

Spline of fifth degree:

$$C^6(z) = \frac{\left(z^2+26z+66+26z^{-1}+z^{-2}\right)\left(z+1+z^{-1}\right)^6}{243\left(z^6+26z^3+66+26z^{-3}+z^{-6}\right)},$$
$$C_1^6(z) = C_{-1}^6(z^{-1}) = \frac{z^3+1018z^2+10678z+14498+29336z^{-1}+32z^{-2}}{243\left(z^2+26z+66+26z^{-1}+z^{-2}\right)}.$$

Convergence: An important fact about TISS S_C^p originated from continuous interpolatory splines is established in [14]. Namely, it was proved that values of the spline $\sigma^p(x)$ at any set of triadic rational points $\sigma^p(k3^{-j})$ can be calculated by successive applications of the CSTIR to the initial data array $\mathbf{f}^0$.

Theorem 4.2 ([14]). *Assume that the spline $\sigma_0^p(x)$ belongs to $\mathfrak{S}^p$, $\sigma_0^p(k) = f_k^0$ and $\mathbf{f}^0 = \left\{f_k^0\right\} \in l_1$. Let $\boldsymbol{C}^p$ be the mask, whose symbol is defined by Eq. (4.2). Assume that for $j \in \mathbb{N}$, $\mathbf{f}^j = \left\{f_k^j\right\}$ is the array whose z-transform is derived from the relation $F^{j+1}(z) = C^p(z)F^j(z^3)$. Then, $\sigma_0^p(k3^{-j}) = f_k^j, \quad j \in \mathbb{N}$.*

Corollary 4.1. *The TISS S_C^p, which is generated by CSTIR with the continuous interpolatory spline of order p, converges to the spline $\sigma_0^p(x)$, which interpolates the initial data array $\sigma_0^p(k) = f_k^0$.*

Recall that splines of order p have $p-2$ continuous derivatives.

Remark If the initial data is the delta sequence $f_k^0 = \delta(k)$ then we get $f_k^j = L^p(k3^{-l})$ where $L^p(x)$ is the fundamental spline in the space $\mathfrak{S}^p$, which, on the other hand, is BLF of the TISS S_C^p. It decays exponentially. Therefore, we can extend the assertion of Theorem 4.2 from splines belonging to $\mathfrak{S}^p$ to splines that interpolate sequences of power growth.

Remark Theorem 4.2 provides an efficient algorithm for fast calculation of the interpolatory spline at triadic rational points.

4.2. *Refinement masks originated from discrete splines*

In this section, we use a special type of the so-called discrete splines for the design of the refinement mask. The discrete splines are defined on the grid $\{k\}_{k\in\mathbb{Z}}$ and they are the counterparts of continuous polynomial splines. The discrete splines were used in [7,5,6] for the construction of wavelets and wavelet frames.

The refinement mask of the subdivision scheme is derived from the following:

Discrete Spline Triadic Insertion Rule (DSTIR): On the grid $\{k\}_{k\in\mathbb{Z}}$, we construct the discrete spline $d_j^p(k)$ of order p with the base $N=3$ such that $d_j^p(3k) = f_k^j,\ k\in\mathbb{Z}$. Then,

$$f_{3k+r}^{j+1} = d_j^p(3k+r), \quad r=-1,0,1,\ k\in\mathbb{Z}. \tag{4.5}$$

4.2.1. *Discrete B-splines*

The sequence

$$B_N^1(j) = \begin{cases} 1 \text{ if } j=0,\ldots,N-1,\ N\in\mathbb{N}, \\ 0 \text{ otherwise} \end{cases}$$

is called the discrete B-spline of the first order, whose base is N. Higher order B-splines are defined as discrete convolutions by the recurrence $B_N^p = B_N^1 * B_N^{p-1}$. Obviously, the z-transform of the B-spline of order p is

$$B_N^p(z) = (1+z^{-1}+z^{-2}+\ldots+z^{-(N-1)})^p, \quad p=1,2,\ldots.$$

Since $N=2M+1$ is odd, then we can introduce the so-called the centered B-spline to be

$$\mathbf{r}_N^p = \{r_N^p(l)\}, \quad r_N^p(l) \triangleq B_N^p(l+Mp) \Longleftrightarrow R_N^p(z) = (z^M+\ldots+z^{-M})^p.$$

4.2.2. *Interpolatory discrete splines*

In our construction, we use discrete splines with the base $N=3$ and drop the index $\cdot_N$ in the notation of the B-spline. Thus, $r^p(l)$ and $R^p(z)$ will stand for

$r_N^p(l)$ and $R_N^p(z)$, respectively. Similarly to continuous splines, a discrete spline $\mathbf{d}^p = \{d^p(l)\}_{l\in\mathbb{Z}}$ of order p on the grid $\{3n\}_{n\in\mathbb{N}}$ is defined as a linear combination with real-valued coefficients of shifts of the centered B-splines:

$$d^p(l) \triangleq \sum_{n=-\infty}^{\infty} h_n r^p(l-3n) \Leftrightarrow D^p(z) = H(z^3)R^p(z).$$

Denote by $r_r^p(l) \triangleq r^p(r+3l),\ r = -1, 0, 1$, the polyphase components of the discrete B-spline. Then, the polyphase components $d_r^p(l) \triangleq d^p(r+3l),\ r = -1, 0, 1$, of the discrete spline $d^p(l)$ are

$$d_r^p(l) \triangleq \sum_{n=-\infty}^{\infty} h_n r_r^p(l-3n) \Leftrightarrow D_r^p(z) = H(z)R_r^p(z),\ r = -1, 0, 1. \tag{4.6}$$

Proposition 4.1 ([7]). *The component $R_0^p(z)$ is symmetric about the inversion $z \to 1/z$ and positive on the unit circle $|z| = 1$. All the components $R_r^p(z)$ have the same value at $z = 1$:*

$$R_r^p(1) = R_0^p(1),\ r = -1, 1. \tag{4.7}$$

The scheme for designing the refinement masks, which uses discrete splines, is similar to the scheme that is based on continuous splines. We construct the discrete spline d^p, which interpolates the data $x = \{x(l)\}_{l\in\mathbb{Z}}$ on the sparse grid $\{3l\}$ and calculate the values of the constructed spline at the points $\{3l \pm 1\}$. Using Eq. (4.6), we find the z-transform of the coefficients of an interpolatory spline to be

$$d_0^p(l) = x(l) \Leftrightarrow D_0^p(z) = H(z)R_0^p(z) = X(z) \Rightarrow H(z) = \frac{X(z)}{R_0^p(z)}. \tag{4.8}$$

The z-transforms of d_r^p is

$$D_r^p(z) = H(z)R_r^p(z) = T_r^p(z)X(z), \quad T_r^p(z) \triangleq \frac{R_r^p(z)}{R_0^p(z)}. \tag{4.9}$$

Equation (4.7) implies that $T_r^p(1) = 1$.

In order to calculate the polyphase components $R_r^p(z)$, we have to solve the following system:

$$\begin{aligned}
R^p(z) &= R_0^p(z^3) + z \cdot R_{-1}^p(z^3) + z^{-1} \cdot R_1^p(z^3)\\
R^p(z \cdot e^{2\pi i/3}) &= R_0^p(z^3) + ze^{2\pi i/3} \cdot R_{-1}^p(z^3) + z^{-1}e^{-2\pi i/3} \cdot R_1^p(z^3)\\
R^p(z \cdot e^{-2\pi i/3}) &= R_0^p(z^3) + ze^{-2\pi i/3} \cdot R_{-1}^p(z^3) + z^{-1}e^{2\pi i/3} \cdot R_1^p(z^3).
\end{aligned}$$

Thus,

$$\begin{aligned} R_0^p(z^3) &= \frac{R^p(z) + R^p(z\cdot e^{2\pi i/3}) + R^p(z\cdot e^{-2\pi i/3})}{3} \\ R_{-1}^p(z^3) &= z^{-1}\frac{R^p(z) + R^p(z\cdot e^{2\pi i/3})e^{-2\pi i/3} + R^p(z\cdot e^{-2\pi i/3})e^{2\pi i/3}}{3} \\ R_1^p(z^3) &= z\frac{R^p(z) + R^p(z\cdot e^{2\pi i/3})e^{2\pi i/3} + R^p(z\cdot e^{-2\pi i/3})e^{-2\pi i/3}}{3}. \end{aligned} \tag{4.10}$$

We define the symbol of the refinement mask of TISS derived from the discrete splines of degree p to be

$$T^p(z) = zT_{-1}^p(z^3) + 1 + z^{-1}T_1^p(z^3)$$

where

$$\begin{aligned} T_{-1}^p(z^3) &= \frac{R_{-1}^p(z^3)}{R_0^p(z^3)} = z^{-1}\frac{R^p(z) + R^p(z\cdot e^{2\pi i/3})e^{-2\pi i/3} + R^p(z\cdot e^{-2\pi i/3})e^{2\pi i/3}}{R^p(z) + R^p(z\cdot e^{2\pi i/3}) + R^p(z\cdot e^{-2\pi i/3})}. \\ T_1^p(z^3) &= \frac{R_1^p(z^3)}{R_0^p(z^3)} = z\frac{R^p(z) + R^p(z\cdot e^{2\pi i/3})e^{2\pi i/3} + R^p(z\cdot e^{-2\pi i/3})e^{-2\pi i/3}}{R^p(z) + R^p(z\cdot e^{2\pi i/3}) + R^p(z\cdot e^{-2\pi i/3})} = T_{-1}^p(z^{-3}). \\ R^p(z) &= (z + 1 + 1/z)^p. \end{aligned}$$

Hence, it follows that

$$\begin{aligned} T^p(z) &= \frac{3R^p(z) + \left(R^p(z\cdot e^{2\pi i/3}) + R^p(z\cdot e^{-2\pi i/3})\right)\left(1 + 2\cos\frac{2\pi}{3}\right)}{R^p(z) + R^p(z\cdot e^{2\pi i/3}) + R^p(z\cdot e^{-2\pi i/3})} \\ &= \frac{3(z+1+1/z)^p}{(z+1+1/z)^p + \left(e^{2\pi i/3}z + 1 + e^{-2\pi i/3}/z\right)^p + \left(e^{-2\pi i/3}z + 1 + e^{2\pi i/3}/z\right)^p}. \end{aligned}$$

Proposition 4.1 implies that the derived subdivision schemes belong to Class **P**.

4.2.3. *Examples of refinement masks derived from discrete splines*

Linear TISS, p=2:

$$\begin{aligned} T_{1,3}^2(z) &= \frac{2+z}{3},\ T_{-1,3}^2(z) = \frac{2+1/z}{3}, \\ T^2(z) &= \frac{(z+1+z^{-1})^2}{3}. \end{aligned}$$

Quadratic TISS, p=3:

$$\begin{aligned} T_{1,3}^3(z) &= \frac{3(2+z)}{z+7+1/z},\ T_{-1,3}^3(z) = \frac{3(2+1/z)}{z+7+1/z}, \\ T^3(z) &= \frac{(z+1+z^{-1})^3}{z^{-3}+7+z^3}. \end{aligned}$$

Cubic TISS, p=4:

$$T^4_{1,3}(z) = \frac{1/z + 16 + 10z}{4z + 19 + 4/z}, \; T^4_{-1,3}(z) = \frac{z + 16 + 10/z}{4z + 19 + 4/z},$$
$$T^4(z) = \frac{(z + 1 + z^{-1})^4}{4z^{-3} + 19 + 4z^3}.$$

5. Implementation of TISS originated from splines

From the signal processing point of view, implementation of TISS is an iterated filtering coupled with the upsampling of the initial dats sequence $\mathbf{f}^0$. The derived symbols of the subdivision masks serve as the transfer functions of the filters. The masks serve as the impulse responses (IR) of the filters. All the derived filters except for the filters derived from the second order splines have infinite IR (IIR). But, since the transfer functions are rational and do not have poles on the unit circle, filtering can be implemented in a fast recursive way. We briefly outline the implementation procedures. In [3], it is discussed in more details.

There are two ways to implement the derived subdivision schemes: polyphase and direct filtering.

5.1. *Polyphase filtering*

One way is to implement the filter $\mathbf{\Phi} = \{\Phi_k\}, \; k \in \mathbb{Z}$, is by using the so-called polyphase representation of the filter:

$$\Phi(z) = z\Phi_{-1}(z^3) + \Phi_0(z^3) + z^{-1}\Phi_1(z^3), \quad \Phi_r(z) \triangleq \sum_{k\in\mathbb{Z}} z^{-k}\Phi_{3k+r}, \; r = -1, 0, 1.$$

Then, the polyphase representation of the array $\mathbf{f}^{j+1}$ is

$$F^{j+1}(z) = zF^{j+1}_{-1}(z^3) + F^{j+1}_0(z^3) + z^{-1}F^{j+1}_1(z^3),$$
$$F^{j+1}_r(z) \triangleq \sum_{k\in\mathbb{Z}} z^{-k} f^{j+1}_{3k+r} = \Phi_r(z)F^j(z), \; r = -1, 0, 1.$$

Thus, in order to retrieve the sub-arrays $\left\{f^{j+1}_{3k\pm1}\right\}$, we have to filter the array $\left\{f^j_k\right\}$ with the filters whose transfer functions are $\Phi_{\pm1}(z)$, respectively. Recall that $\left\{f^{j+1}_{3k} = f^j_k\right\}$.

Example: Implementation of the filter $\mathbf{C}^3$

The z-transform of these filters given in Eq. (4.4) is:

$$C^3(z) = \frac{25z + 46 + z^{-1}}{9\,(z + 6 + z^{-1})} = \frac{\alpha}{9}\frac{25z + 46 + z^{-1}}{(1 + \alpha z)(1 + \alpha z^{-1})},$$

where $\alpha = 3 - 2\sqrt{2} \approx 0.172$. Then, application of the filter $\mathbf{T}_1^3$ to a data array $\mathbf{f} = \{f_k\}$, whose z-transform is $F(z)$, is implemented as a subsequent application of the three filters:

$$\mathbf{C}^3\mathbf{f} = \Psi_l\,\Psi_r\,\Psi \cdot \mathbf{f}.$$

The filters are defined by their z-transforms:

$$\Psi(z) = \frac{\alpha}{9}\left(25z + 46 + z^{-1}\right),\ \Psi_r(z) = \frac{1}{1 + \alpha z^{-1}},\ \Psi_l(z) = \frac{1}{1 + \alpha z}.$$

Thus, filtering is carried out in three steps:

$$F^1(z) = \Psi(z)F(z) \Longleftrightarrow f_k^1 = \frac{\alpha}{9}\left(25f_{k+1} + 46f_k + f_{k-1}\right),$$
$$F^2(z) = \Psi_r(z)F^1(z) \Longleftrightarrow \left(1 + \alpha z^{-1}\right)F^2(z) = F^1(z) \Longleftrightarrow f_k^2 = f_k^1 - \alpha f_{k-1}^2,$$
$$G(z) = \Psi_l(z)F^2(z) \Longleftrightarrow (1 + \alpha z)\,G(z) = F^2(z) \Longleftrightarrow g_k = f_k^2 - \alpha g_{k+1}.$$

The filter Ψ has finite IR (FIR) unlike the filters Ψ_l and Ψ_r. Application of the filter Ψ_r is called a causal recursive filtering. Here, for the calculation of the term f_k^2, the previously derived term f_{k-1}^2 is exploited. Application of Ψ_l is called anti-causal recursive filtering. All these procedures are implemented in a fast way. Computation of splines of higher orders uses filters, which are factorized into longer cascades of the same structure.

How to choose the initial values for recursive filtering was shown in [7]. Here we use this result.

$$f_1^1 \approx f_1 + \sum_{n=1}^{d}(-\alpha)^n f_n$$
$$g_N^1 \approx f_N + \sum_{n=1}^{d}(-\alpha)^n f_{N-n}$$

where d is the prescribed initialization depth.

5.2. *Direct filtering*

Equation (4.2) suggests that, when several steps of the subdivision with continuous splines carried out, a direct application of the filter $\mathbf{T}^p$ is preferable. It follows from Eq. (4.2) that

$$F^{j+1}(z) = C^p(z)\cdot F^j(z^3) = \prod_{l=1}^{j-1} C^p(z^{3^l})\cdot F^0(z^{3^j}) = \frac{U^p(z)\prod_{l=0}^{j-1}(z^{-3^l}+1+z^{3^l})^p}{U^p(z^{3^j})}\cdot F^0(z^{3^j}).$$

For example,

$$F^3(z) = \frac{U^p(z)(z^{-1}+1+z)^p(z^{-3}+1+z^3)^p(z^{-9}+1+z^9)^p}{U^p(z^{27})}\cdot F^0(z^{27}).$$

Thus, the subdivision is implemented via the following steps:

(1) The IIR filter with the transfer function $1/U^p(z)$ is applied to the data array $\mathbf{f}^0$.

(2) The produced array is upsampled[a] and filtered with a FIR filter, whose transfer function is $(z^{-1}+1+z)^p$ (repeated j times).

(3) The produced array is filtered with a FIR filter whose transfer function is $U^p(z)$.

Note that in this case, the IIR filtering is applied only once.

6. Examples of spline-based TISS

In this section, we provide the results about the convergence and smoothness of the TISS that were originated from quadratic and cubic discrete splines.

Quadratic interpolatory discrete spline

The symbol of the TISS S_a is

$$a(z) = \frac{(z+1+z^{-1})^3}{z^3+7+z^{-3}}.$$

The symbol of the appropriate scheme of differences is :

$$q(z) = \frac{a(z)}{z^{-2}+z^{-1}+1}.$$

[a]Upsampling means replacing an array $\{a_k\}$ by the array $\{\tilde{a}_k\}$ such that $\tilde{a}_{3k} = a_k$ and $\tilde{a}_{3k\pm1} = 0$.

The algorithm, which verifies the convergence of TISS, is used. We get $||S_q^1|| = 0.6 < 1$, therefore, the quadratic TISS S_a is convergent. From the algorithm, which verifies the rate of smoothness, we get $||S^1_{q^{[1]}}|| = 0.6 < 1$. Thus, $S_a \in C^1$.

Cubic interpolatory discrete spline

The symbol of the TISS S_a is

$$a(z) = \frac{(z + 1 + z^{-1})^4}{4z^3 + 19 + 4z^{-3}}.$$

The symbol of the appropriate scheme of differences is :

$$q(z) = \frac{a(z)}{z^{-2} + z^{-1} + 1}.$$

Since $||S_q^1|| = 0.52223 < 1$, the cubic TISS S_a is convergent. From the algorithm, which verifies the rate of smoothness, we get $||S^1_{q^{[2]}}|| = 0.81818 < 1$. Thus, $S_a \in C^2$.

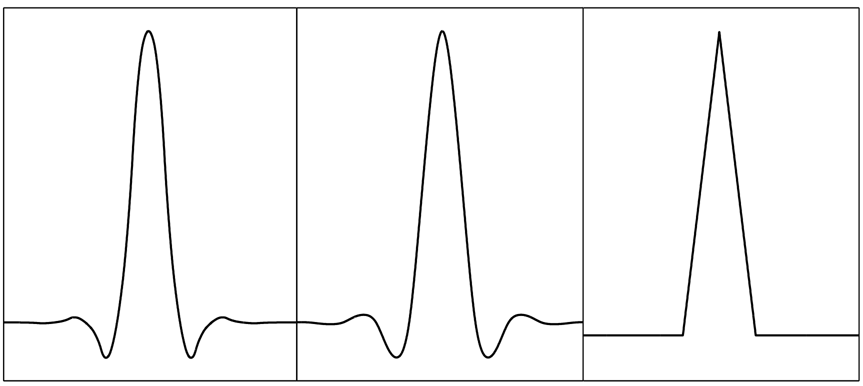

Fig. 6.1. The basic limit functions: Left: quadratic discrete spline TISS. Center: cubic discrete spline TISS. Right: linear spline TISS

Appendix: Evaluation of coefficients of subdivision masks via the discrete Fourier transform

Assume $L = 1$ in Eq. (3.9). $L > 1$ is treated similarly. Assume that $N = 2^p,\ p \in \mathbb{N}$. The discrete Fourier transform (DFT) of an array $x^p = \{x_k^p\}_{k=-N/2}^{N/2}$ and its inverse (IDFT) are

$$\hat{x}_n^p = \sum_{k=-N/2}^{N/2} e^{-2\pi ikn/N} x_k^p \quad \texttt{and} \quad x_k^p = \frac{1}{N} \sum_{k=-N/2}^{N/2} e^{2\pi ikn/N} \hat{x}_k^p. \tag{6.1}$$

We assume that $z = e^{-iw}$. The coefficients of the masks decay exponentially, i.e.

$$|a_k| \le A\gamma^k \Rightarrow \sum_{k=N}^{\infty} |a_k| \le \sum_{k=N}^{\infty} A\gamma^k = A\left(\sum_{k=1}^{\infty} \gamma^k - \sum_{k=1}^{N-1} \gamma^k\right) = A\left(\frac{\gamma}{1-\gamma} - \frac{\gamma(1-\gamma^{N-1})}{1-\gamma}\right) = B\gamma^N \tag{6.2}$$

where $B = A/(1-\gamma),\ 0 < \gamma < 1$ and A is some positive constant. We need to evaluate the sums

$$S_{a,p} = \sum_{k=-\infty}^{\infty} |a_{3k+p}|,\ p \in \{-1, 0, 1\}.$$

Denote

$$A(\theta) = a(e^{-i\theta}) = \frac{T(e^{-i\theta})}{P(e^{-i\theta})} = \sum_{k=-\infty}^{\infty} a_k e^{-i\theta k}.$$

The function A is calculated at the discrete set of points

$$\hat{a}_n = A\Big(\frac{2\pi n}{N}\Big) = \sum_{k=-\infty}^{\infty} e^{-2\pi ikn/N} a_k = \sum_{\substack{k=-\infty \\ l \in \mathbb{Z}}}^{\infty} e^{-2\pi in(k+lN)/N} a_k = \sum_{r=-N/2}^{N/2-1} \Big(\sum_{l=-\infty}^{\infty} e^{-2\pi irn/N} a_{r+lN}\Big)$$

$$= \sum_{r=-N/2}^{N/2-1} e^{-2\pi irn/N} \Big(\sum_{l=-\infty}^{\infty} a_{r+lN}\Big)$$

$$\varphi_r = \sum_{l=-\infty}^{\infty} a_{r+lN} = a_r + \psi_r,\ \psi_r = \sum_{l \in \mathbb{Z} \setminus 0} a_{r+lN}$$

From Eq.(6.2) we get

$$|\psi_r| \le 2B\alpha^N \Rightarrow |a_r| = |\varphi_r| + \alpha_r^N,\ |\alpha_r^N| \le 2B\gamma^N. \tag{6.3}$$

The samples φ_k are derived from the application of IDFT:

$$\varphi_k = \frac{1}{N}\sum_{n=-N/2}^{N/2-1} e^{2\pi ikn/N}\hat{a}_n.$$

By using Eq.(6.3), we can evaluate the sums that we are interested in as follows:

$$S_{a,p} = \sum_{k=-N/4}^{N/4-1}|a_{3k+p}| + 2\sum_{k=N/4}^{\infty}|a_{3k+p}| = \sum_{k=-N/4}^{N/4-1}|\varphi_{3k+p}| + \rho_N,$$

$$\rho_N = \sum_{k=-N/4}^{N/4-1}|\alpha^N_{3k+p}| + 2\sum_{k=N/4}^{\infty}|a_{3k+p}|,\ |\rho_N| \le B(N+2)\gamma^N.$$

Hence, it follows that by doubling N we can approximate the infinite series $S_{a,p}$ by the finite sum $\sigma^N_{a,p} = \sum_{k=-N/4}^{N/4-1}|\varphi_{3k+p}|$, whose terms are derived from the application of the DFT. An appropriate value of N can be derived theoretically using estimations on the roots in the denominator. Practically, we can iterate the calculations by gradually doubling N until the result from calculating $\sigma^{2N}_{a,p}$ becomes identical to $\sigma^N_{a,p}$ (up to a machine precision). The same approach is valid for the evaluation of the sums $\sum_j |q^{(L)}_{3^L\cdot j-i}|$ for any L.

The algorithm that verifies the convergence of S_a with the mask $a(z)$:

(1) `Compute:` $q(z) = a(z)/(z^{-2}+z^{-1}+1)$.

(2) `Set` $\widehat{q}^{(1)}_n = q(e^{-2\pi in/N}),\ N = 2^p,\ p\in\mathbb{N},\ -N/2 \le n \le N/2-1$.

(3) `Verify that` S_{q^L} `is contractive:`

`for` $L = 1, ..., M$

(a) `for` $i = 0, ..., 3^L-1$

`Compute` $\{K_i\}_{i=0}^{3^L-1}:\quad K_i = \sum_j |q^{(L)}_{3^L\cdot j-i}| \simeq \sum_{r=-N/4}^{N/4-1}|\varphi_{3^Lr-i}|$,

`where` $\varphi_k = N^{-1}\sum_{n=-N/2}^{N/2-1} e^{2\pi nki/N}\widehat{q}^{[L]}_n$ `for` N `sufficiently large.`

(b) `If` $\max\{K_i\} < 1,\ 0 \le i \le 3^L-1$

`then` S_q `is contractive, therefore` S_a `is convergent. Stop!`

`else`

`compute` $\widehat{q}^{(L+1)}_n = q^{(L+1)}(e^{-2\pi in/N})$,

`where` $q^{(L+1)}(z) = \prod_{j=1}^{L+1} q(z^{3^{j-1}}),\ N = 2^p,\ p\in\mathbb{N},\ -N/2 \le n \le N/2-1$.

(4) `If after M iterations` S_q `is not contractive. Stop!`

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