**A Note On Cubic Splines**, AMATH 352, March 4, 2002

We would like to use a spline to approximate a function represented by the points $(0,0),(1,0),(3,2)$ and $(4,2)$. The first task is to determine the spacing between the points $h_k$, the slopes $d_k$ and then (though the solution of a system of equations) the second derivatives of the splines $s''(x_k) = m_k$.

$$h_0 = x_1 - x_0 = 1 \quad h_1 = x_2 - x_1 = 2 \quad h_2 = x_3 - x_2 = 1$$
$$d_0 = \frac{y_1 - y_0}{x_1 - x_0} = 0 \quad d_1 = \frac{y_2 - y_1}{x_2 - x_1} = 1 \quad d_2 = \frac{y_3 - y_2}{x_3 - x_2} = 0$$

Next, set up the equations for the second derivatives (the $m_k$) which have been derived by requiring the continuity of the first and second derivatives of the splines at $x_1$ and $x_2$:

$$\begin{aligned} 2(h_0+h_1)m_1 + h_1 m_2 &= 6(d_1-d_0) - h_0 m_0 \\ h_1 m_1 + 2(h_1+h_2)m_2 &= 6(d_2-d_1) - h_2 m_3 \end{aligned}$$

Note that there will be $N-1$ of these equations when the spline interpolates $N+1$ points. Since there are $N+1$ $m_k$'s in this case, we need to apply the end conditions on the splines to determine $m_0$ and $m_3$, the second derivatives at the end points. There are three options that we've introduced in this class:

- **Natural Spline** The second derivatives are zero at the ends:

$$\begin{aligned} m_0 &= 0 \\ h_0 m_0 + 2(h_0+h_1)m_1 + h_1 m_2 &= 6(d_1-d_0) \\ h_1 m_1 + 2(h_1+h_2)m_2 + h_2 m_3 &= 6(d_2-d_1) \\ m_3 &= 0 \end{aligned} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 0 & 2 & 6 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ -6 \\ 0 \end{pmatrix}$$

- **Clamped Spline** The first derivative of the spline is specified at the end points. Evaluating the first derivatives of the splines at $x_0$ and $x_3$ and requiring that they equal $f'(x_0)$ and $f'(x_3)$ yields the following system for the $m_k$:

$$\begin{aligned} 2h_0 m_0 + h_0 m_1 &= 6(d_0 - f'(x_0)) \\ h_0 m_0 + 2(h_0+h_1)m_1 + h_1 m_2 &= 6(d_1-d_0) \\ h_1 m_1 + 2(h_1+h_2)m_2 + h_2 m_3 &= 6(d_2-d_1) \\ h_2 m_2 + 2h_2 m_3 &= 6(f'(x_3) - d_2) \end{aligned} \qquad \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 0 & 2 & 6 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ -6 \\ 0 \end{pmatrix}$$

- **Not-a-knot Spline** Without specifying any extra conditions at the end points (other than that the spline interpolates the data points there), the not-a-knot spline requires that the third derivative of the spline is continuous at $x_1$ and $x_{N-1}$. One can determine the equations for $m_0$ and $m_N$ by requiring that $s_0'''(x_1) = s_1'''(x_1)$. (A similar condition holds at $x_{N-1}$.)

$$\begin{aligned} h_1 m_0 - (h_0+h_1)m_1 + h_0 m_2 &= 0 \\ h_0 m_0 + 2(h_0+h_1)m_1 + h_1 m_2 &= 6(d_1-d_0) \\ h_1 m_1 + 2(h_1+h_2)m_2 + h_2 m_3 &= 6(d_2-d_1) \\ h_2 m_1 - (h_1+h_2)m_2 + h_1 m_3 &= 0 \end{aligned} \qquad \begin{pmatrix} 1 & -3 & 2 & 0 \\ 1 & 6 & 2 & 0 \\ 0 & 2 & 6 & 1 \\ 0 & 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ -6 \\ 0 \end{pmatrix}$$

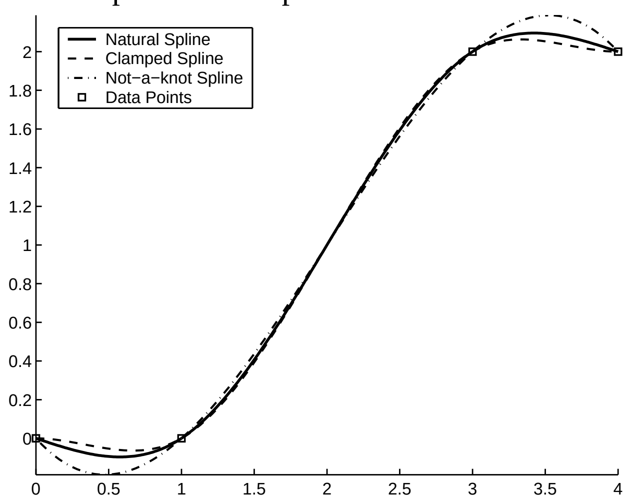
After solving the appropriate system for the end conditions that you have chosen, evaluate the coefficients of the splines:

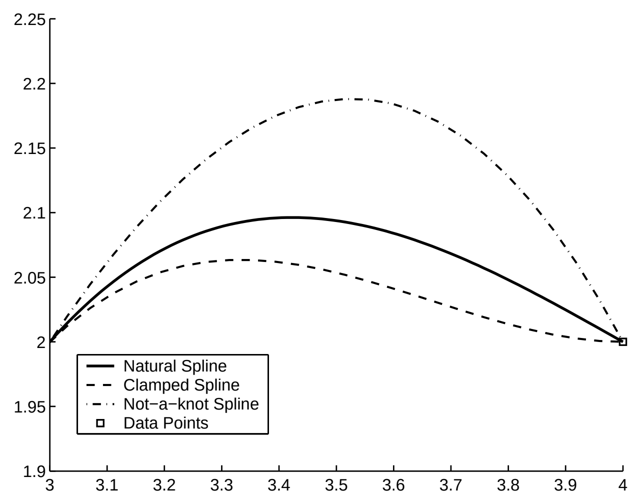
$$s(x) = \begin{cases} s_0(x) &= s_{0,0} + s_{0,1}(x-x_0) + s_{0,2}(x-x_0)^2 + s_{0,3}(x-x_0)^3 & x \in [x_0, x_1] \\ s_1(x) &= s_{1,0} + s_{1,1}(x-x_1) + s_{1,2}(x-x_1)^2 + s_{1,3}(x-x_1)^3 & x \in [x_1, x_2] \\ s_2(x) &= s_{2,0} + s_{2,1}(x-x_2) + s_{2,2}(x-x_2)^2 + s_{2,3}(x-x_2)^3 & x \in [x_2, x_3] \end{cases}$$

where $s_{k,0} = y_k \quad s_{k,1} = d_k - \frac{h_k}{6}(2m_k + m_{k+1}) \quad s_{k,2} = \frac{m_k}{2} \quad s_{k,3} = \frac{m_{k+1} - m_k}{6h_k}$.
With natural end conditions, I found that $m_0 = m_3 = 0$, $m_1 = \frac{3}{2}$ and $m_2 = -\frac{3}{2}$, so that:

$$s(x) = \begin{cases} s_0(x) &= 0 - \frac{1}{4}(x-0) + 0(x-0)^2 + \frac{1}{4}(x-0)^3 & x \in [0,1] \\ s_1(x) &= 0 + \frac{1}{2}(x-1) + \frac{3}{4}(x-1)^2 - \frac{1}{4}(x-1)^3 & x \in [1,3] \\ s_2(x) &= 2 + \frac{1}{2}(x-3) - \frac{3}{4}(x-3)^2 + \frac{1}{4}(x-3)^3 & x \in [3,4] \end{cases}$$

Here is a plot of the natural spline that passes through these four points. At right is a closeup of the splines between $x = 3$ and $x = 4$ to hightlight the difference in the end conditions. Note that the spline appears not to have any curvature at the end points, while the not-a-knot spline does have a non-zero second derivative at the right end in this case. The clamped spline is constructed to have zero slope at the end points.




Note that the `spline` function in MATLAB computes a not-a-knot spline by default. If `X = [0 1 3 4]` and `Y = [0 0 2 2]`, the not-a-knot spline can be computed and plotted in MATLAB with `plot(x,ppval(spline(X,Y),x))`. Specifying additional data points at the beginning and end of the interval will give a clamped spline with those extra values as the slopes at the endpoints of the intervals. The command `plot(x,ppval(spline(X,[0 Y 0]),x))` would give the clamped spline plotted here with $f'(0) = f'(4) = 0$.