Approximate Nonnegative Rank is Equivalent to the Smooth Rectangle Bound

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Talk Plan

- Preliminaries
- Our Result
- Corollaries and Open Questions

Two-Party Communication Complexity [Yao'79]

$f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$

$x \in \{0,1\}^n$

$y \in \{0,1\}^n$

Alice

Bob

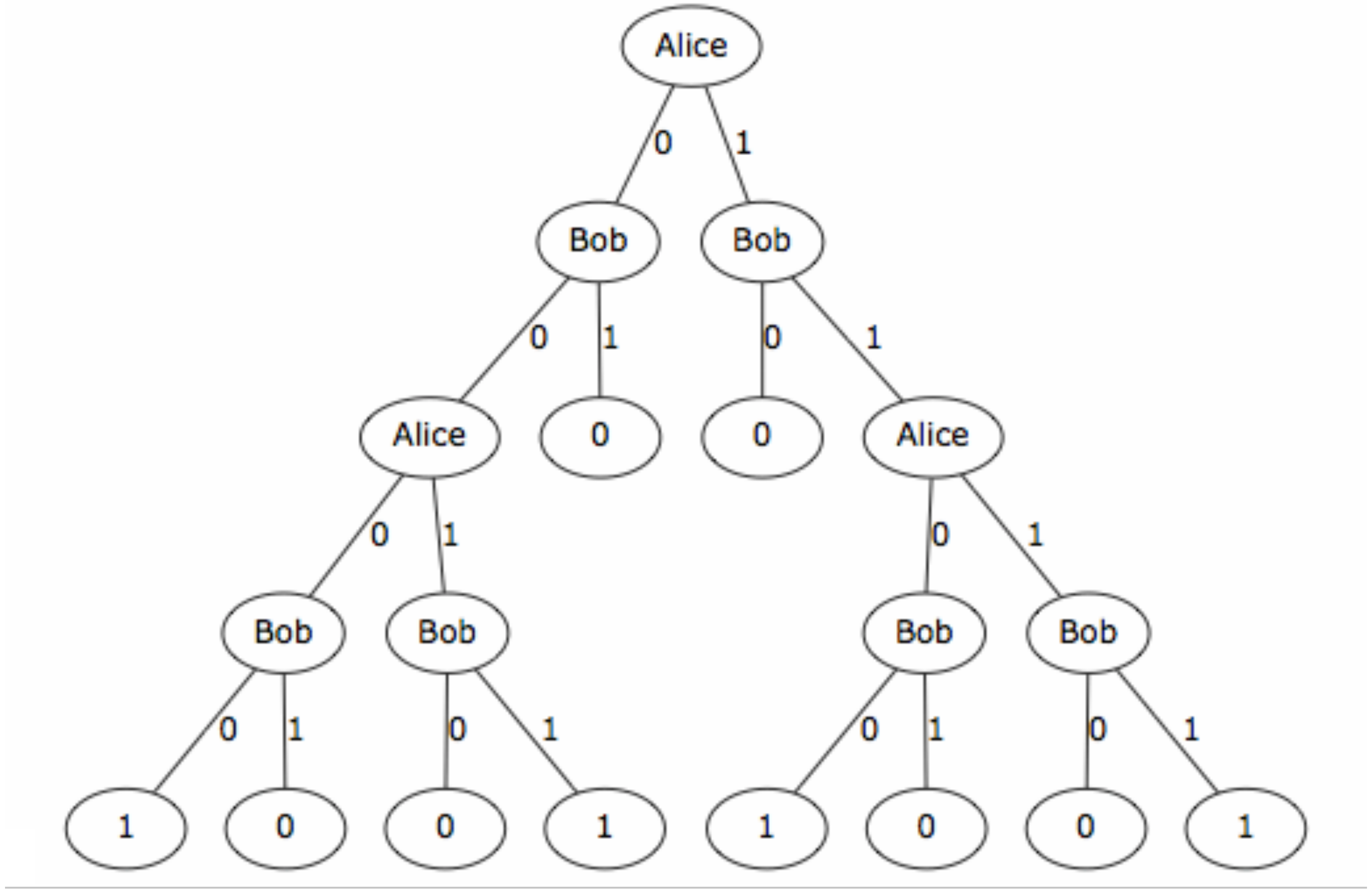
$D(f)$ = The length of the shortest deterministic protocol for f

Important research theme: understanding the relation to combinatorial and algebraic properties of f

Communication Matrix

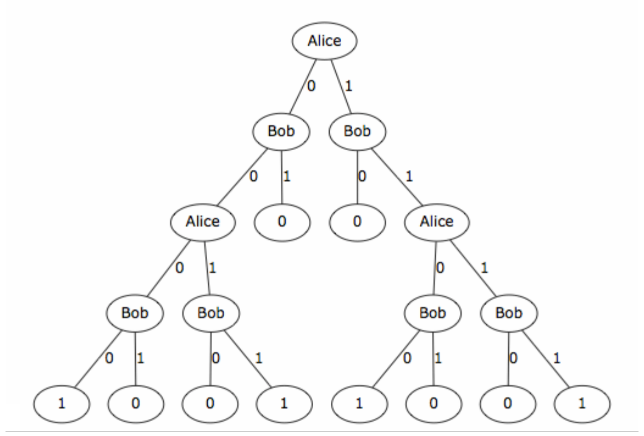
- M_f is $2^n \times 2^n$ matrix, $M_f(x,y) = f(x,y)$
- M_f defined over every field, but we focus on $\mathbb{R}$
- Theorem [Mehlhorm-Schmidt'82]:
  $$\log(\mathrm{rank}(M_f)) \leq D(f) \leq \mathrm{rank}(M_f)+1$$
- Theorem [Lovett'13]: $D(f) \leq \tilde{O}(\sqrt{\mathrm{rank}(M_f)})$
- Log-rank conjecture [Lovász-Saks'88]:
  $$D(f) \leq \mathrm{poly}(\log(\mathrm{rank}(M_f)))$$

Communication protocol



- Set of inputs reaching a vertex/leaf is a rectangle A×B
- Gives partition of 1's of M_f to $\leq 2^{D(f)}$ 1-rectangles
- Each rectangle is a rank-1 matrix
- Hence, $\text{rank}(M_f) \leq 2^{D(f)}$

Going beyond Rank



- 1-rectangles are non-negative rank-1 matrices
- Non-negative rank:
  $\mathrm{rank}^+(M)$ = minimal r s.t. ∃ non-negative matrices $M_1,\ldots,M_r$ of rank-1 with $M = M_1+\ldots+M_r$
- Difference from rank : M_i non-negative
- 1-rectangles partition the non-zero entries of M
- Partition bound:
  $\mathrm{prt}(M)$ = minimal r s.t. ∃ boolean matrices $M_1,\ldots,M_r$ of rank-1 with $M = M_1+\ldots+M_r$

Deterministic Lower Bounds

- Theorem [Yannakakis'91]:
  $\log(\mathrm{rank}(M_f)) \leq \log(\mathrm{rank}^+(M_f)) \leq \log(\mathrm{prt}(M_f)) \leq D(f)$
- Log-rank conjecture [Lovász-Saks'88]:
  $$D(f) \leq \mathrm{poly}(\log(\mathrm{rank}(M_f)))$$
- Log non-negative rank theorem [Lovász'90]:
  $$D(f) \leq \log^2(\mathrm{rank}^+(M_f))$$
- Corollary: $\log(\mathrm{rank}^+(M_f)) \approx \log(\mathrm{part}(M_f)) \approx D(f)$

Lower bounds map

$$\mathsf{D}(f)$$

$$\approx$$

$$\log \mathsf{partition}(f)$$

$$\approx$$

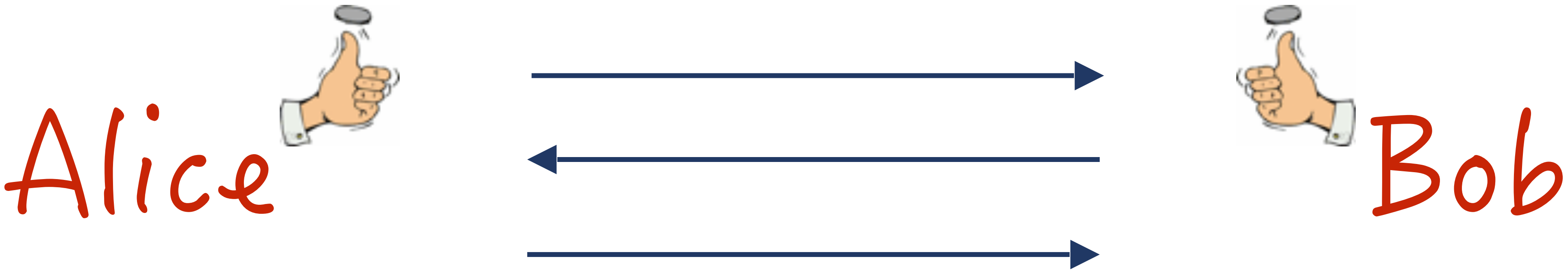
$$\log \mathsf{rank}^{+}(f)$$

$$\rotatebox{90}{$\leq$}$$

$$\log \mathsf{rank}(f)$$

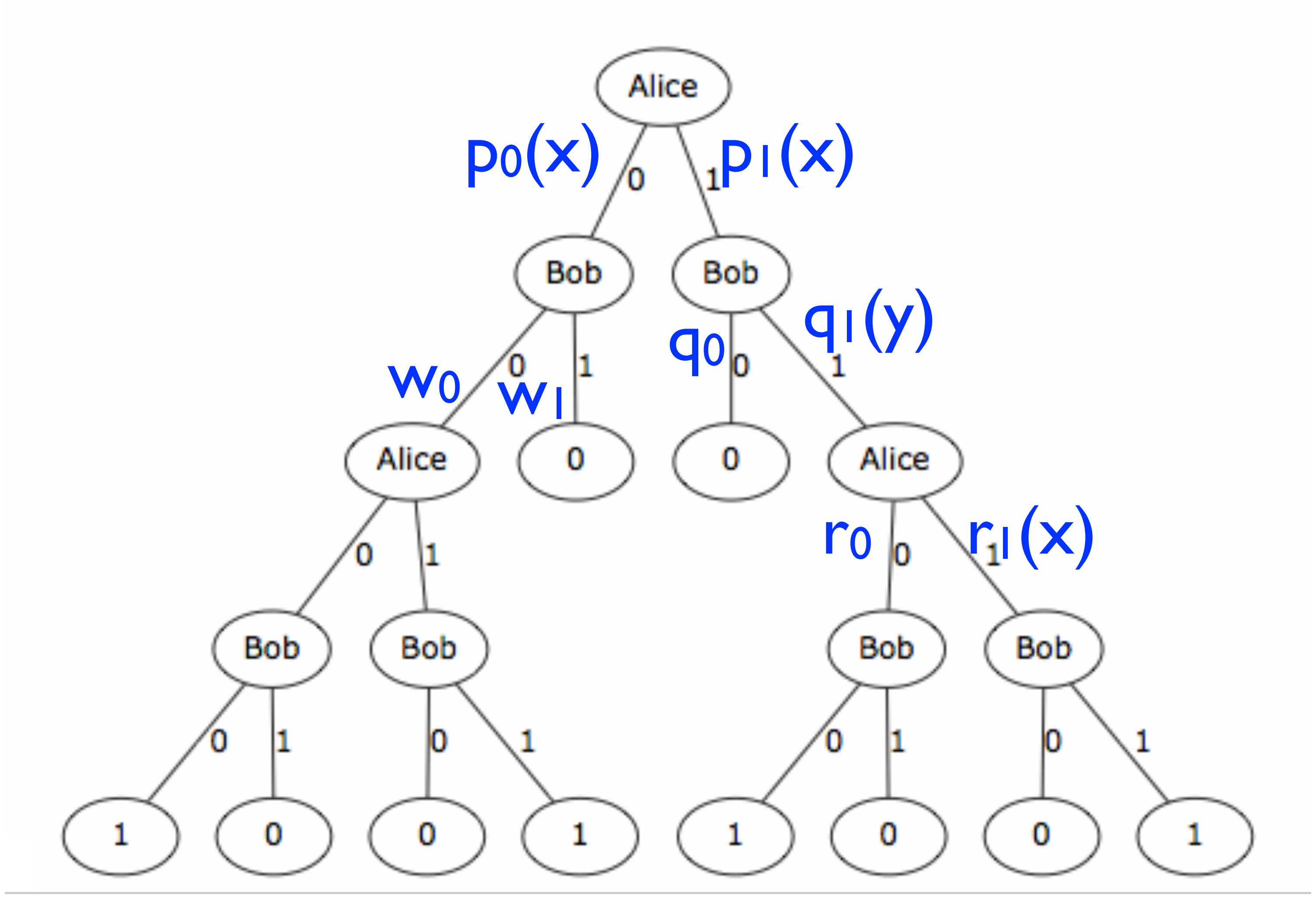
Randomized Communication Complexity

$x \in \{0,1\}^n$ $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ $y \in \{0,1\}^n$



$R_\varepsilon(f)$ = The length of the shortest (wrt to worst input) randomized protocol for f with error probability at most ε

Randomized Communication protocol



- Set of inputs reaching a leaf is a product set A×B
- Leaf corresponds to a rank-1 matrix describing the probability of each input to reach it
- Sum of these matrices ε-approximates M_f

Randomized Lower Bounds

- Approximate rank:
  $\mathrm{rank}_{\varepsilon}(M)$ = min r s.t. ∃ M' of rank r s.t. $||M - M'||_{\infty} \leq \varepsilon$
- Approximate non-negative rank:
  $\mathrm{rank}_{\varepsilon}^{+}(M)$ = min r s.t. ∃ non-negative matrices $M_1, \ldots, M_r$ of rank-1 with $||M - (M_1 + \ldots + M_r)||_{\infty} \leq \varepsilon$
- Equivalently:
  $\mathrm{rank}_{\varepsilon}^{+}(M)$ = min r s.t. ∃ $\mathrm{rank}^{+}(M') = r$ s.t. $||M - M'||_{\infty} \leq \varepsilon$
- Theorem [Krause'96]:
  $$\log(\mathrm{rank}_{\varepsilon}(M_f)) \leq \log(\mathrm{rank}_{\varepsilon}^{+}(M_f)) \leq R_{\varepsilon}(f)$$
- Approximate analog of the partition bound?

Smooth Rectangle Bound [Jain-Klauck'10]

- Smooth rectangle bound ≡ approximate fractional partition number
- $\mathrm{srec}_\varepsilon(M)$ = min $\sum w_i$ s.t. $w_i > 0$ and ∃ rank-1 boolean $\{M_i\}$ (rectangles) with $||M - \sum w_i M_i||_\infty \leq \varepsilon$
- If $w_i \in \{0,1\}$ and $\varepsilon = 0$ then this is $\mathrm{prt}(M)$
- Theorem [Jain-Klauck'10]: $\log(\mathrm{srec}_\varepsilon(f)) \leq R_\varepsilon(f)$

Lower bounds map

$$\begin{tikzcd}
& \mathsf{R}_\epsilon(f) \arrow[dl, phantom, sloped, "\geq"] \arrow[dr, phantom, sloped, "\geq"] \arrow[rrr, phantom, "\leq"] & & & \mathsf{D}(f) \arrow[d, phantom, sloped, "\approx"] \\
\log \mathsf{rank}^+_\epsilon(f) \arrow[d, phantom, sloped, "\geq"] & \approx & \log \mathsf{srec}_\epsilon(f) & & \log \mathsf{partition}(f) \arrow[d, phantom, sloped, "\approx"] \\
\log \mathsf{rank}_\epsilon(f) & & & & \log \mathsf{rank}^+(f) \arrow[d, phantom, sloped, "\geq"] \\
& \text{Main result} \arrow[uu] & & & \log \mathsf{rank}(f)
\end{tikzcd}$$

Lower Bounds Map

	Randomized	**Deterministic**
	$\mathsf{R}_\epsilon(f)$	$\mathsf{D}(f)$
	$\geq$	$\approx$
	$\log \mathsf{srec}_\epsilon(f)$	$\log \mathsf{partition}(f)$
Main result	$\approx$	$\approx$
	$\log \mathsf{rank}^+_\epsilon(f)$	$\log \mathsf{rank}^+(f)$
	$\geq$	$\geq$
	$\log \mathsf{rank}_\epsilon(f)$	$\log \mathsf{rank}(f)$

Main theorem

- Smooth rectangle bound:
  $\mathrm{srec}_\varepsilon(M)$ = min $\sum w_i$ such that $w_i > 0$ and ∃ rectangles M_i with $\|M - \sum w_i M_i\|_\infty \leq \varepsilon$
- Approximate non-negative rank:
  $\mathrm{rank}_\varepsilon^+(M)$ = min r s.t. ∃ non-negative matrices $M_1,\ldots,M_r$ of rank-1 with $\|M - (M_1+\ldots+M_r)\|_\infty \leq \varepsilon$
- Main theorem:
  $\log(\mathrm{srec}_{3\varepsilon}(f)) \leq \log(\mathrm{rank}_\varepsilon^+(f)) \leq 2\log(\mathrm{srec}_{\varepsilon/2}(f))+\log(3n/\varepsilon^2)$

Proof idea

- Smooth rectangle bound:
  $\mathrm{srec}_\varepsilon(f)$ = min $\sum w_i$ such that $w_i > 0$ and ∃ rectangles M_i with $||M_f - \sum w_i M_i||_\infty \leq \varepsilon$
- Approximate non-negative rank:
  $\mathrm{rank}_\varepsilon^+(f)$ = min r s.t. ∃ non-negative matrices $M_1,\ldots,M_r$ of rank-1 with $||M_f - (M_1+\ldots+M_r)||_\infty \leq \varepsilon$

- $\log(\mathrm{rank}_\varepsilon^+(f)) \leq \log(\mathrm{srec}_{\varepsilon/2}(f))$:
- Set $W = \mathrm{srec}_\varepsilon(f) = \sum w_i$
- $\mathbb{E}[W \cdot \mathrm{Rect}(x,y)]$
- Problem: too many rectangles
- Sample $k = O(W^2 n/\varepsilon^2)$ rectangles according to weights
- By Chernoff, point-wise approximation to M_f

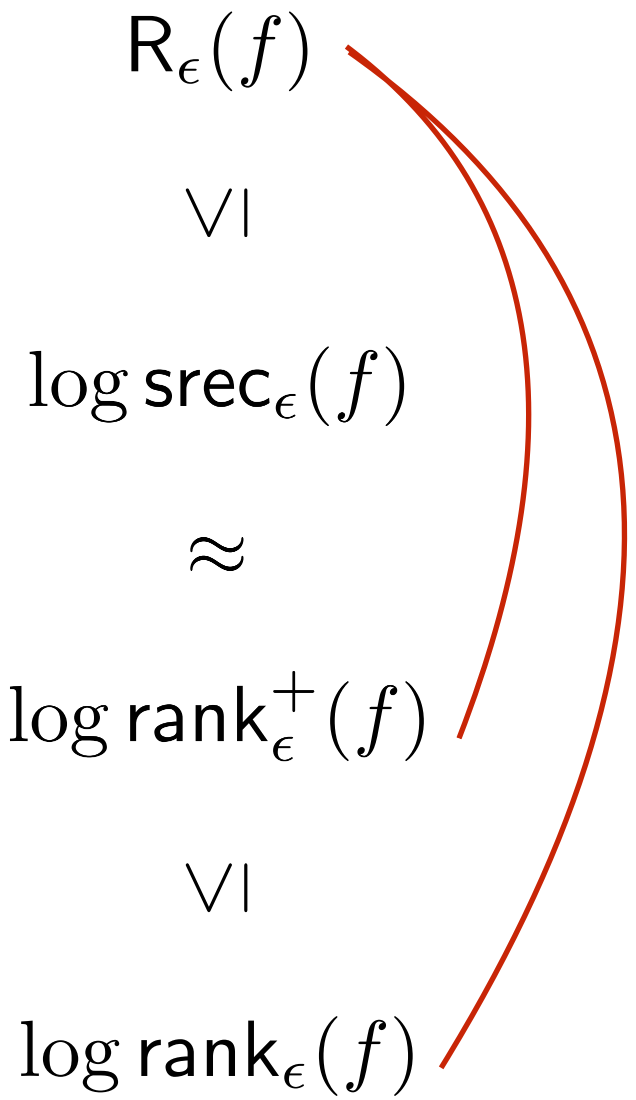
Proof idea

- Smooth rectangle bound:
  $\mathrm{srec}_{\varepsilon}(f) = \min \sum w_i$ such that $w_i > 0$ and ∃ rectangles M_i with $\|M_f - \sum w_i M_i\|_{\infty} \leq \varepsilon$
- Approximate non-negative rank:
  $\mathrm{rank}_{\varepsilon}^{+}(M) = \min r$ s.t. ∃ $\mathrm{rank}^{+}(M')=r$ s.t. $\|M-M'\|_{\infty} \leq \varepsilon$

- $\log(\mathrm{srec}_{3\varepsilon}(f)) \leq \log(\mathrm{rank}_{\varepsilon}^{+}(f))$
- Let $M' = M_1 + \ldots + M_r$, $M_i = u_i^t \cdot v_i$
- Problem: $u_i^t \cdot v_i$ not a rectangle
- Round each u_i, v_i to accuracy ε/r:
- $(\sum u_{ij})^t \cdot (\sum v_{ij}) \approx M_i$, $u_{ij}^t \cdot v_{ik}$ rectangle
- $O(r^3/\varepsilon^2)$ rectangles of weight $(\varepsilon/r)^2$

Open Questions and Corollaries

Log approximate non-negative rank conjecture (LANR)



$$\mathsf{R}_\epsilon(f)$$

$$\geq$$

$$\log \mathsf{srec}_\epsilon(f)$$

$$\approx$$

$$\log \mathsf{rank}^+_\epsilon(f)$$

$$\geq$$

$$\log \mathsf{rank}_\epsilon(f)$$

Conjecture [Lee-Shraibman'09]:
$R_\varepsilon(f) \leq \mathrm{poly}(\log(\mathrm{rank}_\varepsilon(f)))$

[Gavinski-Lovett'12]:
Conjecture implies log-rank conjecture

Weaker conjecture (LANR)[Lee'12]:
$R_\varepsilon(f) = \mathrm{poly}(\log(\mathrm{rank}_\varepsilon^+(f)))$

Recall [Lovász'90]: $D(f) \leq \log^2(\mathrm{rank}^+(M_f))$

Implications of LANR

- LANR conj. : $R_\varepsilon(f) = \text{poly}(\log(\text{rank}_\varepsilon^+(f)))$
- Information complexity $IC_\varepsilon(f)$: amount of information players lean about each other's input in any protocol computing f with ε-error.
- Corollary of thm (using [Kerenidis-Laplante-Lerays-Roland-Xiao'12]):
$$IC_\varepsilon(f) = \Omega(\varepsilon^2 \log(\text{rank}_\varepsilon^+(f)) - \log(n/\varepsilon))$$
- Corollary: LANR ⇒ compressibility of protocols possible!
Information complexity ≈ communication complexity
$$IC_\varepsilon(f) = \text{poly}(R_\varepsilon(f), \log(n/\varepsilon))$$
- [Ganor,Kol,Raz'14,'15]: Compression not possible!

Implications of LANR

- Question [error reduction]: Is it true that

  $\log(\mathrm{rank}_\varepsilon^+(f)) \approx \log(1/\varepsilon) \cdot \log(\mathrm{rank}_{1/3}^+(f))$?

- Satisfied by $R_\varepsilon(f)$ (error amplification) and $\log(\mathrm{rank}_\varepsilon(f))$ [Alon'03]

- LANR gives:

  $\log(\mathrm{rank}_\varepsilon^+(f)) \approx R_\varepsilon(f) = O(\log(1/\varepsilon) \cdot R_{1/3}(f))$

  $\approx \log(1/\varepsilon) \cdot \log(\mathrm{rank}_\varepsilon^+(f)$

- Partial negative answer [Göös-Lovett-Meka-Watson-Zuckerman]:

  $\forall\ 0<\varepsilon<\delta<1/2\ \exists$ partial function f such that

  $\log(\mathrm{rank}_\delta^+(f)) \leq O(\log(n))$ but $\log(\mathrm{rank}_\varepsilon^+(f)) = \Omega(n)$

- Corollary: no point-wise amplification

Implications of LANR

- Question [stability under negation]: is it true that

  $\mathrm{rank}_\varepsilon^+(f) \approx \mathrm{rank}_\varepsilon^+(1\text{-}f)$?

- Satisfied by $R_\varepsilon(f)$, $\mathrm{rank}_\varepsilon(f)$

- LANR gives:

  $\log(\mathrm{rank}_\varepsilon^+(f)) \approx R_\varepsilon(f) = R_\varepsilon(1\text{-}f) \approx \log(\mathrm{rank}_\varepsilon^+(1\text{-}f))$

Approximate Rank vs. Approximate Nonnegative Rank

- Approximate rank:
  $\mathrm{rank}_\varepsilon(M)$ = min r s.t. ∃ M' of rank r s.t. $||M-M'||_\infty \leq \varepsilon$
- Approximate non-negative rank:
  $\mathrm{rank}_\varepsilon^+(M)$ = min r s.t. ∃ non-negative matrices $M_1,\ldots,M_r$ of rank-1 with $||M-(M_1+\ldots+M_r)||_\infty \leq \varepsilon$
- $\mathrm{rank}_\varepsilon(M) \leq \mathrm{rank}_\varepsilon^+(M)$
- How tight can this relation be for boolean matrices?
- Theorem: ∃ f such that $\mathrm{rank}_\varepsilon(M)^2 \leq \mathrm{rank}_\varepsilon^+(M)$
- Open: is this tight?

Summary of open problems

- Log-rank conjecture [Lovász-Saks'88]: $D(f) \leq \text{poly}(\log(\text{rank}(M_f)))$
- Conjecture [Lee-Shraibman'09]: $R_\varepsilon(f) = \text{poly}(\log(\text{rank}_\varepsilon(f)))$
- Conjecture (LANR) [Lee'12]: $R_\varepsilon(f) = \text{poly}(\log(\text{rank}_\varepsilon^+(f)))$
- Question [Braverman'12]: Inf. complexity ≈ comm. complexity:
  $IC_\varepsilon(f) = \text{poly}(R_\varepsilon(f), \log(n/\varepsilon))$?
- Question [error reduction]: $\log(\text{rank}_\varepsilon^+(f)) \approx \log(1/\varepsilon) \cdot \log(\text{rank}_{1/3}^+(f))$?
- Question [stability under negation]: $\text{rank}_\varepsilon^+(f) \approx \text{rank}_\varepsilon^+(1\text{-}f)$?
- Question: what is the largest c s.t ∃ boolean f with $\text{rank}_\varepsilon(M)^c \leq \text{rank}_\varepsilon^+(M)$

More Questions?