Proof Complexity Lower Bounds from Algebraic Circuit Complexity

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Talk overview

- Very short introduction to proof complexity
- Nullstellensatz proof system
- Models of algebraic computations
- Ideal Proof System (IPS)
- Our results
- Some proofs

Refuting CNFs

Question (3SAT)

Given a 3CNF $\phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ prove there is **no** *satisfying assignment to ϕ.*

Question is coNP-hard
NP $\neq$ coNP $\implies$ any proof must be long

holy grail: prove *unconditional* lower bounds on lengths of proofs in every proof system.

note: potentially harder than proving P $\neq$ NP

goal: prove *unconditional* lower bounds on lengths of proofs in strong proof systems

Frege Proof System

Example (Hilbert Propositional Calculus)

Axioms:

$$\phi \to (\psi \to \phi)$$

$$((\phi \to (\psi \to \zeta)) \to ((\phi \to \psi) \to (\phi \to \zeta)))$$

$$(\phi \to \psi) \to (\neg\psi \to \neg\phi)$$

Derivation rules:

$$\textit{Modus ponens:} \qquad \frac{\phi, (\phi \to \psi)}{\psi}$$

Frege proof systems capture the way we write proofs
goal: lower bounds on lengths of proofs in *Frege proof system*
known: lower bounds for *restricted* versions of Frege
this talk: algebraic proof systems

Subset Sum

Question (Subset Sum)

Given $a_1, \ldots, a_n, b \in \mathbb{C}$, *prove there is* **no** *subset* $S \subseteq [n]$ *such that* $\sum_{i \in S} a_i = b$.
Equivalently, prove there are **no** *solutions to*

$$x_1^2 - x_1 = \cdots = x_n^2 - x_n = 0,$$

$$a_1 x_1 + \cdots + a_n x_n - b = 0 \ .$$

Question is coNP-hard
NP $\neq$ coNP $\implies$ any proof must be long

goal: prove *unconditional* lower bounds on lengths of proofs in strong **algebraic** proof systems.

Nullstellensatz Proofs (i)

Let $\overline{f} := (f_1, \ldots, f_m)$ be a system of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$.

Theorem (Hilbert's Nullstellensatz)

The system $f_1(\overline{x}) = \cdots = f_m(\overline{x}) = 0$ *has no solution iff there are* $g_1, \ldots, g_m \in \mathbb{C}[\overline{x}]$ *such that*

$$g_1(\overline{x}) \cdot f_1(\overline{x}) + \cdots + g_m(\overline{x}) \cdot f_m(\overline{x}) = 1 \ .$$

Gives a *sound* and *complete* proof system for unsatisfiability.
complexity: simple unsatisfiable $\overline{f}$ can require $\deg \overline{g} \geq \exp(m)$.
but: coNP-statements concern $\overline{x} \in \{0,1\}^n$ — polynomials over $\{0,1\}^n$ are degree $\leq n$.

Nullstellensatz Proofs (ii)

$\overline{f} := (f_1, \ldots, f_m)$, $\overline{x}^2 - \overline{x} := (x_1^2 - x_1, \ldots, x_n^2 - x_n)$.

Theorem (Boolean Nullstellensatz)

The system $\overline{f} = \overline{0}$ has no solution over $\overline{x} \in \{0,1\}^n$
$\Leftrightarrow$ *the system $\overline{f}, \overline{x}^2 - \overline{x}$ is unsatisfiable*
$\Leftrightarrow$ *there are $g_1, \ldots, g_m, h_1, \ldots, h_n \in \mathbb{C}[\overline{x}]$ such that*

$$\sum_j g_j(\overline{x}) \cdot f_j(\overline{x}) + \sum_i h_i(\overline{x}) \cdot (x_i^2 - x_i) = 1 \, .$$

complexity: $\deg \overline{g}, \overline{h} \leq O(n)$, $\overline{g}, \overline{h}$ have $\leq 2^{O(n)}$ monomials
goal: prove lower bound on the complexity of $\overline{g}, \overline{h}$.
prior work ([BIK+96a, CEI96, BIK+96b, Raz98, Gri98, IPS99, BGIP01, AR01,...]): exhibit simple $\overline{f}$ where

- $\deg \overline{g}, \overline{h} \geq \Omega(n)$
- $\overline{g}, \overline{h}$ require $2^{\Omega(n)}$ monomials

Algebraic Complexity Measures

Degree and number of monomials are *weak* measures for complexity

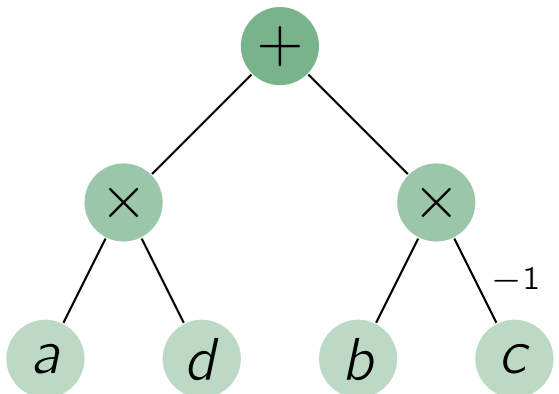
More interesting measure: size of representation in interesting algebraic models

this talk: study Ideal Proof System (IPS) that looks at more interesting measures of complexity

next: Algebraic models and Grochow-Pitassi IPS proof system

Algebraic Formulas

Algebraic formulas are a succinct model of computation for polynomials, e.g. $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$ can be given by



size: number of edges
depth: length of longest input-output path
known: [Kalorkoti85] $\Omega(n^2)$ lower bound for algebraic formulas

The Ideal Proof System (IPS) (i)

$\overline{f} := (f_1, \ldots, f_m)$, $\overline{x}^2 - \overline{x} := (x_1^2 - x_1, \ldots, x_n^2 - x_n)$.

Definition ([GrochowPitassi14])

A *formula*-IPS proof of $\overline{f}, \overline{x}^2 - \overline{x}$ is a pair $(\overline{g}, \overline{h})$ such that

- $\sum g_j f_j + \sum h_i \cdot (x_i^2 - x_i) = 1$
- the size of the proof is the formula size of g_j, h_i

- proof verification: via *Polynomial Identity Testing*, only randomized algorithms known in general.

goal: prove lower bounds for interesting $\overline{f}$

How realistic is this goal?

The Ideal Proof System (IPS) (ii)

Theorem ([GrochowPitassi14])

Let CNF $C = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ be unsatisfiable and encoded by the equations $f_1, \ldots, f_m, \overline{x}^2 - \overline{x}$. Then there are $\overline{g}, \overline{h}$ such that

$$\sum_j g_j \cdot f_j + \sum_i h_i \cdot (x_i^2 - x_i) = 1, \quad \textit{where}$$

- *If there is a size-s Frege proof that C is unsatisfiable, then there are $\overline{g}, \overline{h}$ with $\mathrm{poly}(n, m, s)$-size algebraic formulas.*
- *There are $\overline{g}, \overline{h}$ in $\mathsf{VNP} \approx \{$explicit polynomials$\}$.*

Corollary

- *formula-IPS as powerful as Frege*
- *lower bounds for CNFs $\Longrightarrow$ lower bounds for the permanent*

goal: prove lower bounds for restricted, yet interesting and powerful, algebraic models

The Ideal Proof System (IPS) (iii)

$\overline{f} := (f_1, \ldots, f_m)$, $\overline{x}^2 - \overline{x} := (x_1^2 - x_1, \ldots, x_n^2 - x_n)$.

Definition ([GrochowPitassi14])

For an algebraic model $\mathcal{C}$, a $\mathcal{C}$-IPS proof of $\overline{f}, \overline{x}^2 - \overline{x}$ is a pair $(\overline{g}, \overline{h})$ such that

- $\sum g_j f_j + \sum h_i \cdot (x_i^2 - x_i) = 1$
- the size of the proof is the maximum size of the g_j, h_i as $\mathcal{C}$-computations.

goal: prove lower bounds for $\mathcal{C}$-IPS for interesting $\mathcal{C}$

Multilinear Formulas

Definition

A polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is **multilinear** if the individual degree of each variable x_i is at most 1, that is

$$f(\overline{x}) = \sum_{S \subseteq [n]} \alpha_S \prod_{i \in S} x_i$$

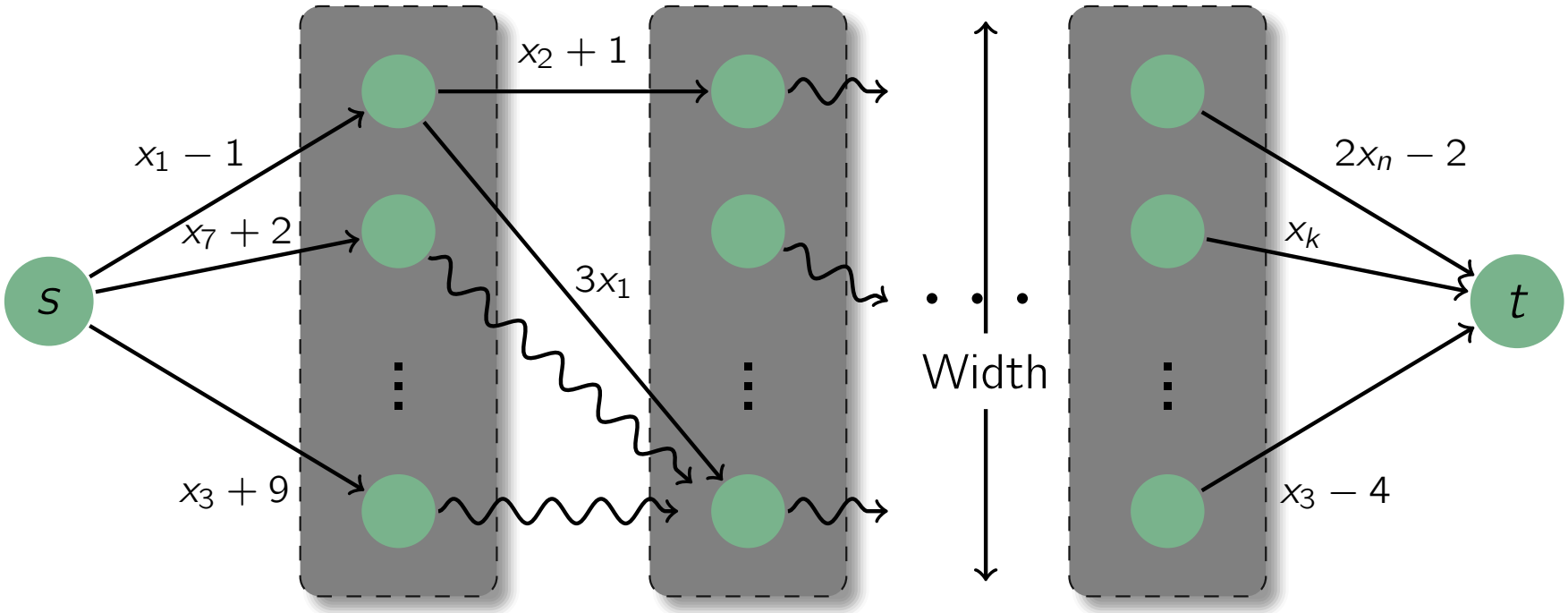
A formula is multilinear if each gate is multilinear

- multilinear polynomials uniquely determined by evaluations over $\{0,1\}^n$

known: [Raz04,RY09]: permanent and determinant require $n^{\Omega(\lg n)}$-size multilinear formulas, $2^{n^{\Omega(1)}}$-size constant-depth multilinear formulas

goal: prove lower bounds for multilinear-formula-IPS

Algebraic Branching Programs



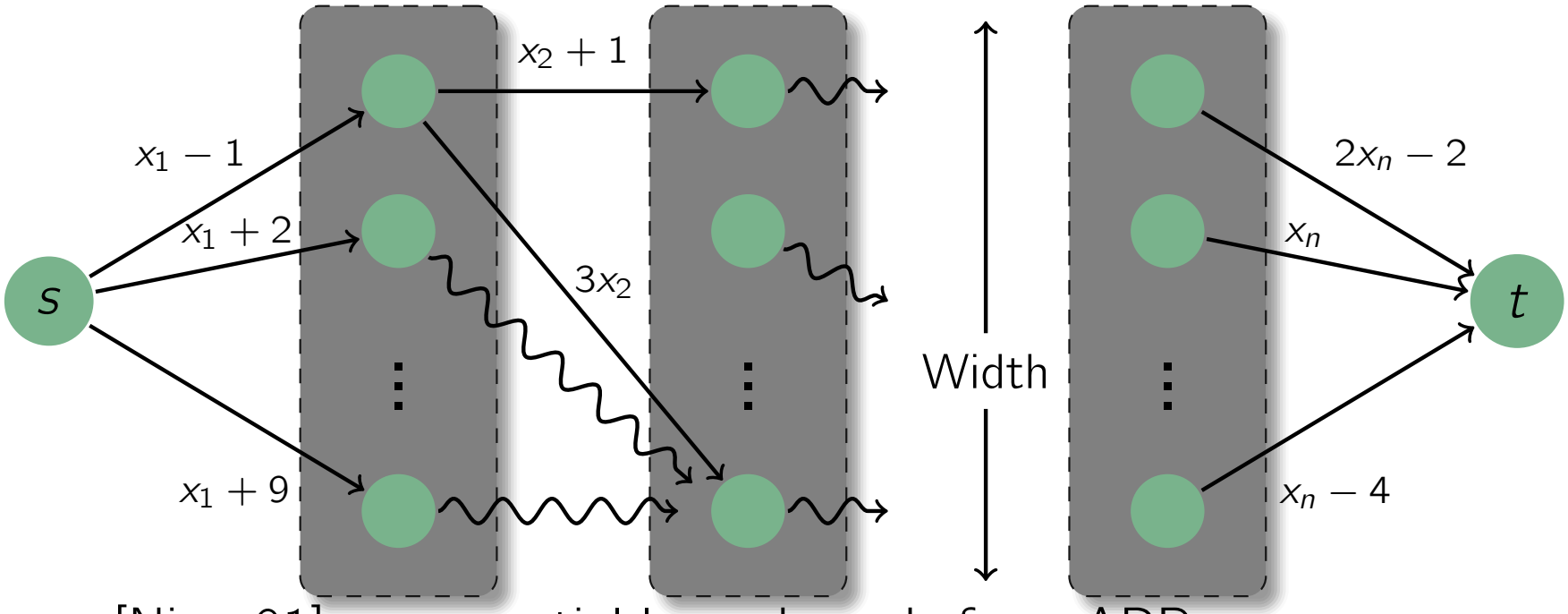
- each $s \to t$ path computes multiplication of edge labels
- program computes the sum of those over all $s \to t$ paths
- output is the $(1,1)$ entry of the matrix product $\prod_{i=1}^{n} M_i(\overline{x})$ (M_i is adjacency matrix of graph on layers $i, i+1$)

goal: prove lower bounds for ABP-IPS

problem: ABPs more expresfull than formulas

Read-Once Oblivious ABPs (i)

read-once oblivious ABP (roABP): every variable appears in one layer



- [Nisan91]: exponential lower bounds for roABPs
- [R**S**05]: polynomial time PIT for roABPs
- [F**S**13,AGKS14,FS**S**14] quasi-poly *black-box* PIT for roABPs

goal: prove lower bounds for roABP-IPS

Recap

Nullstellensatz proof system: proof of unsatisfiability of $\overline{f}, \overline{x}^2 - \overline{x}$ is $\overline{g}, \overline{h}$ such that

$$\sum_j g_j(\overline{x}) \cdot f_j(\overline{x}) + \sum_i h_i(\overline{x}) \cdot (x_i^2 - x_i) = 1$$

IPS: each g_j, h_i has "small" $\mathcal{C}$-computation

Algebraic classes:

- multilinear formulas
- read-once oblivious Algebraic Branching Programs (roABPs)

Next: our results and some proofs

Our Results: Upper Bounds

$x_1 + \cdots + x_n + 1, \overline{x}^2 - \overline{x}$, is unsatisfiable subset-sum instance

Theorem ([ImpagliazzoPudlákSgall99])

$x_1 + \cdots + x_n + 1, \overline{x}^2 - \overline{x}$ *requires Nullstellensatz refutations of*

- *degree* $\geq \Omega(n)$.
- $2^{\Omega(n)}$*-monomials.*

Related to Pigeonhole Principle, well-known "hard" principle

Theorem (Upper Bounds for Subset-Sum)

$x_1 + \cdots + x_n + 1, \overline{x}^2 - \overline{x}$ *has a* poly(n)*-size* $\mathcal{C}$*-IPS proof for* $\mathcal{C} =$

- *depth-3 multilinear formulas*
- *read-once oblivious algebraic branching programs (roABPs)*

Strengthens related upper bounds of [GH03,RT08]

Our Results: Lower Bounds

Theorem (Lower Bounds for Subset-Sum Variants)

$\sum_{i<j} z_{i,j} x_i x_j + 1, \overline{x}^2 - \overline{x}, \overline{z}^2 - \overline{z}$ *requires*

- *multilinear-formula-IPS proofs of* $n^{\Omega(\lg n)}$*-size*
- *constant-depth-multilinear-formula-IPS proofs of* $2^{n^{\Omega(1)}}$*-size*
- *roABP-IPS proofs of* $2^{\Omega(n)}$*-size (in every order)*

First such lower bounds, matches much of the frontier of lower bounds in algebraic complexity theory

Proven via *functional lower bounds*.

Functional Lower Bounds

circuit complexity: single polynomial requires large formulas
proof complexity: *every* proof requires large formulas
idea: if "*unique*" proof then only study single polynomial

Consider an unsatisfiable system $f(\overline{x}), \overline{x}^2 - \overline{x}$, with proof

$$g(\overline{x}) \cdot f(\overline{x}) + \sum_i h_i(\overline{x}) \cdot (x_i^2 - x_i) = 1$$

This implies

$$g(\overline{x}) \cdot f(\overline{x}) = 1, \qquad \overline{x} \in \{0,1\}^n$$
$$g(\overline{x}) = {}^1\!/\!_{f(\overline{x})}, \qquad \overline{x} \in \{0,1\}^n$$

$\Longrightarrow$ g unique as a *function* over $\{0,1\}^n$ or as *multilinear* polynomial

goal: find *easy* $f(\overline{x})$ so **any** g with $g|_{\{0,1\}^n} = \frac{1}{f}|_{\{0,1\}^n}$ is *hard*
need: proof techniques that works in the functional setting and not just for syntactic polynomials (e.g. partial derivative method)

Structure of Proof

Theorem (Lower Bounds for Subset-Sum Variants)

$\sum_{i<j} z_{i,j} x_i x_j + 1, \overline{x}^2 - \overline{x}, \overline{z}^2 - \overline{z}$ *requires*

- *multilinear-formula-IPS proofs of* $n^{\Omega(\lg n)}$*-size*
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- *roABP-IPS proofs of* $2^{\Omega(n)}$*-size (in every order)*

Proof.

- prove functional lower bound for *degree*
- "lift" to functional lower bound for *evaluation dimension*
- conclude functional lower bound for circuit classes via known relations to evaluation dimension
- conclude IPS lower bounds □

Functional Lower Bound — Degree

$x_1 + \cdots + x_n + 1, \overline{x}^2 - \overline{x}$

Proposition

Let $g \in \mathbb{C}[\overline{x}]$ *such that*

$$g(\overline{x}) = \frac{1}{x_1 + \cdots + x_n + 1}, \qquad \overline{x} \in \{0,1\}^n$$

Then $\deg g \geq n$

Tight. Strengthens prior $\deg g \geq n/2$ [IPS99].

Proof.

- multilinearize: $g \mapsto \mathrm{ml}(g)$ with $g|_{\{0,1\}^n} = \mathrm{ml}(g)|_{\{0,1\}^n}$, and $\deg g \geq \deg \mathrm{ml}(g)$
- $\mathrm{ml}(g)$ uniquely determined, compute it explicitly, $\deg \mathrm{ml}(g) = n$ □

Evaluation Dimension

$g \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Study interaction between $\overline{x}$ and $\overline{y}$

Definition ([Nis91,Sap12,FS**13b])**

The **set of evaluations** of g is

$$\mathbf{Eval}_{\overline{x}|\overline{y}}(g) := \{g(\overline{x}, \overline{b})\}_{\overline{b} \in \{0,1\}^n} \subseteq \mathbb{C}[\overline{x}]$$

The **evaluation dimension** of g is $\dim_{\mathbb{C}} \mathbf{Eval}_{\overline{x}|\overline{y}}(g)$

Well-studied complexity measure, (implicitly) used for many lower bounds:

- multilinear formulas [Raz04,RY09,...]
- non-commutative ABPs, roABPs [Nisan91,F**S**13,...]
- depth-3 powering formulas [Saxena08,F**S**13,...]

Functional Lower Bound — Evaluation Dimension

$\sum_{i=1}^{n} x_i y_i + 1, \overline{x}^2 - \overline{x}, \overline{y}^2 - \overline{y}$

Proposition

$g(\overline{x}, \overline{y}) = \frac{1}{\sum_i x_i y_i + 1}$ *for* $\overline{x}, \overline{y} \in \{0,1\}^n$, *then* $\dim \mathbf{Eval}_{\overline{x}|\overline{y}}(g) \geq 2^n$

Proof.

- $\mathbf{Eval}_{\overline{x}|\overline{y}}(g) = \{g(\overline{x}, \overline{b})\}_{\overline{b} \in \{0,1\}^n} = \{g(\overline{x}, \mathbb{1}_S)\}_{S \subseteq \{0,1\}^n}$
- For $\overline{x} \in \{0,1\}^n$, $g(\overline{x}, \mathbb{1}_S) = \frac{1}{\sum_{i \in S} x_i + 1}$
- $\mathrm{ml}(g(\overline{x}, \mathbb{1}_S))$ has degree $|S|$
- $\mathrm{ml}(g(\overline{x}, \mathbb{1}_S)) = \prod_{i \in S} x_i + (\text{lower terms})$
- $\mathrm{ml}(g(\overline{x}, \mathbb{1}_S))$ linearly independent

$$\dim \mathbf{Eval}_{\overline{x}|\overline{y}}(g) \geq \dim \mathrm{ml}(\mathbf{Eval}_{\overline{x}|\overline{y}}(g)) = \dim \{\mathrm{ml}(g(\overline{x}, \mathbb{1}_S))\}_{S \subseteq [n]}$$
$$= \dim \left\{ \prod_{i \in S} x_i + (\text{lower terms}) \right\}_S = 2^n \qquad \square$$

Lower Bounds for IPS

Theorem (Lower Bounds for Subset-Sum Variants)

$\sum_{i<j} z_{i,j} x_i x_j + 1, \overline{x}^2 - \overline{x}, \overline{z}^2 - \overline{z}$ *requires*

- *multilinear-formula-IPS proofs of* $n^{\Omega(\lg n)}$*-size*
- *constant-depth-multilinear-formula-IPS proofs of* $2^{n^{\Omega(1)}}$*-size*
- *roABP-IPS proofs of* $2^{\Omega(n)}$*-size*

Proof.

- degree $\geq n$ functional lower bound for $\frac{1}{\sum_i x_i + 1}$
- $\dim \mathbf{Eval}_{\overline{x}|\overline{y}} \geq 2^n$ functional lower bound for $\frac{1}{\sum_i x_i y_i + 1}$
- $\dim \mathbf{Eval}_{\overline{v}|\overline{u}} \geq 2^{n/2}$ functional lower bounds for $\frac{1}{\sum_{i<j} z_{i,j} x_i x_j + 1}$, for any equipartition $\overline{x} = \overline{v} \cup \overline{u}$
- implies lower bounds for the mentioned circuit classes $\square$

Upper Bounds for IPS (i)

Theorem (Upper Bounds for Subset-Sum)

$x_1 + \cdots + x_n + 1, \overline{x}^2 - \overline{x}$ *has a* poly(n)*-size* $\mathcal{C}$*-IPS proof for* $\mathcal{C} =$

- *depth-3 multilinear formulas*
- *read-once oblivious algebraic branching programs (roABPs)*

Proof.

- Let $p(t) = \prod_{i=0}^{n}(t - i)$
- Then, $p(\sum x_i) = 0$ for $\overline{x} \in \{0,1\}^n$
- Equivalently, $p(\sum x_i) = 0$ modulo $\overline{x}^2 - \overline{x}$
- Note $\frac{p(t)-p(-1)}{t-(-1)} = q(t)$ for some polynomial q
- $q(\sum x_i)(\sum x_i + 1) = p(\sum x_i) - p(-1) \equiv -p(-1) \neq 0$ □

Are we really done?

Upper Bounds for IPS (ii)

Showed that for $p(t) = \prod_{i=0}^{n}(t-i)$ and $q(t) = \frac{p(t)-p(-1)}{t-(-1)}$,

$$q\left(\sum x_i\right)\left(\sum x_i + 1\right) = p\left(\sum x_i\right) - p(-1) \equiv -p(-1) \neq 0$$

To show poly(n)-size $\mathcal{C}$-IPS proofs we need

$$q(\overline{x})\left(\sum x_i + 1\right) + \sum_{i=1}^{n} h_i(\overline{x})(x_i^2 - x_i) = 1 \text{ , where}$$

- $q(\sum x_i)$ has short $\mathcal{C}$-computations
- can efficiently multilinearize $q(\overline{x})\,(\sum x_i + 1)$ in $\mathcal{C}$ (this gives $\overline{h}$)

Proposition

$$q(\sum x_i) = \sum_{i=1}^{\text{poly}(n)} \prod_{j=1}^{n}(a_{i,j}x_j + b_{i,j})$$

Can express this as small roABP and depth-3 multilinear formula and not too difficult to multilinearize □

Conclusions

this talk:

- upper bounds for proving unsatisfiability of $\sum_i x_i + 1, \overline{x}^2 - \overline{x}$
  - depth-3 multilinear formulas
  - read-once oblivious algebraic branching programs (roABPs)
- lower bounds for proving unsatisfiability of $\sum_{i<j} z_{i,j} x_i x_j + 1, \overline{x}^2 - \overline{x}, \overline{z}^2 - \overline{z}$
  - multilinear-formula-IPS proofs of $n^{\Omega(\lg n)}$-size
  - constant-depth-multilinear-formula-IPS proofs of $2^{n^{\Omega(1)}}$-size
  - roABP-IPS proofs of $2^{\Omega(n)}$-size

open question:

- lower bounds for unsatisfiability of $f_1, \ldots, f_m, \overline{x}^2 - \overline{x}$ where each f_i depends on $o(n)$ many variables ?

TOC