Please write your solution in LaTeX and submit them by email to shpilka@cs.technion.ac.il.

1. For a function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ the $k$-th slice is the function

$$f^{(k)}(x) = \begin{cases} 0 & \sum_{i=1}^{n} x_i < k \\ 1 & \sum_{i=1}^{n} x_i > k \\ f(x) & \sum_{i=1}^{n} x_i = k \end{cases}$$

Verify that each $f^{(k)}$ is a monotone function. Prove that for every $f$ and $1 \leq k \leq n$ we have that

$$S^M(f^{(k)}) = O(S(f) + n^2 \log(n)).$$

In other words, there is almost no gap between the monotone complexity and the non-monotone complexity of slice functions.

2. Let $maj(x_1, \ldots, x_n)$ be the majority function.

   (a) Prove that any boolean circuit of depth $d$, where $d$ is a constant, for $maj$ requires size $2^{n^{\Omega(1/d)}}$.

   (b) Prove that for every constant $d$ there exists a depth $d$ boolean circuit of size $2^{\Omega(n^{1/(d-1)})}$ computing $\bigoplus_n$ (the parity on $n$ variables).

   (c) Prove that there exists a depth 2 circuit of polynomial size over the basis $\{\land, \lor, \neg, maj\}$ that computes $\bigoplus_n$.

3. (a) Show that there exist a circuit over the basis $\{\land, \lor, \neg\}$ with only $O(\log(n))$ negation gates (and any number of $\{\land, \lor\}$ gates) that on input $x_1, \ldots, x_n$ computes $\neg x_1, \ldots, \neg x_n$. (the circuit should of course be defined over the positive literals only). Hint: compute the number of 1’s in the input.

   (b) Prove that any such circuit must use $\Omega(\log(n))$ negation gates.