(Non-deterministic) Semantics as a Tool for Analyzing Proof Systems

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“Logic”

1. A formal language $\mathcal{L}$, based on which $\mathcal{L}$-formulas are constructed.
2. A relation $\vdash$ between sets of $\mathcal{L}$-formulas and $\mathcal{L}$-formulas, satisfying:

Reflexivity: if $\psi \in \mathcal{T}$ then $\mathcal{T} \vdash \psi$.
Monotonicity: if $\mathcal{T} \vdash \psi$ and $\mathcal{T} \subseteq \mathcal{T}'$, then $\mathcal{T}' \vdash \psi$.
Transitivity: if $\mathcal{T} \vdash \psi$ and $\mathcal{T}', \psi \vdash \varphi$ then $\mathcal{T}, \mathcal{T}' \vdash \varphi$.

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   Transitivity: if $\mathcal{T} \vdash \psi$ and $\mathcal{T}', \psi \vdash \varphi$ then $\mathcal{T}, \mathcal{T}' \vdash \varphi$.

We can define logics:

- Semantically: $\mathcal{T} \vdash \psi$ if every "model" of $\mathcal{T}$ is a "model" of ψ.
- Syntactically: $\mathcal{T} \vdash \psi$ if ψ has a derivation from $\mathcal{T}$ in a given proof system.

Motivation

Use semantics to:

- *understand* logics defined by new proof systems.
- (co-semi) *decide* such logics.
- prove (or disprove) *proof-theoretic properties* of (families of) proof systems.
  - Proof-theoretic methods are sometimes tedious and error-prone.

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$$\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_m \quad \leftrightsquigarrow \quad \varphi_1 \wedge \ldots \wedge \varphi_n \supset \psi_1 \vee \ldots \vee \psi_m$$

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- Tarskian consequence relations (logics) can obtained by:

V: $\mathcal{T} \vdash^{frm}_{\mathbf{G}} \varphi \quad \Longleftrightarrow \quad \{ \Rightarrow \psi \mid \psi \in \mathcal{T}\} \vdash_{\mathbf{G}} \Rightarrow \varphi$

T: $\mathcal{T} \vdash^{frm}_{\mathbf{G}} \varphi \quad \Longleftrightarrow \quad \vdash_{\mathbf{G}} \Gamma \Rightarrow \varphi$ *for some* $\Gamma \subseteq \mathcal{T}$

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  T: $\mathcal{T} \vdash^{frm}_{\mathbf{G}} \varphi \iff \vdash_{\mathbf{G}} \Gamma \Rightarrow \varphi$ *for some* $\Gamma \subseteq \mathcal{T}$

  We choose V because of its robustness.

LK

Axioms:

$$(id) \quad \varphi \Rightarrow \varphi$$

Structural Rules:

$$(W \Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta} \qquad (\Rightarrow W) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta}$$

$$(cut) \quad \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \Delta}$$

Logical Rules:

$$(\supset \Rightarrow) \quad \frac{\Gamma \Rightarrow \varphi_1, \Delta \quad \Gamma, \varphi_2 \Rightarrow \Delta}{\Gamma, \varphi_1 \supset \varphi_2 \Rightarrow \Delta} \qquad (\Rightarrow \supset) \quad \frac{\Gamma, \varphi_1 \Rightarrow \varphi_2, \Delta}{\Gamma \Rightarrow \varphi_1 \supset \varphi_2, \Delta}$$

$$(\wedge \Rightarrow) \quad \frac{\Gamma, \varphi_1, \varphi_2 \Rightarrow \Delta}{\Gamma, \varphi_1 \wedge \varphi_2 \Rightarrow \Delta} \qquad (\Rightarrow \wedge) \quad \frac{\Gamma \Rightarrow \varphi_1, \Delta \quad \Gamma \Rightarrow \varphi_2, \Delta}{\Gamma \Rightarrow \varphi_1 \wedge \varphi_2, \Delta}$$

Classical Logic

The "Matrix" $\mathbf{M_{LK}}$

- Truth-values: $\{\mathrm{T}, \mathrm{F}\}$
- Truth-tables:

$\widetilde{\supset}$	T	F
T	T	F
F	T	T

$\widetilde{\wedge}$	T	F
T	T	F
F	F	F

- An $\mathbf{M_{LK}}$-valuation is a *model* of a sequent $\Gamma \Rightarrow \Delta$ iff $v(\psi) = \mathrm{F}$ for some $\psi \in \Gamma$ or $v(\psi) = \mathrm{T}$ for some $\psi \in \Delta$.

Soundness and Completeness

$\Omega \vdash_{\mathbf{LK}} s$ iff every $\mathbf{M_{LK}}$-valuation which is a model of every sequent in Ω is also a model of s.

Subformula Property

Notation: $\Omega \vdash_{\mathbf{G}}^{\mathcal{E}} s$ iff there exists a derivation of s from Ω in $\mathbf{G}$ consisting solely of $\mathcal{E}$-sequents (i.e. sequents consisting solely of formulas from $\mathcal{E}$).

Subformula Property

$$\Omega \vdash_{\mathbf{G}} s \qquad \Longrightarrow \qquad \Omega \vdash_{\mathbf{G}}^{sub[\Omega,s]} s$$

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Subformula Property

$$\Omega \vdash_{\mathbf{G}} s \quad\Longrightarrow\quad \Omega \vdash^{sub[\Omega,s]}_{\mathbf{G}} s$$

Q: Can we find "semantics" for $\vdash^{\mathcal{E}}_{\mathbf{LK}}$?

"Semantics" for $\vdash^{\mathcal{E}}_{\mathbf{G}}$

(Stronger) Soundness and Completeness

For every closed set $\mathcal{E}$ of formulas, and set $\Omega \cup \{s\}$ of $\mathcal{E}$-sequents:
$\Omega \vdash^{\mathcal{E}}_{\mathbf{LK}} s$ iff every partial $\mathbf{M_{LK}}$-valuation, defined on $\mathcal{E}$, which is a model of every sequent in Ω is also a model of s.

"Semantics" for $\vdash^{\mathcal{E}}_{\mathbf{G}}$

(Stronger) Soundness and Completeness

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Now, proving the subformula property for **LK** reduces to proving that every partial $\mathbf{M_{LK}}$-valuation (defined on a closed set of formulas) can be extended to a (full) $\mathbf{M_{LK}}$-valuation.

"Semantics" for $\vdash_{\mathbf{G}}^{\mathcal{E}}$

(Stronger) Soundness and Completeness

For every closed set $\mathcal{E}$ of formulas, and set $\Omega \cup \{s\}$ of $\mathcal{E}$-sequents: $\Omega \vdash_{\mathbf{LK}}^{\mathcal{E}} s$ iff every partial $\mathbf{M_{LK}}$-valuation, defined on $\mathcal{E}$, which is a model of every sequent in Ω is also a model of s.

Now, proving the subformula property for **LK** reduces to proving that every partial $\mathbf{M_{LK}}$-valuation (defined on a closed set of formulas) can be extended to a (full) $\mathbf{M_{LK}}$-valuation.

This is trivial. ☺

Cut-Admissibility

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$$\vdash_{\mathbf{G}} S \quad \Longrightarrow \quad \vdash_{\mathbf{G}-(cut)} S$$

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Q: Can we find semantics for $\mathbf{LK} - (cut)$?

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Q: Can we find semantics for $\mathbf{LK} - (cut)$?

- Does not hold in the presence of assumptions, e.g.

$$\Rightarrow p_1 \supset p_2 \vdash_{\mathbf{LK}} \Rightarrow p_1 \supset (p_3 \supset p_2)$$

$$\Rightarrow p_1 \supset p_2 \nvdash_{\mathbf{LK}-(cut)} \Rightarrow p_1 \supset (p_3 \supset p_2)$$

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Theorem

$\vdash^{frm}_{\mathbf{LK}-(cut)}$ *does not have a finite characteristic matrix.*

Non-Deterministic Matrices

- Truth-tables assign non-empty *sets* of truth-values.
- $v(\diamond(\psi_1, \ldots, \psi_n)) \in \widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$ instead of $v(\diamond(\psi_1, \ldots, \psi_n)) = \widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$.

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$\widetilde{\wedge}$	T	F
T	T	F
F	F	F

$\widetilde{\wedge}$	T	F
T	{T}	{F}
F	{F}	{F}

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- Particularly useful to handle *syntactic underspecification*.

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T	T	F
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T	T	F
F	F	F

$\widetilde{\wedge}$	T	F
T	$\{T\}$	$\{F\}$
F	$\{F\}$	$\{F\}$

$$(\wedge \Rightarrow) \quad \frac{\Gamma, \varphi_1, \varphi_2 \Rightarrow \Delta}{\Gamma, \varphi_1 \wedge \varphi_2 \Rightarrow \Delta} \qquad (\Rightarrow \wedge) \quad \frac{\Gamma \Rightarrow \varphi_1, \Delta \quad \Gamma \Rightarrow \varphi_2, \Delta}{\Gamma \Rightarrow \varphi_1 \wedge \varphi_2, \Delta}$$

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- Particularly useful to handle *syntactic underspecification*.

$\widetilde{\wedge}$	T	F
T	T	F
F	F	F

$\widetilde{\wedge}$	T	F
T	{T}	{F}
F	{F}	{F}

$\widetilde{\wedge}$	T	F
T	{T, F}	{F}
F	{F}	{F}

$$(\wedge\Rightarrow)\quad \frac{\Gamma,\varphi_1,\varphi_2\Rightarrow\Delta}{\Gamma,\varphi_1\wedge\varphi_2\Rightarrow\Delta} \qquad (\Rightarrow\wedge)\quad \frac{\Gamma\Rightarrow\varphi_1,\Delta\quad \Gamma\Rightarrow\varphi_2,\Delta}{\Gamma\Rightarrow\varphi_1\wedge\varphi_2,\Delta}$$

Semantics for **LK** – (*cut*)

$$(cut) \quad \frac{\varphi \Rightarrow \qquad \Rightarrow \varphi}{\Rightarrow}$$

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The "NMatrix" $\mathbf{M}_{\mathbf{LK}-(cut)}$

- Truth-values: $\{\langle \text{F}, \text{F}\rangle, \langle \text{T}, \text{T}\rangle, \langle \text{F}, \text{T}\rangle\}$
- Truth-tables:

$\widetilde{\wedge}$	$\langle \text{T}, \text{T}\rangle$	$\langle \text{F}, \text{F}\rangle$	$\langle \text{F}, \text{T}\rangle$
$\langle \text{T}, \text{T}\rangle$	$\{\langle \text{T}, \text{T}\rangle, \langle \text{F}, \text{T}\rangle\}$	$\{\langle \text{F}, \text{F}\rangle, \langle \text{F}, \text{T}\rangle\}$	$\{\langle \text{F}, \text{T}\rangle\}$
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$\langle \text{F}, \text{T}\rangle$	$\{\langle \text{F}, \text{T}\rangle\}$	$\{\langle \text{F}, \text{F}\rangle, \langle \text{F}, \text{T}\rangle\}$	$\{\langle \text{F}, \text{T}\rangle\}$

- An $\mathbf{M}_{\mathbf{LK}-(cut)}$-valuation is a model of a sequent $\Gamma \Rightarrow \Delta$ iff $v_l(\psi) = \text{F}$ for some $\psi \in \Gamma$ or $v_r(\psi) = \text{T}$ for some $\psi \in \Delta$.

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$\widetilde{\wedge}$	$\langle \mathrm{T}, \mathrm{T} \rangle$	$\langle \mathrm{F}, \mathrm{F} \rangle$	$\langle \mathrm{F}, \mathrm{T} \rangle$
$\langle \mathrm{T}, \mathrm{T} \rangle$	$\{\langle \mathrm{T}, \mathrm{T} \rangle, \langle \mathrm{F}, \mathrm{T} \rangle\}$	$\{\langle \mathrm{F}, \mathrm{F} \rangle, \langle \mathrm{F}, \mathrm{T} \rangle\}$	$\{\langle \mathrm{F}, \mathrm{T} \rangle\}$
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$\langle T, T \rangle$	$\{\langle T, T \rangle, \langle F, T \rangle\}$	$\{\langle F, F \rangle, \langle F, T \rangle\}$	$\{\langle F, T \rangle\}$
$\langle F, F \rangle$	$\{\langle F, F \rangle, \langle F, T \rangle\}$	$\{\langle F, F \rangle, \langle F, T \rangle\}$	$\{\langle F, F \rangle, \langle F, T \rangle\}$
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$\widetilde{\supset}$	$\langle T, T\rangle$	$\langle F, F\rangle$	$\langle F, T\rangle$
$\langle T, T\rangle$	$\{\langle T, T\rangle, \langle F, T\rangle\}$	$\{\langle F, F\rangle, \langle F, T\rangle\}$	$\{\langle F, T\rangle\}$
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Semantics for $\mathbf{LK} - (cut)$

Soundness and Completeness

$\Omega \vdash_{\mathbf{LK}-(cut)} s$ iff every $\mathbf{M}_{\mathbf{LK}-(cut)}$-valuation which is a model of every sequent in Ω is also a model of s.

$\hookrightarrow$ New formulation of results of Schütte (1960) and Girard (1987).

Proving Cut-Admissibility for **LK**

Cut-Admissibility for **LK**

$$\vdash_{\mathbf{LK}} s \qquad \Longrightarrow \qquad \vdash_{\mathbf{LK}-(cut)} s$$

Proving Cut-Admissibility for **LK**

Cut-Admissibility for **LK**

$$\vdash_{\mathbf{LK}} s \quad \Longrightarrow \quad \vdash_{\mathbf{LK}-(cut)} s$$

- Reduces to proving that for every $\mathbf{M}_{\mathbf{LK}-(cut)}$-valuation which is not a model of some sequent s, there exists an $\mathbf{M}_{\mathbf{LK}}$-valuation which is not a model of s.
- Simply, by induction on the build-up of formulas.

(Non-deterministic) Semantics as a Tool for Analyzing Proof Systems

Similar ideas can be used to study:

- Systems without (*id*) (in fact, any rule except for weakening, contraction and exchange).
- Concrete *proof-specifications*, specifying which formulas:
  - Are allowed to appear in derivations on each side of the sequent.
  - Are allowed to serve as active formulas of each derivation rule.

(Non-deterministic) Semantics as a Tool for Analyzing Proof Systems

These methods can be applied in broad families of proof systems:

Canonical Systems

$$\frac{\Gamma \Rightarrow \varphi_2, \Delta}{\Gamma \Rightarrow \varphi_1 \rightsquigarrow \varphi_2, \Delta}$$

Labelled Systems

$$\frac{s \cup \{a : \varphi_1\} \quad s \cup \{b : \varphi_2\}}{s \cup \{c : \varphi_1 \star \varphi_2\}}$$

Basic Systems

$$\frac{\Gamma, \varphi_1 \Rightarrow \varphi_2}{\Gamma \Rightarrow \varphi_1 \supset \varphi_2} \qquad \frac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}$$

Canonical Gödel Systems

The System **HIF**

Manipulates single-conclusion hypersequents.

Axioms:

$$\varphi \Rightarrow \varphi$$

Structural Rules:

$$(IW \Rightarrow)\quad \frac{H \mid \Gamma \Rightarrow E}{H \mid \Gamma, \varphi \Rightarrow E} \qquad (\Rightarrow IW)\quad \frac{H \mid \Gamma \Rightarrow}{H \mid \Gamma \Rightarrow \varphi} \qquad (EW)\quad \frac{H}{H \mid \Gamma \Rightarrow E}$$

$$(com)\quad \frac{H \mid \Gamma_1, \Gamma_1' \Rightarrow E_1 \qquad H \mid \Gamma_2, \Gamma_2' \Rightarrow E_2}{H \mid \Gamma_1, \Gamma_2' \Rightarrow E_1 \mid \Gamma_2, \Gamma_1' \Rightarrow E_2} \qquad (cut)\quad \frac{H \mid \Gamma \Rightarrow \varphi \qquad H \mid \Gamma, \varphi \Rightarrow E}{H \mid \Gamma \Rightarrow E}$$

Logical Rules:

$$(\supset \Rightarrow)\quad \frac{H \mid \Gamma \Rightarrow \varphi_1 \qquad H \mid \Gamma, \varphi_2 \Rightarrow E}{H \mid \Gamma, \varphi_1 \supset \varphi_2 \Rightarrow E} \qquad (\Rightarrow \supset)\quad \frac{H \mid \Gamma, \varphi_1 \Rightarrow \varphi_2}{H \mid \Gamma \Rightarrow \varphi_1 \supset \varphi_2}$$

$$(\wedge \Rightarrow)\quad \frac{H \mid \Gamma, \varphi_1, \varphi_2 \Rightarrow E}{H \mid \Gamma, \varphi_1 \wedge \varphi_2 \Rightarrow E} \qquad (\Rightarrow \wedge)\quad \frac{H \mid \Gamma \Rightarrow \varphi_1 \qquad H \mid \Gamma \Rightarrow \varphi_2}{H \mid \Gamma \Rightarrow \varphi_1 \wedge \varphi_2}$$

Semantics - Gödel logic

The "Matrix" $\mathbf{M_{HIF}}$

- Truth-values: $[0, 1]$
- Truth-tables:

$$\widetilde{\supset}(x, y) = \begin{cases} 1 & x \leq y \\ y & x > y \end{cases} \qquad \widetilde{\wedge}(x, y) = \min(x, y)$$

- An $\mathbf{M_{HIF}}$-valuation is a *model*:
  - of a sequent $\Gamma \Rightarrow E$ iff $\min\{v(\psi) \mid \psi \in \Gamma\} \leq \max\{v(\psi) \mid \psi \in E\}$.
  - of a hypersequent H iff it is a model of some $s \in H$.

Soundness and Completeness

$\mathcal{H} \vdash_{\mathbf{HIF}} H$ iff every $\mathbf{M_{HIF}}$-valuation which is a model of every hypersequent in $\mathcal{H}$ is also a model of H.

Semantics for **HIF** – (*cut*)

$$(cut) \quad \frac{\varphi \Rightarrow \qquad \Rightarrow \varphi}{\Rightarrow}$$

Semantics for **HIF** − (*cut*)

$$(cut)\quad \frac{\varphi \Rightarrow \qquad \Rightarrow \varphi}{\Rightarrow}$$

The "NMatrix" $\mathbf{M}_{\mathbf{HIF}-(cut)}$

- Truth-values: $\{\langle x, y\rangle \in [0,1] \times [0,1] \mid x \leq y\}$
- Truth-tables:

$$\widetilde{\supset}(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle) = \left[0, \begin{cases} 1 & y_1 \leq x_2 \\ x_2 & y_1 > x_2 \end{cases}\right] \times \left[\begin{cases} 1 & x_1 \leq y_2 \\ y_2 & x_1 > y_2 \end{cases}, 1\right]$$

$$\widetilde{\wedge}(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle) = [0, \min(x_1, x_2)] \times [\min(y_1, y_2), 1]$$

- An $\mathbf{M}_{\mathbf{HIF}-(cut)}$-valuation is a *model*:
  - of a sequent $\Gamma \Rightarrow E$ iff $\min\{v_l(\psi) \mid \psi \in \Gamma\} \leq \max\{v_r(\psi) \mid \psi \in E\}$.
  - of a hypersequent H iff it is a model of some $s \in H$.

HIF $-$ (cut)

Soundness and Completeness

$\mathcal{H} \vdash_{\mathbf{HIF}-(cut)} H$ iff every $\mathbf{M}_{\mathbf{HIF}-(cut)}$-valuation which is a model of every hypersequent in $\mathcal{H}$ is also a model of H.

- Proving cut-admissibility for **HIF** reduces to proving that for every $\mathbf{M}_{\mathbf{HIF}-(cut)}$-valuation which is not a model of some hypersequent H, there exists an $\mathbf{M}_{\mathbf{HIF}}$-valuation which is not a model of H.

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- Dual construction for **HIF** $-$ (id).
- This method can be generalized for arbitrary canonical derivation rules added to **HIF**.

Conclusions

- Non-deterministic semantics is a useful tool for investigating proof-theoretic properties of logical calculi.
- The semantic tools should complement the usual proof-theoretic ones.

Further Research

- Extensions for first order logics
- Sub-structural calculi

Thank you!