Pure Sequent Calculi: Analyticity and Decision Procedure

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Analyticity, also known as the subformula property, typically guarantees decidability of derivability in propositional sequent calculi. To utilize this fact, two substantial gaps have to be addressed: (i) what makes a sequent calculus analytic? and (ii) how to obtain an efficient decision procedure for derivability in an analytic calculus? In the first part of this paper we answer these questions for *pure calculi*—a general family of fully structural propositional sequent calculi whose rules allow arbitrary context formulas. We provide a sufficient syntactic criterion for analyticity in these calculi, as well as a productive method to construct new analytic calculi from given ones. We further introduce a scalable decision procedure for derivability in analytic pure calculi, by showing that it can be (uniformly) reduced to classical satisfiability. In the second part of the paper, we study the extension of pure sequent calculi with modal operators. We show that such extensions preserve the analyticity of the calculus, and identify certain restricted operators (which we call 'Next' operators) that are also amenable for a general reduction of derivability to classical satisfiability. Our proofs are all semantic, utilizing several strong general soundness and completeness theorems with respect to non-deterministic semantic frameworks: bivaluations (for pure calculi) and Kripke models (for their extension with modal operators).

Additional Key Words and Phrases: Sequent Calculi, Analyticity, Subformula property

1 INTRODUCTION

More than eighty years after its introduction [25], the framework of sequent calculi is by now a mainstream proof theoretic framework. When a given logic is accompanied with a well-behaved sequent calculus, the latter often provides a useful representation of the logic, which allows one to study various properties of it. For propositional logics, which are the focus of this paper, a sequent calculus may be used to establish decidability, and, in turn, to develop a proof search method. To this end, it is typically required that the calculus is *analytic*. Roughly speaking, analyticity (a.k.a. the subformula property) ensures that every derivable sequent $\Gamma \Rightarrow \Delta$ has a derivation that uses only the syntactic material available inside $\Gamma \cup \Delta$.[1]

The current paper is devoted to a *general and uniform* study of propositional sequent calculi, aiming to understand: (i) what guarantees that a given sequent calculus is analytic? and (ii) how analyticity can be utilized to obtain an effective decision procedure for derivability in the calculus? Answering these questions may assist future development of sequent calculi and proof search methods, which are traditionally tailored to specific logics.

Our investigation encompasses a wide family of sequent calculi, called *pure sequent calculi*, as well as the extensions of pure sequent calculi with derivation rules for modal operators. Pure sequent calculi are propositional fully-structural calculi. (By fully-structural, we mean that they include all ordinary structural rules: exchange, contraction and weakening.) In addition, the important restriction on derivation rules in these calculi is that they do not enforce any limitations on the formulas that may be used as context in applications of the rules (following [3], the adjective "pure" stands for this requirement). While being a simple framework, pure calculi were shown

[1] Often, analyticity is obtained as a simple corollary of cut-elimination. Nevertheless, our interest here is on analyticity, which we find more fundamental than cut-elimination in propositional sequent calculi. In fact, there are well-known examples of logics (e.g., the modal logics B and S5) that have a simple analytic sequent calculus, but no (known) cut-free sequent calculus.

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to be adequate for a wide range of logics, including important three and four valued logics and various paraconsistent logics. By further studying the extension of pure calculi with rules for modal operators, our work covers also multi-modal logics.

We first generalize the usual notion of analyticity by employing a parametrized generalization a subformula. The general notion applies to more calculi (e.g., it may allow one to use $\neg\varphi$ in a proof of $\Rightarrow \varphi$), and still inherits the important consequences of analyticity, including decidability. Then, the crux of this paper addresses questions (i) and (ii) above:

- (i) We provide a simple syntactic criterion that ensures analyticity, and present a method for constructing analytic sequent calculi. The latter allows one to obtain an analytic-by-construction calculi by collecting certain instances of logical rules in some given analytic calculus. In particular, this method is useful to obtain calculi for non-classical logics (especially for paraconsistent logics) that are naturally developed as restrictions of classical logic. We also show that the addition of various rules for modal operators preserves analyticity, which provides a uniform approach to analytic calculi for (non-classical) modal logics.
- (ii) We show that derivability in analytic pure calculi can be reduced in polynomial time to (the complement of) SAT—the classical satisfiability problem. While SAT is NP-hard, it is considered "easy" when it comes to real-world applications. Indeed, there are many off-the-shelf SAT solvers, that, despite an exponential worst-case time complexity, are considered extremely efficient (see, e.g., [27]). Our reduction constitutes a scalable uniform decision procedure for logics that can be represented by analytic pure sequent calculi. We further extend this reduction for the extension of such calculi with *Next operators*, a restricted type of modal operators used in temporal and access control logics; and identify a subfamily of calculi for which the reduction generates Horn clauses, leading to a *linear time* decision procedure (using a HORNSAT solver). This provides a systematic approach for developing calculi for particular applications that require extremely efficient decision procedure (as was recently done for an access control logic called “primal infon logic” [19]).

Our main tool to achieve the above is a *semantic* interpretation of sequent calculi. As observed in [12], there is a simple correspondence between pure sequent calculi and two-valued valuation semantics. We utilize this correspondence and extend it for characterizing derivations that are confined to use only a certain set of formulas. Thus, we obtain a purely semantic equivalent definition of analyticity (roughly speaking, analyticity is equivalent to the ability to extend partial countermodels), which is very useful in our general study of sequent calculi. When considering rules for modal operators, we follow a similar approach, and use a correspondence (i.e., a general soundness and completeness theorem) between pure sequent calculi extended with modal operators and certain Kripke-style models.

Related Work

Analyticity in *subfamilies* of pure sequent calculi has been investigated in previous works. A particularly well-behaved subfamily of pure calculi, called *canonical calculi*, was studied by Avron and Lev [10]. For these calculi, it was shown that analyticity and cut-admissibility are equivalent, and both are precisely characterized by a simple and decidable criterion, called *coherence*. A similar criterion was later provided by Avron in [7] for an extended subfamily of *quasi-canonical calculi*. Our results apply for significantly more general family of calculi, allowing us to derive these existing criteria for analyticity as particular instances. In addition, the general framework of Miller and Pimentel [38] allows one to encode all pure calculi in linear logic, and use linear logic to reason about them. Among the pure calculi, it is again only the canonical ones for which a decidable criterion for cut-admissibility is given.

The study of uniform decision procedures parametrized by a given formal calculus is the subject of the LoTREC [24] and MetTeL [43] projects, where the underlying framework is that of Tableuax calculi, rather than sequent calculi, which are the focus of the current work. The decision procedure that we propose here can be seen as a generalization of the one given by Beklemishev and Gurevich [11] for quotations-free primal infon logic. For the case of primal infon logic with quotations, the proposed reduction to SAT produces practically equivalent outputs as the reduction in [13] from this logic to Datalog. A general methodology for translating derivability questions in Hilbertian deductive systems to Datalog was introduced in [14] by Blass and Gurevich. However, this method may produce infinitely many Datalog premises. In contrast, our SAT instances are always finite.

A SAT-based decision procedure for classical modal logics was presented in [26]. The reduction that we present here is different in three aspects. On the one hand, we cover modal logics that are not necessarily classical. On the other hand, the reduction that is presented here is valid only for *Next* operators, while [26] covers other modal operators. Finally, while our decision procedure is obtained by a "one-shot" reduction to SAT, the procedure of [26] generates several SAT-instances in different part of its algorithm.

A leitmotif of this paper is the use of the *semantic* approach, which is particularly useful when cut-elimination is beyond the reach and general families of calculi are studied (see also [39]). The general semantic framework that we use here for pure calculi extends bivaluation semantics [12, 16]. The semantic framework that we use here for pure calculi with modal operators closely follows the one developed by Lahav and Avron in [34], adapted and simplified for our needs. To the best of out knowledge, no previous work considered the preservation of analyticity when extending a calculus with modal operators.

Finally, we note that preliminary short versions of different parts of this paper were included in [35] and [36]. Besides the addition of full proofs, we significantly strengthened the previous results so as to cover a more general notion of analyticity, as well as the extension of pure calculi with various rules for modal operators.

Outline

§2 defines the family of pure sequent calculi and established their semantic interpretation, which plays a major role in subsequent sections. In §3, a generalized analyticity property is defined. §4 describes the reduction of derivability in analytic pure calculi to SAT. Methods for identifying analyticity and for constructing analytic calculi are introduced in §5. Next, §6 extends the theory of pure calculi to accommodate rules for modal operators, and §7 generalizes the reduction to SAT for pure calculi augmented with modal *Next* operators. Finally, §8 includes conclusions and further research questions.

2 PURE SEQUENT CALCULI

In this section we define the family of pure sequent calculi (§2.2), provide a uniform semantic interpretation for them (§2.3), and introduce useful transformations on pure calculi that do not affect the induced derivability relation (§2.4).

2.1 Preliminaries

In what follows, $\mathcal{L}$ denotes an arbitrary *propositional language*, consisting of a countable infinite set of atomic variables $At = \{p_1, p_2, \ldots\}$ and a finite set $\diamond_{\mathcal{L}}$ of propositional connectives. For every $n \geq 0$, the set of all n-ary connectives of $\mathcal{L}$ is denoted by $\diamond^n_{\mathcal{L}}$. Well-formed formulas in a propositional language $\mathcal{L}$ are defined as usual, and we usually identify $\mathcal{L}$ with its set of well-formed formulas (e.g., when writing $\psi \in \mathcal{L}$). Given a set $\mathcal{F} \subseteq \mathcal{L}$, we say that a formula ψ is an *$\mathcal{F}$-formula* if $\psi \in \mathcal{F}$.

A *substitution* is a function from At to some propositional language. A substitution σ is extended to formulas by $\sigma(\diamond(\psi_1, \dots, \psi_n)) = \diamond(\sigma(\psi_1), \dots, \sigma(\psi_n))$ for every connective $\diamond$, and to sets of formulas by $\sigma(\mathcal{F}) = \{\sigma(\psi) \mid \psi \in \mathcal{F}\}$.

A *sequent* is a pair $\langle \Gamma, \Delta \rangle$, denoted $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of formulas. For a sequent $\Gamma \Rightarrow \Delta$, $frm(\Gamma \Rightarrow \Delta) = \Gamma \cup \Delta$. This notation is naturally extended to sets of sequents. A sequent $\Gamma \Rightarrow \Delta$ is called an $\mathcal{F}$*-sequent* if $frm(\Gamma \Rightarrow \Delta) \subseteq \mathcal{F}$. We employ the standard sequent notations, e.g., when writing expressions like $\Gamma, \psi \Rightarrow \Delta$ or $\Rightarrow \psi$. The *union* of sequents is given by $(\Gamma_1 \Rightarrow \Delta_1) \cup (\Gamma_2 \Rightarrow \Delta_2) = (\Gamma_1 \cup \Gamma_2) \Rightarrow (\Delta_1 \cup \Delta_2)$. A sequent $\Gamma_1 \Rightarrow \Delta_1$ is a *subsequent* of a sequent $\Gamma_2 \Rightarrow \Delta_2$, denoted $\Gamma_1 \Rightarrow \Delta_1 \subseteq \Gamma_2 \Rightarrow \Delta_2$, if $\Gamma_1 \subseteq \Gamma_2$ and $\Delta_1 \subseteq \Delta_2$. Substitutions are also extended to sequents by $\sigma(\Gamma \Rightarrow \Delta) = \sigma(\Gamma) \Rightarrow \sigma(\Delta)$ and sets of sequents by $\sigma(S) = \{\sigma(s) \mid s \in S\}$.

2.2 Pure Sequent Calculi

Roughly speaking, pure sequent calculi are propositional fully-structural calculi (sequent calculi that include all the usual structural rules: exchange, contraction, cut, identity and weakening), whose derivation rules do not enforce any limitations on the side formulas. This family of calculi is a prominent proof-theoretic framework, adequate for many propositional logics, including classical logic, many-valued logics, and various paraconsistent logics.

We start by defining *pure rules* and their applications, namely the steps that form derivations in pure calculi. Following [10], we find it convenient to use the object propositional language for specifying derivation rules, instead of meta-variables which are often used to present derivation schemes. Accordingly, applications of rules are obtained by applying a *substitution* on the premises and the conclusion of the rule, and freely adding context formulas.

Definition 2.1. A *pure rule* is a pair $\langle S, s \rangle$, denoted $S \,/\, s$, where S is a finite set of sequents and s is a sequent. The elements of S are called the *premises* of the rule and s is called the *conclusion* of the rule. An *application* of a pure rule $\{s_1, \dots, s_n\} \,/\, s$ is a pair of the form

$$\left\langle \{\sigma(s'_1) \cup c_1, \dots, \sigma(s'_n) \cup c_n\}, \sigma(s) \cup c_1 \cup \dots \cup c_n \right\rangle$$

where σ is a substitution, s'_i is a subsequent of s_i for every $1 \leq i \leq n$, and $c_1, \dots, c_n$ are sequents (called *context sequents*). The sequents $\sigma(s'_i) \cup c_i$ are called the *premises* of the application and the sequent $\sigma(s) \cup c_1 \cup \dots \cup c_n$ is called the *conclusion* of the application. We often denote an application as a derivation step:

$$\frac{\sigma(s'_1) \cup c_1, \dots, \sigma(s'_n) \cup c_n}{\sigma(s) \cup c_1 \cup \dots \cup c_n}$$

Example 2.2 (Pure rules for classical implication). The following is a pure rule (we omit the curly braces to improve readability):

$$p_1 \Rightarrow p_2 \,/\, \Rightarrow p_1 \supset p_2$$

Applications of this rule have the following forms:

$$\frac{\Gamma, \psi_1 \Rightarrow \psi_2, \Delta}{\Gamma \Rightarrow \psi_1 \supset \psi_2, \Delta} \qquad \frac{\Gamma, \psi_1 \Rightarrow \Delta}{\Gamma \Rightarrow \psi_1 \supset \psi_2, \Delta} \qquad \frac{\Gamma \Rightarrow \psi_2, \Delta}{\Gamma \Rightarrow \psi_1 \supset \psi_2, \Delta} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \psi_1 \supset \psi_2, \Delta}$$

Notice that the first application uses the full sequent in its premise, while the others use proper subsequents of the premises.

Applications of the following rules

$$\Rightarrow p_1 \,;\, p_2 \Rightarrow \,/\, p_1 \supset p_2 \Rightarrow \qquad\qquad / \Rightarrow p_1 \supset p_1$$

have respectively the forms:

$$\frac{\Gamma_1 \Rightarrow \psi_1, \Delta_1 \qquad \Gamma_2, \psi_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \psi_1 \supset \psi_2 \Rightarrow \Delta_1, \Delta_2} \qquad\qquad \frac{}{\Rightarrow \psi \supset \psi}$$

In contrast, the usual rule for introducing implication on the right-hand side in intuitionistic logic is not a pure rule, since it allows only *left* context formulas.

Applications of rules are *multiplicative*—allowing a different context sequent in each premise. Since all usual structural rules are assumed, one may equivalently consider *additive* applications, that require one context sequent in all premises. We freely interchange multiplicative or additive applications in the rest of this paper, as they are equivalent, and each is technically convenient in different contexts. Note that we allow applications of pure rules to make use of subsequents of the premises, and not necessarily the full premises (i.e., by defining an application of a rule $s_1, \ldots, s_n \,/\, s$ to have the form $\sigma(s_1) \cup c_1, \ldots, \sigma(s_n) \cup c_n \,/\, \sigma(s) \cup c_1 \cup \ldots \cup c_n$). While this is technically convenient (e.g., in §2.4), again, using the structural rules, both options are equivalent.

Pure sequent calculi are finite sets of pure rules. To make them fully-structural (in addition to defining sequents as pairs of *sets*), the weakening rule, the identity axiom and the cut rule may be used in derivations. A *derivation* in a pure calculus **G** is defined as usual, where in addition to applications of the pure rules of **G**, the following standard application schemes may be used:[2]

$$(\textsc{weak})\ \frac{\Gamma \Rightarrow \Delta}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'} \qquad (\textsc{id})\ \frac{}{\Gamma, \psi \Rightarrow \psi, \Delta} \qquad (\textsc{cut})\ \frac{\Gamma_1 \Rightarrow \psi, \Delta_1 \qquad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Note that the structural rules (CUT), (WEAK) and (ID) can be simulated by pure rules: $\Rightarrow p_1 \,;\, p_1 \Rightarrow \,/\, \Rightarrow$ for (CUT), $\Rightarrow \,/\, \Rightarrow p_1$ and $\Rightarrow \,/\, p_1 \Rightarrow$ for (WEAK), and $/\, p_1 \Rightarrow p_1$ for (ID). However, it is technically convenient to distinguish them from the other pure rules.

Henceforth, unless stated otherwise, we consider only pure rules and pure calculi, and may refer to them simply as *rules* and *calculi*. By an $\mathcal{L}$-rule ($\mathcal{L}$-calculus) we mean a rule (calculus) that mentions only connectives of $\mathcal{L}$. For an $\mathcal{L}$-calculus **G**, a set $\mathcal{F} \subseteq \mathcal{L}$ of formulas, a set S of $\mathcal{F}$-sequents and an $\mathcal{F}$-sequent s, we write $S \vdash^{\mathcal{F}}_{\mathbf{G}} s$ if there is a derivation of s from S in **G** consisting only of $\mathcal{F}$-sequents. For $S \vdash^{\mathcal{L}}_{\mathbf{G}} s$ (i.e., $\mathcal{F} = \mathcal{L}$), we may also write $S \vdash_{\mathbf{G}} s$. As before, we often omit the curly braces, writing, e.g., $\Rightarrow p_1 \,;\, \Rightarrow p_2 \vdash^{\mathcal{F}}_{\mathbf{G}} \Rightarrow p_2$.

Next, we present several examples of pure sequent calculi. The most fundamental example is Gentzen's system for classical logic [25]:

Example 2.3 (Classical logic). The propositional fragment of Gentzen's sequent calculus for classical logic can be directly presented as the following pure calculus, denoted henceforth by **LK**:

$(\neg \Rightarrow)$	$\Rightarrow p_1 \,/\, \neg p_1 \Rightarrow$	$(\Rightarrow \neg)$	$p_1 \Rightarrow \,/\, \Rightarrow \neg p_1$
$(\wedge \Rightarrow)$	$p_1, p_2 \Rightarrow \,/\, p_1 \wedge p_2 \Rightarrow$	$(\Rightarrow \wedge)$	$\Rightarrow p_1 \,;\, \Rightarrow p_2 \,/\, \Rightarrow p_1 \wedge p_2$
$(\vee \Rightarrow)$	$p_1 \Rightarrow \,;\, p_2 \Rightarrow \,/\, p_1 \vee p_2 \Rightarrow$	$(\Rightarrow \vee)$	$\Rightarrow p_1, p_2 \,/\, \Rightarrow p_1 \vee p_2$
$(\supset \Rightarrow)$	$\Rightarrow p_1 \,;\, p_2 \Rightarrow \,/\, p_1 \supset p_2 \Rightarrow$	$(\Rightarrow \supset)$	$p_1 \Rightarrow p_2 \,/\, \Rightarrow p_1 \supset p_2$
$(\bot \Rightarrow)$	$/\, \bot \Rightarrow$	$(\Rightarrow \top)$	$/\, \Rightarrow \top$

Besides **LK** there are many sequent calculi for non-classical logics that fall in this framework. These include calculi for well-known three and four-valued logics, various calculi for paraconsistent logics, and all canonical and quasi-canonical sequent calculi [8–10, 12].

[2] By defining sequents as pairs of *sets* (rather than lists or multisets) we implicitly include the schemes of exchange and contraction in our calculi.

Example 2.4 (Primal infon logic). Primal infon logic [19] was designed to efficiently reason about access control policies, by taking much weaker disjunction and implication, but still expressive enough to describe access control policies. The quotations-free fragment of its sequent calculus [11] can be presented as a pure calculus, which we denote by **P**. It is obtained from the negation–free fragment of **LK** by dismissing the rule $(\vee \Rightarrow)$, and replacing the rule $(\Rightarrow \supset)$ with: $\Rightarrow p_2 \;/\; \Rightarrow p_1 \supset p_2$. Quotations, whose rules are not pure, can be seen as modal operators, and are handled in §6 (in particular, see Example 6.15).

Example 2.5 (The paraconsistent logic C1). The calculus for da Costa's historical paraconsistent logic C1 from [8] is a pure calculus, which we call $\mathbf{C_1}$. It consists of the rules of **LK** except for $(\neg \Rightarrow)$ that is replaced by the following rules:

$$p_1 \Rightarrow \;/\; \neg\neg p_1 \Rightarrow$$

$$\Rightarrow p_1 \,;\, \Rightarrow \neg p_1 \;/\; \neg(p_1 \wedge \neg p_1) \Rightarrow \qquad \neg p_1 \Rightarrow \;;\, \neg p_2 \Rightarrow \;/\; \neg(p_1 \wedge p_2) \Rightarrow$$

$$\neg p_1 \Rightarrow \;;\, p_2, \neg p_2 \Rightarrow \;/\; \neg(p_1 \vee p_2) \Rightarrow \qquad p_1, \neg p_1 \Rightarrow \;;\, \neg p_2 \Rightarrow \;/\; \neg(p_1 \vee p_2) \Rightarrow$$

$$p_1 \Rightarrow \;;\, p_2, \neg p_2 \Rightarrow \;/\; \neg(p_1 \supset p_2) \Rightarrow \qquad p_1, \neg p_1 \Rightarrow \;;\, \neg p_2 \Rightarrow \;/\; \neg(p_1 \supset p_2) \Rightarrow$$

Example 2.6 (Łukasiewicz three-valued logic). A sequent calculus for Łukasiewicz three-valued logic was presented in [5]. This calculus, which we call **Ł3**, can be directly presented as a pure calculus. For example, the rules involving implication are the following:

$$\neg p_1 \Rightarrow \;;\, p_2 \Rightarrow \;;\; \Rightarrow p_1, \neg p_2 \;/\; p_1 \supset p_2 \Rightarrow \qquad p_1 \Rightarrow p_2 \,;\, \neg p_2 \Rightarrow \neg p_1 \;/\; \Rightarrow p_1 \supset p_2$$

$$p_1, \neg p_2 \Rightarrow \;/\; \neg(p_1 \supset p_2) \Rightarrow \qquad \Rightarrow p_1 \,;\, \Rightarrow \neg p_2 \;/\; \Rightarrow \neg(p_1 \supset p_2)$$

Next, we present a useful lemma that establishes structural properties of pure sequent calculi.

Lemma 2.7. *If* $S \vdash^{\mathcal{F}}_{\mathbf{G}} s$, *then:*

(1) $\sigma(S) \vdash^{\sigma(\mathcal{F})}_{\mathbf{G}} \sigma(s)$ *for every substitution* σ.

(2) $\{s' \cup c \mid s' \in S\} \vdash^{\mathcal{F} \cup frm(c)}_{\mathbf{G}} s \cup c$ *for every sequent* c.

Proof. By induction on the length of derivations. □

2.3 Semantics

In this section we introduce a semantic interpretation of pure calculi, based on (possibly non-deterministic) two-valued valuation functions. This semantics will be used to characterize analyticity in pure calculi and provide a decision procedure for analytic pure calculi.

Our semantics follows [12] and uses *bivaluations*—functions assigning a binary truth value to each formula. The simple framework of bivaluations is applicable to a wide variety of propositional logics. The price for this simplicity and generality is the loss of truth-functionality: the truth value assigned to a compound formula is not always uniquely determined by the truth values assigned to its subformulas. Accordingly, it does not suffice to define bivaluations over atomic formulas, as done in ordinary truth-tables semantics.

For the purpose of characterizing analyticity, we extend the bivaluation framework by considering also *partial* bivaluations that assign truth values to *some* formulas. These allows us to have finite models which are essential in semantic decision procedures. Next, we precisely define (partial) bivaluations, and provide a general soundness and completeness theorem, supplying each pure calculus **G** and a set $\mathcal{F}$ of formulas with a set of partial bivaluations for which **G** is sound and complete when only $\mathcal{F}$-formulas may appear in derivations.

Definition 2.8. A *bivaluation* is a function v from some set of propositional formulas, denoted $dom(v)$, to $\{0, 1\}$. A bivaluation v is extended to $dom(v)$-sequents by: $v(\Gamma \Rightarrow \Delta) = 1$ iff $v(\varphi) = 0$ for some $\varphi \in \Gamma$ or $v(\varphi) = 1$ for some $\varphi \in \Delta$. A bivaluation v is extended to sets (of $dom(v)$-formulas or of sequents) by $v(X) = \min\{v(x) \mid x \in X\}$, where $\min \emptyset = 1$. Given a set $\mathcal{F}$ of formulas, by an *$\mathcal{F}$-bivaluation* we refer to a bivaluation v with $dom(v) = \mathcal{F}$.

To relate sequent calculi and bivaluations, we simply read pure rules as semantic constraints on bivaluations. This is formally defined as follows:

Definition 2.9. A bivaluation v *respects* a rule $s_1, \dots, s_n \,/\, s$ if $v(\{\sigma(s'_1), \dots, \sigma(s'_n)\}) \leq v(\sigma(s))$ for every subsequents $s'_1 \subseteq s_1, \dots, s'_n \subseteq s_n$ and substitution σ such that $\sigma(frm(\{s'_1, \dots, s'_n, s\})) \subseteq dom(v)$. A bivaluation v is called **G**-*legal* for a calculus **G** if it respects all rules of **G**.

Example 2.10. A $\{p_1, \neg\neg p_1\}$-bivaluation v respects the rule $p_1 \Rightarrow \;/\; \neg\neg p_1 \Rightarrow$ iff either $v(p_1) = v(\neg\neg p_1) = 0$ or $v(p_1) = 1$. A $\{p_1, p_1 \vee p_2\}$-bivaluation v respects the rule $\Rightarrow p_1, p_2 \,/\, \Rightarrow p_1 \vee p_2$ iff either $v(p_1) = 0$ or $v(p_1 \vee p_2) = 1$ (note that $p_2 \notin dom(v)$). It is easy to verify that **LK**-legal $\mathcal{L}$-bivaluations (where $\mathcal{L}$ is the language of **LK**) coincide with the well-known classical valuations.

Next, we prove a general soundness and completeness theorem, that ties the domain of bivaluations to the set of formulas that are allowed to appear in derivations.

Theorem 2.11 (Soundness and Completeness). *$S \vdash^{\mathcal{F}}_{\mathbf{G}} s$ iff $v(S) \leq v(s)$ for every* **G**-*legal $\mathcal{F}$-bivaluation v.*

Proof. To prove soundness, let v be a **G**-legal $\mathcal{F}$-bivaluation, such that $v(S) = 1$. We prove that $v(s) = 1$ by induction on the length of the derivation of s from S in **G** (which consists only of $\mathcal{F}$-sequents). If $s \in S$, or s is the conclusion of an application of (id), (cut), or (weak), then this is straightforward. If s is the conclusion of an application of some rule $s_1, \dots, s_n \,/\, s_0 \in \mathbf{G}$, then there are subsequents $s'_1 \subseteq s_1, \dots, s'_n \subseteq s_n$, a substitution σ, and $\mathcal{F}$-sequents $c_1, \dots, c_n$ such that $s = \sigma(s_0) \cup c_1 \cup \dots \cup c_n$, $\sigma(frm(\{s'_1, \dots, s'_n, s_0\})) \subseteq \mathcal{F}$, and $S \vdash^{\mathcal{F}}_{\mathbf{G}} \sigma(s'_i) \cup c_i$ for every $1 \leq i \leq n$. By the induction hypothesis, $v(\sigma(s'_i) \cup c_i) = 1$ for every $1 \leq i \leq n$. If $v(c_i) = 1$ for some $1 \leq i \leq n$, then $v(\sigma(s_0) \cup c_1 \cup \dots \cup c_n) = 1$. Otherwise, for every $1 \leq i \leq n$, $v(\sigma(s'_i)) = 1$. Since v is **G**-legal, $v(\sigma(s_0)) = 1$ and hence $v(\sigma(s_0) \cup c_1 \cup \dots \cup c_n) = 1$.

To prove completeness, assume that $S \nvdash^{\mathcal{F}}_{\mathbf{G}} s$. We construct a **G**-legal $\mathcal{F}$-bivaluation v such that $v(S) = 1$ and $v(s) = 0$. Since $\mathcal{F}$ may be infinite, this construction requires the following generalization of sequents: An *ω-sequent* is a pair $\langle L, R \rangle$, denoted $L \Rightarrow R$, where L and R are (possibly infinite) subsets of $\mathcal{F}$. We write $S \vdash^{\mathcal{F}}_{\mathbf{G}} L \Rightarrow R$ if $S \vdash^{\mathcal{F}}_{\mathbf{G}} \Gamma \Rightarrow \Delta$ for some finite $\Gamma \subseteq L$ and $\Delta \subseteq R$. Other definitions and notations for sequents are adapted for ω-sequents in the obvious way.

Call an ω-sequent $L \Rightarrow R$ *maximal unprovable* if the following hold:

- $L \cup R \subseteq \mathcal{F}$
- $S \nvdash^{\mathcal{F}}_{\mathbf{G}} L \Rightarrow R$
- $S \vdash^{\mathcal{F}}_{\mathbf{G}} L, \varphi \Rightarrow R$ for every $\varphi \in \mathcal{F} \setminus L$
- $S \vdash^{\mathcal{F}}_{\mathbf{G}} L \Rightarrow \varphi, R$ for every $\varphi \in \mathcal{F} \setminus R$

It is routine to extend s to a maximal unprovable ω-sequent $L \Rightarrow R$. Using (cut), it can be easily shown that $L \cup R = \mathcal{F}$. Then, a countermodel v is defined by $v(\varphi) = 1$ if $\varphi \in L$, and $v(\varphi) = 0$ if $\varphi \in R$. Clearly, $v(S) = 1$ and $v(s) = 0$. It remains to show that v is **G**-legal. Let $r = \Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_n \Rightarrow \Delta_n \,/\, \Gamma_0 \Rightarrow \Delta_0$ be a rule of **G**, $\Gamma'_1 \Rightarrow \Delta'_1, \dots, \Gamma'_n \Rightarrow \Delta'_n$ respective subsequents of $\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_n \Rightarrow \Delta_n$, and σ a substitution, such that $\sigma(frm(\{\Gamma'_1 \Rightarrow \Delta'_1, \dots, \Gamma'_n \Rightarrow \Delta'_n, \Gamma_0 \Rightarrow \Delta_0\})) \subseteq \mathcal{F}$ and $v(\sigma(\Gamma'_i \Rightarrow \Delta'_i)) = 1$ for every $1 \leq i \leq n$. We prove that $v(\sigma(\Gamma_0 \Rightarrow \Delta_0)) = 1$. By our assumption, for every $1 \leq i \leq n$, there exists either $\varphi \in \Gamma'_i$ such that $v(\sigma(\varphi)) = 0$ (and then $\sigma(\varphi) \in R$) or $\varphi \in \Delta'_i$ such that $v(\sigma(\varphi)) = 1$ (and then $\sigma(\varphi) \in L$). We construct a sequent $\Gamma \Rightarrow \Delta$ as follows. For every $1 \leq i \leq n$, we include in Γ a formula $\sigma(\varphi)$ for some $\varphi \in \Delta'_i$ such that $v(\sigma(\varphi)) = 1$, or, if such φ does not exist, we include in Δ a formula $\sigma(\varphi)$ for some $\varphi \in \Gamma_i'$ such that $v(\sigma(\varphi)) = 0$. Then, we have $(\Gamma \Rightarrow \Delta) \subseteq (L \Rightarrow R)$. In addition, using (ID), we have $S \vdash_{\mathbf{G}}^{\mathcal{F}} \sigma(\Gamma_i'), \Gamma \Rightarrow \sigma(\Delta_i'), \Delta$ for every $1 \leq i \leq n$. By applying the rule r with $\Gamma \Rightarrow \Delta$ as a context sequent, we obtain that $S \vdash_{\mathbf{G}}^{\mathcal{F}} \sigma(\Gamma_0), \Gamma \Rightarrow \sigma(\Delta_0), \Delta$. Since $S \nvdash_{\mathbf{G}}^{\mathcal{F}} L \Rightarrow R$, we have $\sigma(\Gamma_0 \Rightarrow \Delta_0) \nsubseteq L \Rightarrow R$, and so either $v(\psi) = 0$ for some $\psi \in \sigma(\Gamma_0)$ or $v(\psi) = 1$ for some $\psi \in \sigma(\Delta_0)$. Either way, we have $v(\sigma(\Gamma_0 \Rightarrow \Delta_0)) = 1$. □

Well-known soundness and completeness theorems from the literature can be obtained as particular instances of Thm. 2.11, by taking $\mathcal{F}$ to be the entire propositional language. Examples include, e.g., soundness and completeness of **LK** with respect to the classical truth tables, and soundness and completeness of **P** (Example 2.4) with respect to the non-deterministic semantics from [19].

2.4 Streamlining Pure Calculi

In many cases, two calculi allow for exactly the same sequents to be derived, although they employ different derivation rules. In this section we present several useful streamlining transformations that transform one calculus into another, without affecting the induced derivability relation.

Definition 2.12 (Equivalent calculi and rules). Two calculi $\mathbf{G}_1$ and $\mathbf{G}_2$ are called *equivalent* if $\vdash_{\mathbf{G}_1}^{\mathcal{F}} = \vdash_{\mathbf{G}_2}^{\mathcal{F}}$ for every set $\mathcal{F}$ of formulas. Equivalence is naturally defined also between single rules (and between a rule and a calculi) by identifying a rule r with the calculus $\{r\}$.

Lemma 2.13 (Basic equivalences). *The following hold:*

(1) *$S \,/\, \Gamma \Rightarrow \psi, \Delta$ is equivalent to $S\,;\psi \Rightarrow \,/\, \Gamma \Rightarrow \Delta$.*
(2) *$S \,/\, \Gamma, \psi \Rightarrow \Delta$ is equivalent to $S\,; \Rightarrow \psi \,/\, \Gamma \Rightarrow \Delta$.*
(3) *$\{S\,; s_1 \,/\, s\ ,\ S\,; s_2 \,/\, s\}$ is equivalent to $S\,; s_1 \cup s_2 \,/\, s$.*

Proof. All claims are handled similarly. We show only the left-to-right direction of the third claim. Using Thm. 2.11, it suffices to show that every bivaluation that respects the rule $S\,; s_1 \cup s_2 \,/\, s$ also respects the rules $S; s_1 \,/\, s$ and $S; s_2 \,/\, s$. Let v be a bivaluation that respects $S; s_1 \cup s_2 \,/\, s$. We prove, w.l.o.g., that it respects the rule $S\,; s_1 \,/\, s$. Let $S = \{q_1, \dots, q_n\}$. Let $q_1' \subseteq q_1, \dots, q_n' \subseteq q_n$, and let s' be a subsequent of s_1, and σ a substitution such that $\sigma(\mathit{frm}\{q_1', \dots, q_n', s', s\}) \subseteq \mathit{dom}(v)$. Suppose that $v(\sigma(q_i')) = 1$ for every $1 \leq i \leq n$ and that $v(\sigma(s')) = 1$. Clearly, $s' \subseteq s_1 \cup s_2$. Since v respects $S\,; s_1 \cup s_2 \,/\, s$, we have that $v(\sigma(s)) = 1$. □

Note that the use of subsequents in applications of pure rules is essential for the poof of Lemma 2.13.

We call a rule *axiomatic* if it has an empty set of premises. In turn, call a calculus *axiomatic* if it consists solely of axiomatic rules (the non-axiomatic schemes (WEAK) and (CUT) are allowed). We point out a useful application of Lemma 2.13, which allows us to convert every calculus to an axiomatic one. In particular, it will allow us to consider only axiomatic calculi in §4 and §7.

Example 2.14. The rule $\Rightarrow p_1\,;\ \Rightarrow p_2\ /\ \Rightarrow p_1 \wedge p_2$ of **LK** is equivalent to the axiomatic rule $\emptyset \,/\, p_1, p_2 \Rightarrow p_1 \wedge p_2$, and the rule $\neg p_1 \Rightarrow\ ; p_2 \Rightarrow\ ;\ \Rightarrow p_1, \neg p_2 \,/\, p_1 \supset p_2 \Rightarrow$ of **Ł3** (Example 2.6) is equivalent to the axiomatic rules $\emptyset \,/\, p_1, p_1 \supset p_2 \Rightarrow \neg p_1, p_2$ and $\emptyset \,/\, \neg p_2, p_1 \supset p_2 \Rightarrow \neg p_1, p_2$.

Theorem 2.15. *Every calculus is equivalent to an axiomatic calculus.*

Proof. Without loss of generality, we assume that no rule of **G** includes an empty premise. If such a rule exists, it can be simply omitted. Consider the following transformations of pure rules (all of them are instances of the equivalences in Lemma 2.13):

(1) $S\,; \psi \Rightarrow\ /\, \Gamma \Rightarrow \Delta \longmapsto S \,/\, \Gamma \Rightarrow \psi, \Delta$

(2) $S\,;\ \Rightarrow \psi \,/\, \Gamma \Rightarrow \Delta \longmapsto S \,/\, \Gamma, \psi \Rightarrow \Delta$
(3) $S\,; \Gamma, \psi \Rightarrow \Delta \,/\, s \longmapsto \{S\,; \Gamma \Rightarrow \Delta \,/\, s\,,\ S\,; \psi \Rightarrow\ \,/\, s\}$ for $\Gamma \cup \Delta \neq \emptyset$ and $\psi \notin \Delta$
(4) $S\,; \Gamma \Rightarrow \psi, \Delta \,/\, s \longmapsto \{S\,; \Gamma \Rightarrow \Delta \,/\, s\,,\ S\,;\ \Rightarrow \psi \,/\, s\}$ for $\Gamma \cup \Delta \neq \emptyset$ and $\psi \notin \Gamma$

Given a calculus **G**, we apply these four transformations on the rules of **G** as long as it is possible. By Lemma 2.13, each step in this process results in a calculus which is equivalent to **G**. Observing that at least one transformation is applicable to any non-axiomatic rule, it remains to establish termination. For each rule $S \,/\, s$, let $\|S \,/\, s\| = \sum_{\Gamma \Rightarrow \Delta \in S}(|\Gamma| + |\Delta|)$. For every set R of rules, we associate the multiset M_R, given by $M_R = \lambda n \in \mathbb{N}.\ |\{r \in R \mid \|r\| = n\}|$. We prove that if R_2 is obtained from R_1 by one of the transformations, then $M_{R_2} \prec M_{R_1}$, where $\prec$ is the Dershowitz-Manna well founded ordering over multisets of natural numbers [21]. Clearly, $R_2 = (R_1 \setminus \{r\}) \cup R$ for some set R that is obtained from r by one of the transformations. If the transformation is 1 or 2, then, w.l.o.g., r has the form $S \uplus \{\psi \Rightarrow\ \} \,/\, \Gamma \Rightarrow \Delta$ and R has the form $\{S \,/\, \Gamma \Rightarrow \psi, \Delta\}$. This means that M_{R_2} is obtained from M_{R_1} by replacing one copy of $\|S\,; \psi \Rightarrow\ \,/\, \Gamma \Rightarrow \Delta\|$ with a new copy of $\|S\,; \psi \Rightarrow\ \,/\, \Gamma \Rightarrow \Delta\| - 1$. If the transformation is 3 or 4, then, w.l.o.g., r has the form $S \uplus \{\Gamma \Rightarrow \psi, \Delta\} \,/\, s$ where $\psi \notin \Delta$ and $\Gamma \cup \Delta \neq \emptyset$, and R has the form $\{S\,; \Gamma \Rightarrow \Delta \,/\, s, S\,;\ \Rightarrow \psi \,/\, s\}$. This means that M_{R_2} is obtained from M_{R_1} by replacing a copy of $\|S\,; \Gamma \Rightarrow \psi, \Delta \,/\, s\|$ with a copy of $\|S\,; \Gamma \Rightarrow \Delta \,/\, s\|$ and a copy of $\|S\,;\ \Rightarrow \psi \,/\, s\|$. Both are smaller than $\|S\,; \Gamma \Rightarrow \psi, \Delta \,/\, s\|$. □

Transformations 3 and 4 in the proof of Theorem 2.15 replace one rule by two rules, and thus, translating a calculus into an equivalent axiomatic calculus may require exponential time.

Using the rewriting rules in the proof of Theorem 2.15, written as pure rules, (CUT) can be translated into (ID). This, however, has nothing to do with cut-elimination, as our notion of equivalence (Definition 2.12) allows the use of (CUT).

3 ANALYTICITY

Analyticity is a crucial property of proof systems. In the case of fully-structural propositional sequent calculi, analyticity often implies their decidability and consistency (the fact that the empty sequent is not derivable). Roughly speaking, a calculus is analytic if whenever a sequent s is derivable in it from a set S of sequents, s can be proven using only the "syntactic material available inside $S \cup \{s\}$". This "material" is usually taken to consist of all subformulas occurring in $S \cup \{s\}$, and then analyticity amounts to the subformula property. However, weaker restrictions on the formulas that are allowed to appear in derivations of a given sequent may also suffice for decidability and consistency. For example, in $\mathbf{C_1}$ and **Ł3** (Examples 2.5 and 2.6), there are sequents whose derivations require not only subformulas, but also of negations of subformulas of the derived sequent.

In this section we provide a generalized definition of analyticity, which is parametrized by a distinguished set of unary connectives and a natural number. This generalized notion holds for a larger family of calculi and still suffices to ensure decidability and consistency. We then equip this definition with a semantic characterization, which, in addition to providing another viewpoint of (generalized) analyticity, is our main tool for proving this property.

In what follows, $\circledcirc$ denotes an arbitrary subset of unary connectives in $\diamond^1_{\mathcal{L}}$ and k denotes an arbitrary positive integer. We denote the set of strings over $\circledcirc$ of length at most k by $\circledcirc^{\leq k}$ (e.g., $\{\neg, \circ\}^{\leq 2} = \{\epsilon, \neg, \circ, \neg\neg, \circ\circ, \neg\circ, \circ\neg\}$, where ϵ denotes the empty string). For convenience, we use the following notations: for a unary connective $\circ$ and a set $\mathcal{F}$ of formulas, $\circ\mathcal{F} = \{\circ\varphi \mid \varphi \in \mathcal{F}\}$, and for a set $\circledcirc$ of unary connectives, $\circledcirc\varphi = \{\circ\varphi \mid \circ \in \circledcirc\}$, $\circledcirc\mathcal{F} = \bigcup_{\circ \in \circledcirc} \circ\mathcal{F}$, $\circledcirc^{\leq k}\varphi = \{\bar{\circ}\varphi \mid \bar{\circ} \in \circledcirc^{\leq k}\}$, and $\circledcirc^{\leq k}\mathcal{F} = \bigcup_{\varphi \in \mathcal{F}} \circledcirc^{\leq k}\varphi$.

Definition 3.1. A formula φ is an *immediate* $\circledcirc$-k-*subformula* of a formula ψ if either $\psi \in \circledcirc\varphi$, or $\psi = \diamond(\psi_1, \ldots, \psi_n)$ and $\varphi \in \circledcirc^{\leq k}\psi_i$ for some n-ary connective $\diamond \notin \circledcirc$, formulas $\psi_1, \ldots, \psi_n$, and

$1 \leq i \leq n$. The $\circledcirc$-k-*subformula* relation is the reflexive transitive closure of the immediate $\circledcirc$-k-subformula relation. We denote the set of $\circledcirc$-k-subformulas of a formula ψ by $sub_k^{\circledcirc}(\psi)$. This notation is naturally extended to sequents, sets of sequents, etc.

Intuitively, $\circledcirc$-k-subformulas of formulas whose main connective is not in $\circledcirc$ are obtained by prefixing ordinary subformulas with a sequence of $\circledcirc$-elements of length $\leq k$. For formulas whose main connective is in $\circledcirc$, we only take usual subformulas (until we reach a connective outside of $\circledcirc$).

Note that $\circledcirc = \emptyset$ (and so $\circledcirc^{\leq k} = \{\epsilon\}$ for any k), the $\circledcirc$-k-subformula relation amounts to the usual subformula relation. In this case we call φ a *subformula* of ψ.

Example 3.2.

$$sub_1^{\{\neg\}}(\neg(p_1 \supset p_2)) = \{p_1, p_2, \neg p_1, \neg p_2, p_1 \supset p_2, \neg(p_1 \supset p_2)\}$$
$$sub_2^{\{\neg\}}(\circ p_1) = \{p_1, \neg p_1, \neg\neg p_1, \circ p_1\}$$
$$sub_2^{\{\neg,\circ\}}(\circ p_1) = \{p_1, \circ p_1\}$$

Before defining analyticity, we study the properties of this generalized subformula relation. The first step is to define an adequate complexity measure cc on formulas. For every $\psi \in \mathcal{L}$, denote by $\bar{\circ}_\psi$ the longest (possibly empty) prefix of ψ consisting of $\circledcirc$-elements, and by b_ψ the formula in $\mathcal{L} \setminus \circledcirc\mathcal{L}$ for which $\psi = \bar{\circ}_\psi b_\psi$. Let $c : \mathcal{L} \to \mathbb{N}$ be a usual complexity measure on formulas (so that $c(\varphi) < c(\psi)$ whenever φ is a proper subformula of ψ). The function $cc : \mathcal{L} \to (\mathbb{N} \times \mathbb{N})$ is then given by $cc(\psi) = \langle c(b_\psi), |\bar{\circ}_\psi| \rangle$, where $|\bar{\circ}_\psi|$ denotes the length of $\bar{\circ}_\psi$.

Proposition 3.3. *$cc(\varphi) < cc(\psi)$ whenever φ is a proper $\circledcirc$-k-subformula of ψ (where $<$ is the standard lexicographic order over $\mathbb{N} \times \mathbb{N}$).*

Proof. We consider the case that φ is an immediate $\circledcirc$-k-subformula of ψ. The claim then follows by standard induction. First, if $\psi = \diamond(\psi_1, \dots, \psi_n)$ and $\varphi \in \circledcirc^{\leq k}\psi_i$ for some $1 \leq i \leq n$ and $\diamond \notin \circledcirc$, then $c(b_\varphi) = c(b_{\psi_i}) \leq c(\psi_i) < c(\psi) = c(b_\psi)$, and so $cc(\varphi) < cc(\psi)$. Second, if $\psi = \circ\varphi$ for $\circ \in \circledcirc$, then $\bar{\circ}_\psi = \circ\bar{\circ}_\varphi$, and $b_\psi = b_\varphi$. Hence, $c(b_\psi) = c(b_\varphi)$, but $|\bar{\circ}_\psi| = |\bar{\circ}_\varphi| + 1$, and so $cc(\varphi) < cc(\psi)$. □

Using this complexity measure, it easily follows that the $\circledcirc$-k-subformula relation is anti-symmetric. Since every formula has finitely many immediate $\circledcirc$-k-subformulas, it also follows (by König's lemma) that $sub_k^{\circledcirc}(\psi)$ is finite for every $\psi \in \mathcal{L}$.

In addition, we have the following useful property of the generalized relation:

Lemma 3.4. *$\sigma(sub_k^{\circledcirc}(\psi)) \subseteq sub_k^{\circledcirc}(\sigma(\psi))$ for every formula ψ and substitution σ.*

Next, we define our generalized notion of analyticity.

Definition 3.5 (Analyticity). A calculus $\mathbf{G}$ is called *$\circledcirc$-k-analytic* if $S \vdash_{\mathbf{G}} s$ implies $S \vdash_{\mathbf{G}}^{sub_k^{\circledcirc}(S \cup \{s\})} s$ for every set S of sequents and a sequent s.

Just like the usual subformula property, $\circledcirc$-k-analyticity of a pure calculus entails its decidability.[3] Formally:

Definition 3.6. The *derivability problem* for an $\mathcal{L}$-calculus $\mathbf{G}$ is given by:
Input: A finite set S of $\mathcal{L}$-sequents and an $\mathcal{L}$-sequent s.
Question: Does $S \vdash_{\mathbf{G}} s$?

[3]Obviously, one cannot expect to have decision procedures for derivability in every pure calculus. Indeed, any Hilbert calculus H (without side conditions on rule applications) can be translated to a pure sequent calculus $\mathbf{G}_H$, by taking a rule of the form $\Rightarrow \psi_1 ; \dots ; \Rightarrow \psi_n \,/\, \Rightarrow \psi$ for each Hilbert-style derivation rule that derives ψ from $\psi_1, \dots, \psi_n$ (where $n = 0$ for axioms). It is easy to show that ψ is derivable from Γ in H iff $\vdash_{\mathbf{G}_H} \Gamma \Rightarrow \psi$.

PROPOSITION 3.7. *The derivability problem is decidable for every $\circledcirc$-k-analytic pure calculus.*

PROOF. For every $\circledcirc$-k-analytic calculus **G**, finite set S of sequents, and sequent s, we have that $S \vdash_{\mathbf{G}} s$ iff $S \vdash_{\mathbf{G}}^{\mathcal{F}} s$ for $\mathcal{F} = sub_k^{\circledcirc}(S \cup \{s\})$. Since $\mathcal{F}$ is finite, the latter can be checked by an exhaustive search for derivations of s from S in **G** that include only $\mathcal{F}$-formulas. □

Moreover, $\circledcirc$-k-analyticity guarantees the consistency of the calculus provided that the calculus is not trivial:

PROPOSITION 3.8. *The empty sequent is not derivable in any $\circledcirc$-k-analytic calculus that does not include the rule $\emptyset$ / $\Rightarrow$.*

PROOF. A proof of the empty sequent in a $\circledcirc$-k-analytic calculus would entail the existence of a proof that includes no formulas at all. This is only possible in the presence of the rule $\emptyset$ / $\Rightarrow$. □

$\emptyset$-k-analytic calculi are calculi that enjoy the usual subformula property. We call such calculi simply *analytic*. Note that whenever two calculi are equivalent (as defined in Def. 2.12), then one is $\circledcirc$-k-analytic iff the other is.

Analyticity of a given calculus is traditionally proved as a corollary of cut-admissibility. Indeed, if all rules in a pure calculus (except for (CUT)) admit the local subformula property (i.e., the premises of each rule consist only of subformulas of the formulas its conclusion), then cut-admissibility implies analyticity. This argument can be easily generalized for $\circledcirc$-k-analyticity. For example, the calculi **LK**, **P**, $\mathbf{C_1}$ and **Ł3** (Examples 2.3 to 2.6) admit cut-elimination. Taking into account the structure of their rules, this entails that **LK** and **P** are analytic, and that $\mathbf{C_1}$ and **Ł3** are $\{\neg\}$-1-analytic.

There are cases, however, in which a sequent calculus does not enjoy cut-admissibility, although it is analytic. Examples include, e.g., sequent calculi for the modal logics $S5$ and B [41, 44], bi-intuitionistic logic [40], and several calculi for paraconsistent logics [6]. Other methods for proving $\circledcirc$-k-analyticity (independent of cut-admissibility) are thus needed.

Next, we provide a semantic characterization of analyticity that is independent of cut-admissibility. Roughly speaking, to apply this criterion, one has to consider *partial* bivaluations and show that the existence of a countermodel in the form of such a partial bivaluation entails the existence of an (infinite) full countermodel.

THEOREM 3.9. *An $\mathcal{L}$-calculus* **G** *is $\circledcirc$-k-analytic iff every* **G***-legal bivaluation v can be extended to a* **G***-legal $\mathcal{L}$-bivaluation, provided that $dom(v)$ is finite and closed under $\circledcirc$-k-subformulas.*

PROOF. Suppose that $S \vdash_{\mathbf{G}} s$ but $S \nvdash_{\mathbf{G}}^{\mathcal{F}} s$ for $\mathcal{F} = sub_k^{\circledcirc}(S \cup \{s\})$. By Thm. 2.11, there exists a **G**-legal $\mathcal{F}$-bivaluation v such that $v(S) = 1$ and $v(s) = 0$, but $u(S) \leq u(s)$ for every **G**-legal $\mathcal{L}$-bivaluation u. Therefore, v cannot be extended to a **G**-legal $\mathcal{L}$-bivaluation. In addition, $dom(v) = \mathcal{F}$ is finite and closed under $\circledcirc$-k-subformulas.

For the converse, suppose that v is a **G**-legal bivaluation, $dom(v)$ is finite and closed under $\circledcirc$-k-subformulas, and v cannot be extended to a **G**-legal $\mathcal{L}$-bivaluation. Let $s = \Gamma \Rightarrow \Delta$, where $\Gamma = \{\psi \in dom(v) \mid v(\psi) = 1\}$ and $\Delta = \{\psi \in dom(v) \mid v(\psi) = 0\}$. Then, $dom(v) = frm(s) = sub_k^{\circledcirc}(s)$ and $v(s) = 0$. We show that $u(s) = 1$ for every **G**-legal $\mathcal{L}$-bivaluation u. Indeed, every such u does not extend v, and so $u(\psi) \neq v(\psi)$ for some $\psi \in dom(v)$. Then, $u(\psi) = 0$ if $\psi \in \Gamma$, and $u(\psi) = 1$ if $\psi \in \Delta$. In either case, $u(s) = 1$. By Thm. 2.11, $\nvdash_{\mathbf{G}}^{sub_k^{\circledcirc}(s)} s$ and $\vdash_{\mathbf{G}} s$. □

Notice that the proof of Theorem 3.9 does not rely on any particular property of the $sub_k^{\circledcirc}$ operator, except for the facts that $sub_k^{\circledcirc}(\varphi)$ is always finite and $\varphi \in sub_k^{\circledcirc}(\varphi)$.

Often, a slightly weaker notion of analyticity is employed, by considering only cases where $S = \emptyset$. We say that a calculus **G** is *weakly $\circledcirc$-k-analytic* if $\vdash_{\mathbf{G}} s$ implies $\vdash_{\mathbf{G}}^{sub_k^{\circledcirc}(s)} s$ for every sequent

s. The proof of Thm. 3.9 shows that this seemingly weaker notion is actually equivalent to the stronger one for pure calculi. Indeed, the second direction of the proof actually shows that if **G** is *weakly* $\circledcirc$-k-analytic then every **G**-legal bivaluation v can be extended to a **G**-legal $\mathcal{L}$-bivaluation, provided that $dom(v)$ is finite and closed under $\circledcirc$-k-subformulas.[4]

Example 3.10. Consider a calculus **G** consisting of the following rules:

$$p_1 \Rightarrow \ / \circ p_1 \Rightarrow \qquad\qquad p_1 \Rightarrow \ / \ \Rightarrow \circ p_1$$

The **G**-legal bivaluation v defined by $dom(v) = \{p_1\}$ and $v(p_1) = 0$ cannot be extended to a full **G**-legal bivaluation: the first rule forces $v(\circ p_1) = 0$, while the second requires $v(\circ p_1) = 1$. Indeed, **G** is not analytic, as the sequent $\Rightarrow p_1$ is derivable in it, but only using (a cut on) the formula $\circ p_1$.

In the next section (§4), we show that $\circledcirc$-k-analyticity also allows for a uniform SAT-based decision procedure. Section 5 studies methods for constructing $\circledcirc$-k-analytic calculi.

4 SAT-BASED DECISION PROCEDURE

As shown in §3, the derivability problem of a given calculus is decidable whenever the calculus is $\circledcirc$-k-analytic for some $\circledcirc$ and k. However, the mere decidability of this problem does not provide an efficient decision procedure. A great deal of ingenuity is often required for developing efficient proof-search algorithms for sequent calculi (see, e.g., [20]).

In this section we show that for $\circledcirc$-k-analytic pure calculi it is possible to replace proof-search by SAT solving. This is done using a polynomial-time reduction of the derivability problem to the complement of SAT. While SAT is NP-complete, it is considered "easy" when it comes to real-world applications. Indeed, there are many off-the-shelf SAT solvers, that, despite an exponential worst-case time complexity, are considered extremely efficient (see, e.g., [27]). Indeed, our implementation of the reduction, once integrated with a modern SAT solver has good performance.

To achieve the above, we utilize the semantic view of pure sequent calculi (see §2.3), that allows us to reduce the derivability problem in a given analytic sequent calculus to small countermodel search, which can be be easily given in terms of a SAT instance. We start by precisely defining the reduction, proceed by proving its correctness, and its polynomial time complexity. Then we briefly describe the implementation.

SAT instances are taken to be CNFs represented as sets of clauses, where clauses are sets of literals (that is, atomic variables and their negations, denoted by overlines). The set $\{x_\psi \mid \psi \in \mathcal{L}\}$ is used as the set of atomic variables in the SAT instances. The translation of sequents to SAT instances is naturally given by:

$$\mathsf{SAT}^+(\Gamma \Rightarrow \Delta) = \{\{\overline{x_\psi} \mid \psi \in \Gamma\} \cup \{x_\psi \mid \psi \in \Delta\}\}$$
$$\mathsf{SAT}^-(\Gamma \Rightarrow \Delta) = \{\{x_\psi\} \mid \psi \in \Gamma\} \cup \{\{\overline{x_\psi}\} \mid \psi \in \Delta\}$$

This translation captures the semantic interpretation of sequents. Indeed, given an $\mathcal{L}$-bivaluation v and a classical assignment u that assigns true to x_ψ iff $v(\psi) = 1$, we have that for every $\mathcal{L}$-sequent s: $v(s) = 1$ iff u satisfies $\mathsf{SAT}^+(s)$, and $v(s) = 0$ iff u satisfies $\mathsf{SAT}^-(s)$. Now, for a bivaluation to be **G**-legal for some calculus **G**, it should satisfy the semantic restrictions arising from the rules of **G** (recall Def. 2.9). These restrictions can be directly encoded as SAT instances (as done, e.g., in [31] for the classical truth tables).

In the following reduction, we assume that the given calculus is axiomatic. If it is not, it can be transformed into an equivalent axiomatic calculus (see Thm. 2.15).

[4]The equivalence of strong and weak analyticity in pure calculi was also proved in [33] by a syntactic argument, similar to the one in [4] that shows the equivalence of weak and strong cut-admissibility.

Definition 4.1. The SAT instance associated with a given axiomatic $\mathcal{L}$-calculus $\mathbf{G}$, a subset $\circledcirc$ of $\diamond^1_{\mathcal{L}}$, a natural number $k \geq 0$, a set of $\mathcal{L}$-sequents S and an $\mathcal{L}$-sequent s, denoted $\mathsf{SAT}^{\circledcirc}_k(\mathbf{G}, S, s)$, consists of the following clauses:

(1) $\mathsf{SAT}^+(s')$ for every $s' \in S$
(2) $\mathsf{SAT}^-(s)$
(3) $\mathsf{SAT}^+(\sigma(s'))$ for every rule $\emptyset / s'$ of $\mathbf{G}$ and substitution σ such that $\sigma(frm(s')) \subseteq sub^{\circledcirc}_k(S \cup \{s\})$

Example 4.2. Consider the $\{\neg\}$-1-analytic calculus **Ł3** for Łukasiewicz three-valued logic (Example 2.6). Its axiomatic version, $Ax(\mathbf{Ł3})$, contains the rules $\emptyset / p_1, p_1 \supset p_2 \Rightarrow \neg p_1, p_2$ and $\emptyset / \neg p_2, p_1 \supset p_2 \Rightarrow \neg p_1, p_2$ (Example 2.14). Accordingly, $\mathsf{SAT}^{\{\neg\}}_1(Ax(\mathbf{Ł3}), S, s)$ includes the clauses $\{\overline{x_{\psi_1}}, \overline{x_{\psi_1 \supset \psi_2}}, x_{\neg\psi_1}, x_{\psi_2}\}$ and $\{\overline{x_{\neg\psi_2}}, \overline{x_{\psi_1 \supset \psi_2}}, x_{\neg\psi_1}, x_{\psi_2}\}$ for every formula of the form $\psi_1 \supset \psi_2$ in $sub^{\{\neg\}}_1(S \cup \{s\})$.

The correctness of this reduction directly follows from our definitions and Thm. 2.11:

Theorem 4.3. *For any $\circledcirc$-k-analytic axiomatic $\mathcal{L}$-calculus* $\mathbf{G}$, *we have $S \vdash_{\mathbf{G}} s$ iff $\mathsf{SAT}^{\circledcirc}_k(\mathbf{G}, S, s)$ is unsatisfiable.*

Proof. Suppose that $S \nvdash_{\mathbf{G}} s$. By Thm. 2.11, there exists a $\mathbf{G}$-legal $\mathcal{L}$-bivaluation v such that $v(S) > v(s)$. The classical assignment u that assigns true to x_ψ iff $v(\psi) = 1$ satisfies $\mathsf{SAT}^{\circledcirc}_k(\mathbf{G}, S, s)$.

For the converse, let u be a classical assignment satisfying $\mathsf{SAT}^{\circledcirc}_k(\mathbf{G}, S, s)$. Let $\mathcal{F} = sub^{\circledcirc}_k(S \cup \{s\})$. Consider the $\mathcal{F}$-bivaluation v defined by $v(\psi) = 1$ iff u assigns true to x_ψ. v is $\mathbf{G}$-legal, and $v(S) > v(s)$. By Thm. 2.11, $S \nvdash^{\mathcal{F}}_{\mathbf{G}} s$. Since $\mathbf{G}$ is $\circledcirc$-k-analytic, we may conclude that $S \nvdash_{\mathbf{G}} s$. □

Now, we show that this reduction is computable in *polynomial* time.

Definition 4.4. The *$\circledcirc$-k-complexity* of an axiomatic rule $\emptyset / s$, denoted $c^{\circledcirc}_k(\emptyset / s)$, is the minimal size of a set $\Gamma \subseteq frm(s)$ such that $frm(s) \subseteq sub^{\circledcirc}_k(\Gamma)$. The *$\circledcirc$-$k$-complexity* of an axiomatic calculus $\mathbf{G}$, denoted $c^{\circledcirc}_k(\mathbf{G})$, is given by $\max\{c^{\circledcirc}_k(r) \mid r \in \mathbf{G}\}$. If $\circledcirc = \emptyset$, we denote $c^{\circledcirc}_k$ by c.

Example 4.5. $c(\emptyset / p_1, p_2 \Rightarrow p_1 \wedge p_2) = 1$, $c(\emptyset / p_1, p_1 \supset p_2 \Rightarrow \neg p_1, p_2) = 2$ and $c^{\{\neg\}}_1(\emptyset / p_1, p_1 \supset p_2 \Rightarrow \neg p_1, p_2) = 1$. By similar calculations for the other rules of **LK** and **Ł3**, we obtain that $c(Ax(\mathbf{LK})) = 1$, $c(Ax(\mathbf{Ł3})) = 2$, and $c^{\{\neg\}}_1(Ax(\mathbf{Ł3})) = 1$.

Theorem 4.6. *Let* $\mathbf{G}$ *be an axiomatic $\mathcal{L}$-calculus. Given S and s, the SAT instance $\mathsf{SAT}^{\circledcirc}_k(\mathbf{G}, S, s)$ is computable in $O(n^m)$ time, where n is the length of the string representing S and s and $m = c^{\circledcirc}_k(\mathbf{G})$.*

Proof. The following algorithm computes $\mathsf{SAT}^{\circledcirc}_k(\mathbf{G}, S, s)$:

(1) Build a parse tree for the input using standard techniques. As usual, every node represents an occurrence of some subformula in $S \cup \{s\}$.
(2) Using, e.g., the linear time algorithm from [15], compress the parse tree into an ordered dag by maximally unifying identical subtrees. After the compression, the nodes of the dag represent subformulas of $S \cup \{s\}$, rather than occurrences. Hence we may identify nodes with their corresponding formulas.
(3) Traverse the dag. For every $\bar{\circ} \in \circledcirc^{\leq k}$ and node v that has a parent that is labeled with an element from $\diamond_{\mathcal{L}} \setminus \circledcirc$, add a new path ending with v, such that the concatenation of the path is $\bar{\circ}$, if such a path does not exist. To do so it is possible to maintain in each node v a constant-size list of all elements of $\circledcirc^{\leq k}$ that end with v. Note that after these additions, the nodes of the dag one-to-one correspond to $sub^{\circledcirc}_k(S \cup \{s\})$.
(4) $\mathsf{SAT}^-(s)$ is obtained by traversing the dag and generating $\{x_\psi\}$ for every ψ on the left-hand side of s and $\{\overline{x_\psi}\}$ for every ψ on the right-hand side of s.

(5) For every $s' \in S$, $\mathsf{SAT}^+(s')$ is obtained similarly.
(6) $\bigcup \left\{\mathsf{SAT}^+(\sigma(s')) \mid \emptyset \,/\, s' \in \mathbf{G}, \sigma(\mathit{frm}(s')) \subseteq \mathit{sub}_k^{\circledcirc}(S \cup \{s\})\right\}$ is generated by iterating over the rules of $\mathbf{G}$. For each rule $\emptyset \,/\, s'$, let $\varphi_1, \ldots, \varphi_{m'}$ $(m' \leq m)$ be formulas such that $\mathit{frm}(s')$ consists only of $\circledcirc$-k-subformulas of $\varphi_1, \ldots, \varphi_{m'}$. Go over all m'-tuples of nodes in the dag. For each m' nodes $v_1, \ldots, v_{m'}$ check whether $v_1, \ldots, v_{m'}$ match the pattern given by $\varphi_1, \ldots, \varphi_{m'}$, and if so, construct a mapping h from the formulas in $\mathit{sub}_k^{\circledcirc}(s')$ to their matching nodes. Then, construct a clause consisting of a literal $\overline{x_{h(\varphi)}}$ for every φ on the left-hand side of s', and a literal $x_{h(\varphi)}$ for every φ on the right-hand side of s'. Note that only a constant depth of the sub-dags rooted at $v_1, \ldots, v_{m'}$ is considered—that is the complexity of $\varphi_1, \ldots, \varphi_{m'}$, in addition to nodes on paths that represent elements of $\circledcirc^{\leq k}$. These are independent of the input $S \cup \{s\}$. To see that we generate exactly all required clauses, note that a substitution σ satisfies $\sigma(\mathit{frm}(s')) \subseteq \mathit{sub}_k^{\circledcirc}(S \cup \{s\})$ iff $\sigma(\{\varphi_1, \ldots, \varphi_{m'}\}) \subseteq \mathit{sub}_k^{\circledcirc}(S \cup \{s\})$. Thus, there exists a substitution σ satisfying $\sigma(\mathit{frm}(s')) \subseteq \mathit{sub}_k^{\circledcirc}(S \cup \{s\})$ iff there are m' nodes matching the patterns given by $\varphi_1, \ldots, \varphi_{m'}$.

Steps 1,2,3,4 and 5 require linear time. Each pattern matching in step 6 is done in constant time, and so handling a rule r with $c_k^{\circledcirc}(r) \leq m$ takes $O(n^m)$ time. Thus step 6 requires $O(n^m)$ time. □

Remark 1. We employ the same standard computation model of analysis of algorithms used in [19]. An efficient implementation of this algorithm cannot afford the variables x_ψ to literally include a full string representation of ψ. Thus we assume that each node has a key that can be printed and manipulated in constant time (e.g., its memory address).

Corollary 4.7. *For any $\circledcirc$-k-analytic calculus $\mathbf{G}$, the derivability problem for $\mathbf{G}$ is in co-NP.*

4.1 Linear Time Decision Procedure

Theorem 4.6 shows that the SAT instance $\mathsf{SAT}_k^{\circledcirc}(\mathbf{G}, S, s)$ can be efficiently generated. Thus, it is natural to identify calculi whose corresponding SAT instances can be also efficiently decided. For example, when the generated clauses are all *Horn clauses*, satisfiability (HORNSAT) can be decided in linear time [22]. This is the case for *Horn calculi*, as defined next.

Definition 4.8. A rule is called a *Horn rule* if the sum of the number of formulas in the right-hand side of the conclusion and the number of premises with a non-empty left-hand side is at most one. A calculus is called a *Horn calculus* if each of its rules is a Horn rule.

Example 4.9. All rules of **LK** except for $(\Rightarrow \neg)$, $(\vee \Rightarrow)$ and $(\Rightarrow \supset)$ are Horn.

Definition 4.8 captures the structure of a calculus that ensures that its equivalent axiomatic calculus consists solely of single-conclusion sequents (sequents with at most one formula on the right-hand side). In turn, the corresponding SAT-instances (see Def. 4.1) are all Horn clauses:

Proposition 4.10. *Let $\mathbf{G}$ be a Horn calculus, S a set of single-conclusion sequents, and s a sequent. Then $\mathsf{SAT}_k^{\circledcirc}(\mathbf{G}, S, s)$ consists solely of Horn clauses.*

Proof. By Thm. 2.15, there exists an axiomatic calculus $\mathbf{G}'$ that is equivalent to $\mathbf{G}$. It is easy to verify that $\mathbf{G}'$ is also a Horn calculus, and that when S is a set of single-conclusion sequents, $\mathsf{SAT}_k^{\circledcirc}(\mathbf{G}', S, s)$ consists solely of Horn clauses. □

As a corollary, we obtain a $O(n^{c_k^{\circledcirc}(\mathbf{G})})$-time decision procedure for the derivability problem for every $\circledcirc$-k-analytic Horn calculus G. When $c_k^{\circledcirc}(\mathbf{G}) = 1$ (that is, when each rule r has some 'main' formula φ so that all other formulas that appear in r are $\circledcirc$-k-subformulas of the φ), a linear time decision procedure is obtained.

Example 4.11. [11] presents a reduction from the derivability problem for **P** to HORNSAT. This reduction is a particular instance of the reduction presented above, and produces a linear time decision procedure for this logic. One may also require that the disjunction of **P** is symmetric by adding the pure axiomatic rule $r = \emptyset \,/\, p_1 \vee p_2 \Rightarrow p_2 \vee p_1$. The obtained calculus is also analytic and Horn. However, $c(r) = 2$, and so the resulting calculus will no longer have a complexity measure of 1, but of 2. The algorithm described in Thm. 4.6 will then require quadratic time, and thus the entire decision procedure will also require quadratic time.

Below (see Example 5.14) we also consider an extension of **P**, called **EP**, that is still a Horn calculus with $c(\mathbf{EP}) = 1$, and thus, as **P**, it admits a linear-time HORNSAT-based decision procedure. Another example presented below is the Horn calculus **DY** (Example 5.8) for the Dolev-Yao model of intruder deductions, which again admits a linear-time HORNSAT-based decision procedure.

Example 4.12. The linear time decision procedure for dual-Horn clauses can be utilized as well. For example, consider the analytic calculus $\mathbf{P}_d$ that consists of the rules $(\vee \Rightarrow)$, $(\Rightarrow \vee)$, $(\wedge \Rightarrow)$ of **LK** and the following ones for "dual primal implication":

$$(\prec\, \Rightarrow) \qquad p_1 \Rightarrow \,/\, p_1 \prec p_2 \Rightarrow \qquad\qquad (\Rightarrow \prec) \qquad \Rightarrow p_1 \,;\, p_2 \Rightarrow \,/\, \Rightarrow p_1 \prec p_2$$

Clearly, $c(\mathbf{P}_d) = 1$. In addition, for any sequent s and a set S of "single-assumption sequents" (sequents of the form $\Gamma \Rightarrow \Delta$ with $|\Gamma| \leq 1$), $\mathsf{SAT}_k^{\emptyset}(\mathbf{P}_d, S, s)$ consists of dual-Horn clauses (for any k). Thus the derivability problem for $\mathbf{P}_d$ can be decided in linear time.

4.2 Implementation

We have implemented our reduction in a tool called *Gen2sat*, available at http://www.cs.tau.ac.il/research/yoni.zohar/gen2sat.html, and described and evaluated in [46]. *Gen2sat* is implemented in Java, and uses the SAT-solver sat4j [37]. For a given pure calculus **G** (possibly augmented by *Next*-operators as described in §7) and an input sequent s, *Gen2sat* decides whether s is derivable in **G**. If s is not derivable, the tool provides a countermodel. If it is derivable, the tool provides a sub-calculus in which s is already derivable (using the explanation for the lack of a countermodel given by sat4j). The input to *Gen2sat* can also be the output of a tool called *Paralyzer* that transforms Hilbert calculi of a certain general form into equivalent analytic sequent calculi [17]. *Gen2sat* was recently used for educational purposes in a logic course for Information Systems graduate students at the University of Haifa [45].

5 IDENTIFYING AND CONSTRUCTING ANALYTIC CALCULI

Proof theory reveals a wide mosaic of possibilities for non-classical logics, and in particular, for sub-classical logics (logics that are strictly contained in classical logic). By choosing a subset of derivation rules that are derivable in (a proof system for) classical logic, one easily obtains a (proof system for a) sub-classical logic. Various important and useful non-classical logics can be formalized in this way, with the most prominent example being intuitionistic logic. In general, the resulting logics come at first with no semantics, and might be unusable for computational purposes, since the new calculi might not be analytic. This is evident within the framework of Hilbert-style calculi, which are rarely analytic. But, even for Gentzen-type sequent calculi, where the initial proof system for classical logic **LK** is analytic, there is no guarantee that an arbitrary collection of classically derivable sequent rules constitutes an analytic sequent calculus.

The purpose of this section is to provide a simple criterion for a given calculus to be $\circledcirc$-k-analytic, as well as a method for constructing new $\circledcirc$-k-analytic calculi. (Once such calculi are obtained, they are, of course, subject to the reduction to SAT presented in §4.)

While the semantic characterization of $\circledcirc$-k-analyticity from Thm. 3.9 provides meaningful insights on this property, it is not effective for determining $\circledcirc$-k-analyticity, as in order to use it, one needs to go over an infinite set of bivaluations, and check whether they can be fully extended. Therefore, a decidable *syntactic* criterion for $\circledcirc$-k-analyticity is desired.

In §5.1 we generalize the result of [10] in order to provide a sufficient syntactic criterion for $\circledcirc$-k-analyticity. Calculi that admit this criterion are then used in §5.2 for providing a method to construct $\circledcirc$-k-analytic calculi. Section 5.3 includes a detailed proof of the key lemma that is required for these results.

5.1 Sufficient Criterion for Analyticity

In this section we generalize the coherence condition from [10], which was given for canonical calculi, and show that the generalized condition ensures analyticity. Roughly speaking, canonical calculi are pure calculi in which each rule introduces exactly one connective in the conclusion, and all premises include only atomic formulas. Here we relax these requirements, and allow several connectives to be mentioned in one conclusion, as well as non atomic formulas in the premises. We require that all premises include only $\circledcirc$-k-subformulas of the conclusion, and that only one formula appears in the conclusion of the rule.

Definition 5.1. A rule r is called $\circledcirc$-k*-ordered* if every formula in its premises is a proper $\circledcirc$-k-subformula of some formula in its conclusion. Further, r is called $\circledcirc$-k*-directed* if it is $\circledcirc$-k-ordered, and its conclusion has the form $\Rightarrow \varphi$ or $\varphi \Rightarrow$ for some formula φ. A calculus is called $\circledcirc$-k*-directed* if it consists of $\circledcirc$-k-directed rules. We call a $\emptyset$-k-directed rule (calculus) *directed* (for any k).

Example 5.2. The calculi **LK** and **P** (Examples 2.3 and 2.4) are directed, while the calculi $\mathbf{C_1}$ and **Ł3** (Examples 2.5 and 2.6) are $\{\neg\}$-1-directed.

In [10], a *coherence* property was defined for canonical calculi, and was shown to be a necessary and sufficient condition for their analyticity. Roughly speaking, a canonical calculus is *coherent* if whenever two rules share the same formula in their conclusion, but on different sides, the empty sequent is derivable from their premises using only (cut). We generalize this requirement for the case of $\circledcirc$-k-directed calculi:

Definition 5.3. A $\circledcirc$-k-directed calculus **G** is called *coherent* if for every two rules of **G** of the forms $S_1 \,/\, \Rightarrow \varphi_1$ and $S_2 \,/\, \varphi_2 \Rightarrow$, and two substitutions σ_1, σ_2, if $\sigma_1(\varphi_1) = \sigma_2(\varphi_2)$, then the empty sequent is derivable from $\sigma_1(S_1) \cup \sigma_2(S_2)$ using only (cut).

For canonical calculi, this definition coincides with that of [10]. Also, it is decidable whether a given calculus is coherent or not: for each pair of rules $S_1 \,/\, \Rightarrow \varphi_1$ and $S_2 \,/\, \varphi_2 \Rightarrow$, one can first rename the atomic variables so that no atomic variable occurs in both rules, and then it suffices to check the above condition for the most general unifier of φ_1 and φ_2.

Example 5.4. **LK**, **P** and **Ł3** are coherent, while $\mathbf{C_1}$ is not. Indeed, for the rules $p_1 \Rightarrow \,/\, \Rightarrow \neg p_1$ and $p_1 \Rightarrow \,/\, \neg\neg p_1 \Rightarrow$ of $\mathbf{C_1}$, if $\sigma_1(p_1) = \neg p_1$ and $\sigma_2(p_1) = p_1$, we have $\sigma_1(\neg p_1) = \sigma_2(\neg\neg p_1)$, but the empty sequent cannot be derived from $\neg p_1 \Rightarrow$ and $p_1 \Rightarrow$ using only (cut).

Our notion of coherence suffices for $\circledcirc$-k-analyticity in $\circledcirc$-k-directed calculi:

Theorem 5.5. *Every coherent $\circledcirc$-k-directed calculus is $\circledcirc$-k-analytic.*

This theorem is obtained as a corollary of Thm. 5.10 below. Before turning to Thm. 5.10 and deriving Thm. 5.5, we present some examples and applications.

Example 5.6. **LK** and **P** are coherent and directed, and hence they are analytic. **Ł3** is coherent and $\{\neg\}$-1-directed, and hence it is $\{\neg\}$-1-analytic. Similarly, every canonical system (as defined in [10]) is directed, and hence every coherent canonical system is analytic.

Example 5.7 (Hierarchy of double negations). The paper [29] studies an infinite family, denoted $\{L2^{n+2} \mid n \in \mathbb{N}\}$, of pure sequent calculi for non-classical logics that admit the double negation principle as well as its weaker forms (e.g., $\neg\neg\neg\psi \leftrightarrow \neg\psi$). For example, the calculus $L4$, whose $\{\neg, \wedge, \vee\}$-fragment captures the relevance logic of first-degree entailment [1], is obtained by augmenting $\mathbf{LK} \setminus \{(\neg \Rightarrow), (\Rightarrow \neg)\}$ with the following rules:

$$p_1, \neg p_2 \Rightarrow \;/\; \neg(p_1 \supset p_2) \Rightarrow \qquad \Rightarrow p_1 \;;\; \Rightarrow \neg p_2 \;/\; \Rightarrow \neg(p_1 \supset p_2)$$
$$\neg p_1 \Rightarrow \;;\; \neg p_2 \Rightarrow \;/\; \neg(p_1 \wedge p_2) \Rightarrow \qquad \Rightarrow \neg p_1, \neg p_2 \;/\; \Rightarrow \neg(p_1 \wedge p_2)$$
$$\neg p_1, \neg p_2 \Rightarrow \;/\; \neg(p_1 \vee p_2) \Rightarrow \qquad \Rightarrow \neg p_1 \;;\; \Rightarrow \neg p_2 \;/\; \Rightarrow \neg(p_1 \vee p_2)$$
$$p_1 \Rightarrow \;/\; \neg\neg p_1 \Rightarrow \qquad \Rightarrow p_1 \;/\; \Rightarrow \neg\neg p_1$$

This calculus is coherent and $\{\neg\}$-1-directed, and hence, by Thm. 5.5, it is $\{\neg\}$-1-analytic. Moreover, it can be easily observed that for every n, $L2^{n+2}$ is coherent and $\{\neg\}$-$n+1$-directed, and thus, it is $\{\neg\}$-$n+1$-analytic.

Example 5.8 (Dolev-Yao intruder deductions). In [18], a formal deductive system for the Dolev-Yao intruder model was presented. Its language consists of two binary connectives: pairing, denoted $\langle\cdot,\cdot\rangle$, and encryption, denoted $[\cdot]_{\cdot}$ (where the argument in the subscript represents the key). Formulated as an Hilbert calculus, which we call $\mathcal{H}$, this system includes the rules of the first column in the following table:

	$\mathcal{H}$	$\mathbf{G}(\mathcal{H})$	**DY**
Pairing	$p_1 \,;\, p_2 \,/\, \langle p_1, p_2\rangle$	$\Rightarrow p_1 \,;\, \Rightarrow p_2 \,/\, \Rightarrow \langle p_1, p_2\rangle$	$\Rightarrow p_1 \,;\, \Rightarrow p_2 \,/\, \Rightarrow \langle p_1, p_2\rangle$
Unpairing	$\langle p_1, p_2\rangle \,/\, p_1$	$\Rightarrow \langle p_1, p_2\rangle \,/\, \Rightarrow p_1$	$p_1 \Rightarrow \,/\, \langle p_1, p_2\rangle \Rightarrow$
	$\langle p_1, p_2\rangle \,/\, p_2$	$\Rightarrow \langle p_1, p_2\rangle \,/\, \Rightarrow p_2$	$p_2 \Rightarrow \,/\, \langle p_1, p_2\rangle \Rightarrow$
Encryption	$p_1 \,;\, p_2 \,/\, [p_1]_{p_2}$	$\Rightarrow p_1 \,;\, \Rightarrow p_2 \,/\, \Rightarrow [p_1]_{p_2}$	$\Rightarrow p_1 \,;\, \Rightarrow p_2 \,/\, \Rightarrow [p_1]_{p_2}$
Decryption	$[p_1]_{p_2} \,;\, p_2 \,/\, p_1$	$\Rightarrow [p_1]_{p_2} \,;\, \Rightarrow p_2 \,/\, \Rightarrow p_1$	$p_1 \Rightarrow \,;\, \Rightarrow p_2 \,/\, [p_1]_{p_2} \Rightarrow$

The middle column of the table provides a pure sequent calculus, denoted $\mathbf{G}(\mathcal{H})$, that is obtained from $\mathcal{H}$ (as sketched in Footnote 3). The right column includes a calculus, which we call **DY**, obtained from $\mathbf{G}(\mathcal{H})$ by streamlining (see Lemma 2.13). **DY** is coherent and directed, and thus by Thm. 5.5, it is analytic.

5.2 Constructing Analytic Calculi

While Thm. 5.5 allows us to prove that many calculi are $\circledcirc$-k-analytic (by observing that they are $\circledcirc$-k-directed and coherent), some calculi are left out. For example, $\mathbf{C_1}$ (Example 2.5) is $\{\neg\}$-1-analytic, but it is not coherent. To capture $\mathbf{C_1}$ and other useful calculi, we introduce a more general method to prove $\circledcirc$-k-analyticity, which is, in fact, a method for obtaining calculi that are *analytic by construction.*

As a motivating example, consider the atomic paraconsistent logic P_1 from [42], that allows contradictions on atomic formulas, but forbids them on compound ones. That is, in P_1 we have that every formula φ follows from $\{\psi, \neg\psi\}$ when ψ is compound, but not from $\{p, \neg p\}$. Since the explosion principle is manifested in **LK** through the rule $(\neg \Rightarrow)$, a natural way to design a sequent calculus for P_1 is to allow applications of $(\neg \Rightarrow)$ only on compound formulas. This is achieved by the following calculus, denoted **P1**, obtained from **LK** by replacing $(\neg \Rightarrow)$ with several weaker variants of it, namely, with its following *applications*:

$$\Rightarrow \neg p_1 \;/\; \neg\neg p_1 \Rightarrow \qquad\qquad \Rightarrow p_1 \wedge p_2 \;/\; \neg(p_1 \wedge p_2) \Rightarrow$$
$$\Rightarrow p_1 \vee p_2 \;/\; \neg(p_1 \vee p_2) \Rightarrow \qquad\qquad \Rightarrow p_1 \supset p_2 \;/\; \neg(p_1 \supset p_2) \Rightarrow$$

As we shall see in what follows, this type of construction is subject to the criterion that we propose in this section. Thus, the analyticity of **P1** is established in Example 5.11 below (note that **P1** is directed and coherent, and so one could also use Thm. 5.5 above).

The general construction of $\circledcirc$-k-analytic calculi that we present is obtained by joining applications of rules of a certain basic coherent $\circledcirc$-k-directed calculus. The derivable rules that are collected to create new calculi will be *applications* of existing rules. Note that, following our definitions, every pure rule is an application of itself (using the identity substitution and the empty context sequents), and every application of a pure rule constitutes a new, perhaps weaker, pure rule. In particular, we may apply Def. 5.1 to applications of rules, and speak about $\circledcirc$-k-ordered applications (i.e., an application in which every formula that occurs in the premises is a proper $\circledcirc$-k-subformula of some formula that occurs in the conclusion). Also observe that an application $\langle \sigma(s_1) \cup c_1, \dots, \sigma(s_n) \cup c_n \;/\; \sigma(s) \cup c_1 \cup \dots \cup c_n \rangle$ of a rule $s_1, \dots, s_n \;/\; s$ is $\circledcirc$-k-ordered iff every formula of the context sequents $c_1, \dots, c_n$ is a proper $\circledcirc$-k-subformula of some formula in $\sigma(s)$.

Example 5.9. The following are ordered, $\{\neg\}$-1-ordered and $\{\neg\}$-2-ordered applications of the rule $(\supset \Rightarrow)$ of **LK** (respectively):

$$\frac{p_2 \Rightarrow p_1 \wedge p_2 \qquad p_1, p_2 \Rightarrow}{p_1, p_2, (p_1 \wedge p_2) \supset p_2 \Rightarrow} \qquad\qquad \frac{\neg p_1 \Rightarrow p_1 \wedge p_2 \qquad \neg p_1, p_2 \Rightarrow}{\neg p_1, (p_1 \wedge p_2) \supset p_2 \Rightarrow}$$

$$\frac{\neg\neg p_3 \Rightarrow p_1 \wedge p_2 \qquad \neg(p_2 \supset p_3) \Rightarrow \neg(p_1 \wedge p_2)}{\neg\neg p_3, (p_1 \wedge p_2) \supset (p_2 \supset p_3) \Rightarrow \neg(p_1 \wedge p_2)}$$

Our main result for this section is the following theorem that provides a method for constructing $\circledcirc$-k-analytic calculi.

Theorem 5.10. *Let* $\mathbf{G_B}$ *be a* $\circledcirc$*-*k*-directed coherent calculus. Then, every calculus consisting of rules that are* $\circledcirc$*-*k*-ordered applications of rules of* $\mathbf{G_B}$ *is* $\circledcirc$*-*k*-analytic.*

First, observe that Thm. 5.5 is obtained as a corollary:

Proof of Theorem 5.5. Every rule of $\mathbf{G_B}$ is a trivial $\circledcirc$-k-ordered application of itself, and, by Thm. 5.10, $\mathbf{G_B}$ itself is $\circledcirc$-k-analytic. □

Before proving Thm. 5.10, we present several examples. For these examples we collect applications of **LK** (i.e., we take $\mathbf{G_B} = \mathbf{LK}$), which is coherent and $\circledcirc$-k-directed for every $\circledcirc$ and k.

Example 5.11 (Atomic paraconsistent logic). The calculus **P1** described above for Sette's atomic paraconsistent logic can be constructed using the method of Thm. 5.10. Begin with $\mathbf{LK} \setminus \{(\neg \Rightarrow)\}$, and add the above ordered applications of $(\neg \Rightarrow)$ to allow left-introduction of negation only for compound formulas. By Thm. 5.10, this calculus is analytic. Note that **P1** is equivalent to the calculus given in [2] for this logic.

In some cases, when adding a new rule r to an existing calculus **G**, some premises of r are already derivable in **G**. For example, consider augmenting **P** (Example 2.4) with the rule $\bot \Rightarrow p_1 \;/\; \Rightarrow \bot \supset p_1$, which is an application of $(\Rightarrow \supset)$. Since $\bot \Rightarrow p_1$ is derivable in **P**, it is a redundant premise: one can alternatively add the rule $\emptyset \;/\; \Rightarrow \bot \supset p_1$. The next proposition is used for omitting such redundant premises in the following examples.

Proposition 5.12. *Let* $\mathbf{G}$ *be a pure calculus, and let* $r = S \,/\, s$ *be a rule of* $\mathbf{G}$ *such that* $\vdash_{\mathbf{G}} s'$ *for some* $s' \in S$. *Let* $\mathbf{G}' = (\mathbf{G} \setminus \{r\}) \cup \{r'\}$, *where* $r' = (S \setminus \{s'\}) \,/\, s$. *Then* $\vdash_{\mathbf{G}} = \vdash_{\mathbf{G}'}$, *and if* $\mathbf{G}$ *is* $\circledcirc$-k-*analytic then so is* $\mathbf{G}'$.

Proof. To see that $\vdash_{\mathbf{G}} = \vdash_{\mathbf{G}'}$, note that every derivation in $\mathbf{G}$ is also a derivation in $\mathbf{G}'$, and every derivation in $\mathbf{G}'$ can be turned into a derivation in $\mathbf{G}$, by first deriving $\sigma(s')$ for an appropriate σ, and then applying r instead of r'. Now, suppose that $\mathbf{G}$ is $\circledcirc$-k-analytic. We prove that so is $\mathbf{G}'$. Let S_0 be a set of sequents and s_0 a sequent such that $S_0 \vdash_{\mathbf{G}'} s_0$. Let $\mathcal{F} = sub_k^{\circledcirc}(S_0 \cup \{s_0\})$. We show that $S_0 \vdash_{\mathbf{G}'}^{\mathcal{F}} s_0$. Since $\vdash_{\mathbf{G}} = \vdash_{\mathbf{G}'}$, we have $S_0 \vdash_{\mathbf{G}} s_0$. Since $\mathbf{G}$ is $\circledcirc$-k-analytic, it follows that $S_0 \vdash_{\mathbf{G}}^{\mathcal{F}} s_0$. The derivation of s_0 from S_0 in $\mathbf{G}$ that uses only $\mathcal{F}$-formulas is also a derivation in $\mathbf{G}'$, and thus we have $S_0 \vdash_{\mathbf{G}'}^{\mathcal{F}} s_0$. □

Example 5.13. In [8], it was shown that $\mathbf{C_1}$ (Example 2.5) is $\{\neg\}$-1-analytic, as a corollary of cut-admissibility. Using the methods of this section, we provide a simpler proof of the $\{\neg\}$-1-analyticity of $\mathbf{C_1}$. For this purpose, we construct a calculus which is equivalent to $\mathbf{C_1}$, which we call $\mathbf{C_1}'$. Take $\mathbf{G_B}$ to be $\mathbf{LK}$, and $\mathbf{G}$ to be $\mathbf{LK} \setminus \{(\neg \Rightarrow)\}$. By Thm. 5.5, $\mathbf{G}$ is $\{\neg\}$-1-analytic. $\mathbf{C_1}'$ is obtained by augmenting $\mathbf{G}$ with the following rules:

$$\emptyset \,/\, \neg\neg p_1 \Rightarrow p_1$$
$$\emptyset \,/\, p_1, \neg p_1, \neg(p_1 \wedge \neg p_1) \Rightarrow \qquad \emptyset \,/\, \neg(p_1 \wedge p_2) \Rightarrow \neg p_1, \neg p_2$$
$$\emptyset \,/\, \neg(p_1 \vee p_2) \Rightarrow \neg p_1, p_2 \qquad \emptyset \,/\, \neg(p_1 \vee p_2) \Rightarrow \neg p_1, \neg p_2$$
$$\emptyset \,/\, \neg(p_1 \vee p_2) \Rightarrow p_1, \neg p_2 \qquad \emptyset \,/\, \neg(p_1 \supset p_2) \Rightarrow p_1, p_2$$
$$\emptyset \,/\, \neg(p_1 \supset p_2) \Rightarrow p_1, \neg p_2 \qquad \emptyset \,/\, \neg(p_1 \supset p_2) \Rightarrow \neg p_1, \neg p_2$$

Every rule here has the form $\emptyset \,/\, s$, where s is the conclusion of a $\{\neg\}$-1-ordered application of the rule $(\neg \Rightarrow)$ of $\mathbf{G_B}$, whose premises are all derivable in $\mathbf{G}$. For example, $\neg(p_1 \wedge p_2) \Rightarrow \neg p_1, \neg p_2$ is the conclusion of the following $\{\neg\}$-1-ordered application of $(\neg \Rightarrow)$, whose premise is derivable in $\mathbf{G}$:

$$\frac{\Rightarrow p_1 \wedge p_2, \neg p_1, \neg p_2}{\neg(p_1 \wedge p_2) \Rightarrow \neg p_1, \neg p_2}$$

By Thm. 5.10 and Prop. 5.12, $\mathbf{C_1}'$ is $\{\neg\}$-1-analytic. Using Lemma 2.13, it is easy to see that $\mathbf{C_1}'$ is equivalent to $\mathbf{C_1}$, and furthermore, its $\{\neg\}$-1-analyticity entails the $\{\neg\}$-1-analyticity of $\mathbf{C_1}$.

Example 5.14. The calculus $\mathbf{P}$ (Example 2.4) is analytic, as shown in Example 5.6. Its analyticity also follows from the fact that it consists of ordered applications of rules of $\mathbf{LK}$ (the only rule in $\mathbf{P}$ which is not in $\mathbf{LK}$ is $\Rightarrow p_2 \,/\, \Rightarrow p_1 \supset p_2$, which is an ordered application of $p_1 \Rightarrow p_2 \,/\, \Rightarrow p_1 \supset p_2$). It is also possible to augment $\mathbf{P}$ with additional rules in order to make it somewhat closer to $\mathbf{LK}$, without compromising its analyticity. For example, an extended calculus, which we denote by $\mathbf{EP}$, is obtained by augmenting $\mathbf{P}$ with the following set of rules, which recover some natural properties of the classical connectives (none of them is derivable in $\mathbf{P}$):

$$\emptyset \,/\, \Rightarrow \bot \supset p_1 \qquad \emptyset \,/\, p_1 \vee p_1 \Rightarrow p_1 \qquad \emptyset \,/\, \Rightarrow p_1 \supset p_1$$
$$\emptyset \,/\, \bot \vee p_1 \Rightarrow p_1 \qquad \emptyset \,/\, p_1, \neg p_1 \Rightarrow \qquad \emptyset \,/\, \Rightarrow (p_1 \wedge p_2) \supset p_1$$
$$\emptyset \,/\, p_1 \vee \bot \Rightarrow p_1 \qquad \emptyset \,/\, p_1 \vee (p_1 \wedge p_2) \Rightarrow p_1 \qquad \emptyset \,/\, \Rightarrow (p_1 \wedge p_2) \supset p_2$$
$$\emptyset \,/\, (p_1 \wedge p_2) \vee p_1 \Rightarrow p_1 \qquad \emptyset \,/\, \Rightarrow p_2 \supset (p_1 \supset p_2)$$

Each of these rules has the form $\emptyset \,/\, s$, where s is the conclusion of an ordered application of a rule of $\mathbf{LK}$, whose premises are all derivable in $\mathbf{P}$. By Thm. 5.10 and a repeated application of Prop. 5.12, augmenting $\mathbf{P}$ with these axiomatic rules results in an analytic calculus.

5.3 Proof of Theorem 5.10

Let $\mathbf{G}$ be a calculus that consists of $\circledcirc$-k-ordered applications of rules of a $\circledcirc$-k-directed coherent calculus $\mathbf{G_B}$. We prove that $\mathbf{G}$ is $\circledcirc$-k-analytic. Using Thm. 3.9, it suffices to prove that every $\mathbf{G}$-legal bivaluation v can be extended to a $\mathbf{G}$-legal $\mathcal{L}$-bivaluation, provided that $dom(v)$ is finite and closed under $\circledcirc$-k-subformulas. Thus, in what follows, we fix an arbitrary $\mathbf{G}$-legal bivaluation v such that $dom(v)$ is finite and closed under $\circledcirc$-k-subformulas.

We extend v iteratively: in each step we add a single formula to the domain of v. Thus, we construct a sequence of $\mathbf{G}$-legal bivaluations that extend v, and use this sequence in order to define a $\mathbf{G}$-legal $\mathcal{L}$-bivaluation that extends v.

Since the $\circledcirc$-k-subformula relation is a partial order, $sub_k^{\circledcirc}(\psi)$ is finite for every ψ, and $dom(v)$ is finite, there exists an enumeration $\psi_1, \psi_2, \ldots$ of $\mathcal{L}$ such that:

(1) If $\psi_i \in dom(v)$ and $\psi_j \notin dom(v)$ then $i < j$.
(2) If ψ_i is a $\circledcirc$-k-subformula of ψ_j then $i \leq j$.

We define a sequence $v_0, v_1, \ldots$ of bivaluations inductively by:

(1) $v_0 = v$.
(2) For every $i > 0$, v_i is defined over $dom(v) \cup \{\psi_1, \ldots, \psi_i\}$ as follows:
  (a) $v_i(\varphi) = v_{i-1}(\varphi)$ for every $\varphi \in dom(v_{i-1})$.
  (b) If $\psi_i \notin dom(v_{i-1})$, then $v_i(\psi_i) = 1$ iff there exists a rule of the form $s_1, \ldots, s_n \,/ \Rightarrow \varphi$ in $\mathbf{G_B}$, sequents $s'_1 \subseteq s_1, \ldots, s'_n \subseteq s_n$, and a substitution σ such that $\sigma(frm(\{s'_1, \ldots, s'_n\})) \subseteq dom(v_{i-1})$, $\sigma(\varphi) = \psi_i$ and $v_{i-1}(\sigma(s'_j)) = 1$ for every $1 \leq j \leq n$. Otherwise, $v_i(\psi_i) = 0$.

The following lemma is needed in order to show that each bivaluation in the sequence is $\mathbf{G}$-legal.

Lemma 5.15. *Let $\hat{r} = \left\langle \{\sigma(s'_1) \cup c_1, \ldots, \sigma(s'_n) \cup c_n\}, \sigma(s) \cup c_1 \cup \ldots \cup c_n \right\rangle$ be a $\circledcirc$-k-ordered application of a $\circledcirc$-k-directed rule $r = s_1, \ldots, s_n / s$, and let φ_s be the single formula in $frm(s)$. Then, all formulas in $sub_k^{\circledcirc}(\sigma(s'_i) \cup c_i)$ are proper $\circledcirc$-k-subformulas of $\sigma(\varphi_s)$ for every $1 \leq i \leq n$. In particular,*

$$sub_k^{\circledcirc}(\{\sigma(s'_1) \cup c_1, \ldots, \sigma(s'_n) \cup c_n, \sigma(s)\}) \subseteq sub_k^{\circledcirc}(\sigma(s)).$$

Proof. Let ψ be a $\circledcirc$-k-subformula of some $\varphi \in \sigma(frm(s'_i)) \cup frm(c_i)$. We show that φ is a proper $\circledcirc$-k-subformula of $\sigma(\varphi_s)$. Since ψ is a $\circledcirc$-k-subformula of φ, it would then follow that ψ is also a proper $\circledcirc$-k-subformula of $\sigma(\varphi_s)$. If $\varphi = \sigma(\varphi')$ for some $\varphi' \in frm(s'_i)$, then since r is $\circledcirc$-k-directed, φ' is a proper $\circledcirc$-k-subformula of φ_s. By Lemma 3.4, φ is a proper $\circledcirc$-k-subformula of $\sigma(\varphi_s)$. Otherwise, $\varphi \in frm(c_i)$, and since $\hat{r}$ is $\circledcirc$-k-ordered, φ is a proper $\circledcirc$-k-subformula of some formula in $frm(\sigma(s) \cup c_1 \cup \ldots \cup c_n)$. If φ is a proper $\circledcirc$-k-subformula of some formula in $frm(\sigma(s))$, then this formula must be $\sigma(\varphi_s)$. Otherwise, let θ be a formula in $frm(c_1 \cup \ldots \cup c_n)$ such that φ is a proper $\circledcirc$-k-subformula of θ, and θ has a maximal number of connectives. Since $\hat{r}$ is $\circledcirc$-k-ordered, θ is also a proper $\circledcirc$-k-subformula of some formula $\theta' \in frm(\sigma(s) \cup c_1 \cup \ldots \cup c_n)$. By the maximality of θ, we have that $\theta' \in frm(\sigma(s))$, which means that $\theta' = \sigma(\varphi_s)$. Since φ is a proper $\circledcirc$-k-subformula of θ, we also have that φ is a proper $\circledcirc$-k-subformula of $\sigma(\varphi_s)$. □

Next, we show by induction on i, that each v_i is $\mathbf{G}$-legal. For $i = 0$, this holds by our assumption regarding v. Let $i > 0$, and r be a rule of $\mathbf{G}$. Then, there exist a rule $s_1, \ldots, s_n / s$ of $\mathbf{G_B}$, sequents $s'_1 \subseteq s_1, \ldots, s'_n \subseteq s_n$, a substitution α, and sequents $c_1, \ldots, c_n$ such that $r = \alpha(s'_1) \cup c_1, \ldots, \alpha(s'_n) \cup c_n \,/\, \alpha(s) \cup c_1 \cup \ldots \cup c_n$. Let $s''_1 \subseteq s'_1, \ldots, s''_n \subseteq s'_n$, $c'_1 \subseteq c_1, \ldots, c'_n \subseteq c_n$ and σ be a substitution such that $\sigma(frm(\{\alpha(s''_1) \cup c'_1, \ldots, \alpha(s''_n) \cup c'_n, \alpha(s) \cup c_1 \cup \ldots \cup c_n\})) \subseteq dom(v_i)$. We show that $v_i(\{\sigma(\alpha(s''_j) \cup c'_j) \mid 1 \leq j \leq n\}) \leq v_i(\sigma(\alpha(s) \cup c_1 \cup \ldots \cup c_n))$. If $\psi_i \notin \sigma(frm(\{\alpha(s''_1) \cup c'_1, \ldots, \alpha(s''_n) \cup c'_n, \alpha(s) \cup c_1 \cup \ldots \cup c_n)\})$ or $\psi_i \in dom(v_{i-1})$, then $\sigma(frm(\{\alpha(s''_1) \cup c'_1, \ldots, \alpha(s''_n) \cup c'_n, \alpha(s) \cup c_1 \cup \ldots \cup c_n\})) \subseteq dom(v_{i-1})$, and hence this holds by the induction hypothesis. Assume now that $\psi_i \in \sigma(frm(\{\alpha(s''_1) \cup c'_1, \ldots, \alpha(s''_n) \cup c'_n, \alpha(s) \cup c_1 \cup \ldots \cup c_n)\})$ and $\psi_i \notin dom(v_{i-1})$. Let φ_s be the single formula in $frm(s)$. We first prove that $\psi_i = \sigma(\alpha(\varphi_s))$. Otherwise, $\sigma(\alpha(\varphi_s)) \in dom(v_{i-1})$. By Lemma 5.15, the set of formulas that occur in r is contained in $sub_k^{\circledcirc}(\alpha(\varphi_s))$, and by Lemma 3.4, we also have that for every formula φ that occurs in r, $\sigma(\varphi) \in \sigma(sub_k^{\circledcirc}(\alpha(\varphi_s))) \subseteq sub_k^{\circledcirc}(\sigma(\alpha(\varphi_s)))$. $dom(v_{i-1})$ is closed under $\circledcirc$-k-subformulas, and $\sigma(\alpha(\varphi_s)) \in dom(v_{i-1})$. Thus we have $\psi_i \in dom(v_{i-1})$, which is a contradiction.

Similarly, we show that $\sigma(frm(\alpha(s''_j) \cup c'_j)) \subseteq dom(v_{i-1})$ for every $1 \leq j \leq n$. Indeed, let $\varphi \in \sigma(frm(\alpha(s''_j) \cup c'_j))$ and let $\varphi' \in frm(\alpha(s''_j) \cup c'_j)$ such that $\varphi = \sigma(\varphi')$. By Lemma 5.15, φ' is a proper $\circledcirc$-k-subformula of $\alpha(\varphi_s)$, and hence by Lemma 3.4, φ is a proper $\circledcirc$-k-subformula of $\psi_i = \sigma(\alpha(\varphi_s))$. In particular, $\varphi \neq \psi_i$. Since $\sigma(frm(\alpha(s''_j) \cup c'_j)) \subseteq dom(v_i)$, it follows that $\varphi \in dom(v_{i-1})$.

Now, suppose that $v_i(\sigma(\alpha(s''_j) \cup c'_j)) = 1$ for every $1 \leq j \leq n$. We prove that $v_i(\sigma(\alpha(s) \cup c_1 \cup \ldots \cup c_n)) = 1$. If $v_i(\sigma(c'_1 \cup \ldots \cup c'_n)) = 1$, then we are clearly done. Assume otherwise. Hence, we have $v_i(\sigma(\alpha(s''_j))) = 1$ for every $1 \leq j \leq n$. Since $\sigma(\alpha(frm(s''_j))) \subseteq dom(v_{i-1})$ for every $1 \leq j \leq n$, we have $v_{i-1}(\sigma(\alpha(s''_j))) = 1$ for every such j. Distinguish two cases:

- $s = \;\Rightarrow \varphi_s$: Since $\sigma(\alpha(frm(s''_j))) \subseteq dom(v_{i-1})$ for every $1 \leq j \leq n$, $\sigma(\alpha(\varphi_s)) = \psi_i$, and $v_{i-1}(\sigma(\alpha(s''_j))) = 1$ for every $1 \leq j \leq n$, by the definition of v_i we have $v_i(\psi_i) = 1$, and so $v_i(\sigma(\alpha(s))) = 1$.
- $s = \varphi_s \Rightarrow\;$: To prove that $v_i(\sigma(\alpha(s))) = 1$, we show that $v_i(\psi_i) = 0$. By the definition of v_i, it suffices to prove that for every rule of the form $q_1, \ldots, q_m \,/\, \Rightarrow \varphi'$ in $\mathbf{G_B}$, sequents $q'_1 \subseteq q_1, \ldots, q'_m \subseteq q_m$ and substitution σ' such that $\sigma'(frm(q'_j)) \subseteq dom(v_{i-1})$ for every $1 \leq j \leq m$ and $\sigma'(\varphi') = \psi_i$, we have $v_{i-1}(\sigma'(q'_j)) = 0$ for some $1 \leq j \leq m$. Let $q_1, \ldots, q_m \,/\, \Rightarrow \varphi'$ and σ' as above. Since $\mathbf{G_B}$ is coherent, the empty sequent is derivable from $\{\sigma(\alpha(s_1)), \ldots, \sigma(\alpha(s_n)), \sigma'(q_1), \ldots, \sigma'(q_m)\}$ using only (cut). It can be shown by induction on this derivation that the same holds for $\{\sigma(\alpha(s'_1)), \ldots, \sigma(\alpha(s'_n)), \sigma'(q'_1), \ldots, \sigma'(q'_m)\}$, and in particular, we have $\sigma(\alpha(s'_1)), \ldots, \sigma(\alpha(s'_n)), \sigma'(q'_1), \ldots, \sigma'(q'_m) \vdash_{\mathbf{G}}^{dom(v_{i-1})} \;\Rightarrow$. By Thm. 2.11, since v_{i-1} is $\mathbf{G}$-legal and $v_{i-1}(\sigma(\alpha(s'_j))) = 1$ for every $1 \leq j \leq n$, we have $v_{i-1}(\sigma'(q'_j)) = 0$ for some $1 \leq j \leq m$.

Finally, let v' be the $\mathcal{L}$-bivaluation given by $v'(\psi_i) = v_i(\psi_i)$ for every $i > 0$. Clearly, v' extends v. To see that it is $\mathbf{G}$-legal, let $s_1, \ldots, s_n \,/\, s \in \mathbf{G}$, $s'_1 \subseteq s_1, \ldots, s'_n \subseteq s_n$, and σ be a substitution. Let $j = \max\{i \mid \psi_i \in \sigma(frm(\{s'_1, \ldots, s'_n, s\}))\}$. Then, $v'(\psi) = v_j(\psi)$ for every $\psi \in \sigma(frm(\{s'_1, \ldots, s'_n, s\}))$. Since v_j is $\mathbf{G}$-legal, $v'(\{\sigma(s'_i) \mid 1 \leq i \leq n\}) = \min\{v_j(\sigma(s'_i)) \mid 1 \leq i \leq n\} \leq v_j(\sigma(s)) = v'(\sigma(s))$. □

6 ADDING MODAL OPERATORS TO PURE SEQUENT CALCULI

Useful non-classical logics are beyond the reach of $\circledcirc$-k-analytic pure calculi. For example, the usual sequent rules for the modal operators $\Box$ and $\Diamond$ in modal logics (e.g., K, KTB, S5 etc.) limit the context sequents, and thus are not pure. In this section, we consider the extensions of pure sequent calculi with rules for introducing modal operators. Our investigation is not limited to a single modal operator, and thus the systems that we study are *multimodal*. Moreover, the base logic need not be classical, and can be any logic that is described by a pure calculus. In §6.2, we prove a soundness and completeness theorem for the resulting calculi with respect to a Kripke-style semantics that generalizes the bivaluation semantics of §2.3. This semantics is then used in §6.3 in order prove the following result: if a pure calculus is $\circledcirc$-k-analytic, then it remains so when rules for modal operators are added. The main lemma that is used in the proof of this result is proved in Lemma 6.20. The semantics is also used in the next section, where we extend the reduction of §4 to pure calculi that are augmented with a special kind of modal operators. Note that we focus here on positive, Box-like modal operators. An investigation of Diamond-like modal operators, and of negative modalities (see, e.g., [23]) is left for future research.

$$(\textsc{k})\ \frac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi} \qquad \underset{\textsc{functional}}{(\textsc{pf})}\ \frac{\Gamma \Rightarrow \varphi, \Delta}{\Box\Gamma \Rightarrow \Box\varphi, \Box\Delta} \qquad \underset{\textsc{transitive}}{(4)}\ \frac{\Box\Gamma_1, \Gamma_2 \Rightarrow \varphi}{\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box\varphi}$$

$$\underset{\substack{\textsc{transitive}\\\textsc{euclidean}}}{(45)}\ \frac{\Box\Gamma_1, \Gamma_2 \Rightarrow \varphi, \Box\Delta}{\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box\varphi, \Box\Delta} \qquad \underset{\textsc{symmetric}}{(\textsc{b})}\ \frac{\Gamma \Rightarrow \varphi, \Box\Delta}{\Box\Gamma \Rightarrow \Box\varphi, \Delta} \qquad \underset{\substack{\textsc{symmetric}\\\textsc{transitive}}}{(\textsc{b}4)}\ \frac{\Box\Gamma_1, \Gamma_2 \Rightarrow \varphi, \Box\Delta_1, \Box\Delta_2}{\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box\varphi, \Box\Delta_1, \Delta_2}$$

Additional reflexivity rule:

$$(\textsc{t})\ \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \Box\varphi \Rightarrow \Delta}$$

Additional corresponding seriality rules:

$$(\textsc{d}_{\textsc{k}})\ \frac{\Gamma \Rightarrow}{\Box\Gamma \Rightarrow} \qquad (\textsc{d}_{\textsc{pf}})\ \frac{\Gamma \Rightarrow \Delta}{\Box\Gamma \Rightarrow \Box\Delta} \qquad (\textsc{d}_4)\ \frac{\Box\Gamma_1, \Gamma_2 \Rightarrow}{\Box\Gamma_1, \Box\Gamma_2 \Rightarrow}$$

$$(\textsc{d}_{45})\ \frac{\Box\Gamma_1, \Gamma_2 \Rightarrow \Box\Delta}{\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box\Delta} \qquad (\textsc{d}_{\textsc{b}})\ \frac{\Gamma \Rightarrow \Box\Delta}{\Box\Gamma \Rightarrow \Delta} \qquad (\textsc{d}_{\textsc{b}4})\ \frac{\Box\Gamma_1, \Gamma_2 \Rightarrow \Box\Delta_1, \Box\Delta_2}{\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box\Delta_1, \Delta_2}$$

Fig. 1. Application schemes of sequent rules for a modal operator $\Box$

Let $\boxdot$ be a finite set of unary connectives, called *modal operators*, such that $\boxdot \cap \diamond_{\mathcal{L}} = \emptyset$. We denote by $\mathcal{L}_{\boxdot}$ the propositional language obtained by augmenting $\mathcal{L}$ with the modal operators in $\boxdot$. The notations $\Box\mathcal{F}$ and $\boxdot\mathcal{F}$ are similar to the notations from §3, and are extended to sequents and sets of sequents in the obvious way.

Unlike the connectives of $\mathcal{L}$, which may appear in any pure rule, the modal operators are manipulated according to a predefined set of rules given in Fig. 1. These sequent rules were previously shown to correspond to the classical modal logic axioms [30, 34, 44]. With the exception of (T), these are not pure rules, as their applications do not allow arbitrary context sequents. To keep the discussion modular, we assume a given *specification function* M specifying the derivation rules for every $\Box \in \boxdot$. For every $\Box \in \boxdot$, $\mathsf{M}(\Box)$ is either a singleton consisting of one of the rules from the first part of Fig. 1, or a pair consisting of such a rule (X) together with either (T) or a matching ($\textsc{d}_{\textsc{x}}$) rule. (Note that there is no need to consider the combination of both (T) and a ($\textsc{d}_{\textsc{x}}$)-rule, since, by possibly using cuts, all ($\textsc{d}_{\textsc{x}}$)-rules are derivable in the presence of (T)). We exclude the combination of (PF) and (T), as we were unable to find an appropriate semantic condition for this combination.[5] Thus, there are $6 + 6 + 5 = 17$ options for rules manipulating each modal operator.

Given a pure calculus **G** for $\mathcal{L}$, we obtain the calculus $\mathbf{G}_{\mathsf{M}}$ for $\mathcal{L}_{\boxdot}$ by augmenting **G** with the rules determined by $\mathsf{M}(\Box)$ for each $\Box \in \boxdot$. For a set $\mathcal{F} \subseteq \mathcal{L}_{\boxdot}$ of formulas, we write $S \vdash^{\mathcal{F}}_{\mathbf{G}_{\mathsf{M}}} s$ (or $S \vdash_{\mathbf{G}_{\mathsf{M}}} s$ when $\mathcal{F} = \mathcal{L}_{\boxdot}$) if there is a derivation of a sequent s from a set S of sequents in $\mathbf{G}_{\mathsf{M}}$ consisting only of $\mathcal{F}$-sequents.

Example 6.1. Sequent calculi for classical modal logics are obtained by taking $\boxdot = \{\Box\}$, and augmenting **LK** with the appropriate rules for the modal operators. For example, calculi for the modal logics K and KD are obtained by respectively taking $\mathsf{M}(\Box) = \{(\textsc{k})\}$ and $\mathsf{M}(\Box) = \{(\textsc{k}), (\textsc{d}_{\textsc{k}})\}$. The logics $S4$ and $S5$ are captured by respectively taking $\mathsf{M}(\Box) = \{(4), (\textsc{t})\}$ and $\mathsf{M}(\Box) = \{(\textsc{b}4), (\textsc{t})\}$.

[5]If classical negation is definable, the meaning of $\Box$ in the presence of both (PF) and (T) becomes trivial: on the one hand, (T) easily gives us the derivability of $\Box\varphi \Rightarrow \varphi$. On the other hand, $\varphi \Rightarrow \Box\varphi$ can be proved using the rules ($\neg \Rightarrow$) and ($\Rightarrow \neg$) of **LK**, together with (T), (PF) and (CUT).

6.1 Equivalence and Admissible Rules

Interestingly, whether or not different specifications for rules of the modal operators yield "observationally distinct" calculi might depend on the underlying pure calculus. That is, there are (families of) pure calculi for which the addition of different rules for modal operators induces the same derivability relation. In this section, we present two such cases. The first establishes the equivalence of {(PF), (D_{PF})} and {(K), (D_K)} when added to a Horn calculus (see Def. 4.8):

Proposition 6.2. *Suppose that $M(\Box) = \{(PF), (D_{PF})\}$ and $M'(\Box) = \{(K), (D_K)\}$ for every $\Box \in \boxdot$. Let* **G** *be a Horn calculus. Then, $S \vdash_{\mathbf{G}_M} s$ iff $S \vdash_{\mathbf{G}_{M'}} s$ for every set S of of single-conclusion sequents and sequent s.*

Proof. The right-to-left direction is trivial. For the left-to-right direction, we prove that if $S \vdash_{\mathbf{G}_M} \Gamma \Rightarrow \Delta$, then $S \vdash_{\mathbf{G}_M} \Gamma \Rightarrow E$ for some singleton or empty set $E \subseteq \Delta$. This allows us to replace any application of (PF) or (D_{PF}) with an application of either (K) or (D_K) on a single-conclusion subsequent, and then obtaining the original conclusion using weakening. We do so by induction on the length of the derivation of $\Gamma \Rightarrow \Delta$.

If $\Gamma \Rightarrow \Delta \in S$ or $\Gamma \Rightarrow \Delta$ is the conclusion of an application of (ID), (WEAK), (CUT), (K), or (D_K) then this is obvious. We consider the case that $\Gamma \Rightarrow \Delta$ is the conclusion of an application of some pure rule $\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_n \Rightarrow \Delta_n \,/\, \Gamma_0 \Rightarrow \Delta_0$ of **G**. Then, there exist a substitution σ and sequents $\Gamma_1' \Rightarrow \Delta_1', \Gamma_1'' \Rightarrow \Delta_1'', \dots, \Gamma_n' \Rightarrow \Delta_n', \Gamma_n'' \Rightarrow \Delta_n''$ such that for every $1 \le i \le n$, $\Gamma_i' \Rightarrow \Delta_i' \subseteq \Gamma_i \Rightarrow \Delta_i$, $\Gamma \Rightarrow \Delta = \Gamma_1'', \dots, \Gamma_n'', \sigma(\Gamma_0) \Rightarrow \sigma(\Delta_0), \Delta_1'', \dots, \Delta_n''$, and $S \vdash_{\mathbf{G}_M} \Gamma_i'', \sigma(\Gamma_i') \Rightarrow \sigma(\Delta_i'), \Delta_i''$ with shorter derivations for every $1 \le i \le n$. Since **G** is Horn, one of the following holds:

(1) For every $1 \le i \le n$, $\Gamma_i' = \emptyset$ and $|\sigma(\Delta_0)| \le 1$: In this case, $S \vdash_{\mathbf{G}_M} \Gamma_i'' \Rightarrow \sigma(\Delta_i'), \Delta_i''$ for every $1 \le i \le n$. By the induction hypothesis, for every $1 \le i \le n$, $S \vdash_{\mathbf{G}_M} \Gamma_i'' \Rightarrow E_i$ for some singleton or empty set $E_i \subseteq \sigma(\Delta_i') \cup \Delta_i''$. If $E_i \subseteq \Delta_i''$ for some $1 \le i \le n$, then using (WEAK) we are done. Otherwise, for every $1 \le i \le n$, there exists $\varphi_i \in \sigma(\Delta_i')$ such that $E_i = \{\varphi_i\}$. Hence for every $1 \le i \le n$, $S \vdash_{\mathbf{G}_M} \Gamma_i'' \Rightarrow \varphi_i$. Now, we may apply the rule with context sequents $\Gamma_i'' \Rightarrow$ and get that $S \vdash_{\mathbf{G}_M} \Gamma_1'', \dots, \Gamma_n'', \sigma(\Gamma_0) \Rightarrow \sigma(\Delta_0)$, which means that $S \vdash_{\mathbf{G}_M} \Gamma \Rightarrow \sigma(\Delta_0)$.

(2) There exists a single $1 \le i \le n$ such that $\Gamma_i' \ne \emptyset$, and $\Delta_0 = \emptyset$: By the induction hypothesis, there exists a singleton or empty set $E_i \subseteq \sigma(\Delta_i') \cup \Delta_i''$ such that $S \vdash_{\mathbf{G}_M} \Gamma_i'', \sigma(\Gamma_i') \Rightarrow E_i$. Also by the induction hypothesis, for every $j \ne i$, there exists a singleton or empty set $E_j \subseteq \Delta_j'' \cup \sigma(\Delta_j')$ and $S \vdash_{\mathbf{G}_M} \Gamma_j'' \Rightarrow E_j$. If $E_j \subseteq \Delta_j''$ for some $j \ne i$, then using (WEAK), we get that and $S \vdash_{\mathbf{G}_M} \Gamma \Rightarrow E_j$ (and $E_j \subseteq \Delta$). Otherwise, for every $j \ne i$ there exists $\varphi_j \in \sigma(\Delta_j')$ such that $E_j = \{\varphi_j\}$. Apply the rule with context sequents $\Gamma_i'' \Rightarrow E_i$ and $\Gamma_j'' \Rightarrow$ for every $j \ne i$ and obtain $S \vdash_{\mathbf{G}_M} \Gamma_1'', \dots, \Gamma_n'', \sigma(\Gamma_0) \Rightarrow E_i$, and so $S \vdash_{\mathbf{G}_M} \Gamma \Rightarrow E_i$ (for $E_i \subseteq \Delta$). □

Next, we provide sufficient conditions for the admissibility of the seriality rules (D_K), (D_4), (D_{45}) and (D_{PF}). Recall that Fig. 1 associates each modal rule (X) with its own seriality rule (D_X). Such a rule is needed, for instance, to derive $\Box\bot \Rightarrow$, which should be derivable when the accessibility relation is required to be serial. For a certain family of pure calculi, that we call *definite calculi*, we can prove that (D_K), (D_4), (D_{45}) and (D_{PF}) are redundant:

Definition 6.3. A rule is called *definite* if at least one of its premises has an empty right-hand side whenever the conclusion has an empty right-hand side. A calculus is called *definite* if each of its rules is definite.

Example 6.4. All rules of **LK** except for ($\neg \Rightarrow$) and ($\bot \Rightarrow$) are definite.

Proposition 6.5. *Suppose that $M'(\Box) = M(\Box) \setminus \{(\textsc{d}_K), (\textsc{d}_4), (\textsc{d}_{45}), (\textsc{d}_{PF})\}$ for every $\Box \in \boxdot$. Let* **G** *be a definite calculus. Suppose that $S \vdash_{\mathbf{G}_M} s$ for some set S of sequents with non-empty right-hand sides and sequent s. Then, $S \vdash_{\mathbf{G}_{M'}} s$.*

Proof. First, since **G** is definite, using induction on the length of the derivation, it can be shown that all sequents in the derivation of s from S in $\mathbf{G}_{\mathsf{M}}$ have a non-empty right-hand side. Such derivations cannot use ($\textsc{d}_K$) or ($\textsc{d}_4$). Moreover, any application of ($\textsc{d}_{PF}$) whose premise has a non-empty right-hand side is also an application of (pf). Finally, consider an application $\langle \Box\Gamma_1 \cup \Gamma_2 \Rightarrow \Box\Delta, \Box\Gamma_1 \cup \Box\Gamma_2 \Rightarrow \Box\Delta \rangle$ of ($\textsc{d}_{45}$). Since $\Box\Delta \neq \emptyset$, we can use (weak) to obtain $\Box\Gamma_1, \Gamma_2 \Rightarrow \psi, \Box\Delta$ for some $\psi \in \Delta$, and using (45) we get $\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box\psi, \Box\Delta$, which is s. □

Note that ($\textsc{d}_B$) and ($\textsc{d}_{B4}$) are not admissible under the conditions above. Indeed, when augmenting the empty (pure) calculus with these rules, the sequent $\Box\Box p_1 \Rightarrow p_1$ is derivable using only (id) and either ($\textsc{d}_B$) or ($\textsc{d}_{B4}$), while it is not derivable using (b) or (b4).

6.2 Kripke Semantics for Modal Operators

In this section we generalize the bivaluations semantics from §2.3 and elevate it to a Kripke-style semantics. Given a pure calculus and a specification of rules for the modal operators, the semantics of the original connectives is governed by the bivaluation semantics in each possible world, while the semantics of the modal connectives follows their usual meaning in Kripke models. As in the case of bivaluations, we consider *partial* Kripke models in order to achieve a semantic counterpart of analyticity.

Definition 6.6. A *biframe* for M is a tuple $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V} \rangle$ where:

(1) W is a set of elements called *worlds*. Henceforth, we may identify $\mathcal{W}$ with this set (e.g., when writing $w \in \mathcal{W}$ instead of $w \in W$).
(2) $\mathcal{R}$ is a function assigning a binary relation on W (called *accessibility relation*) to every $\Box \in \boxdot$. We write $\mathcal{R}_\Box$ instead of $\mathcal{R}(\Box)$, and $\mathcal{R}_\Box[w]$ for $\{w' \in W \mid w\mathcal{R}_\Box w'\}$. For every $\Box \in \boxdot$, the relation $\mathcal{R}_\Box$ should have particular properties according to $\mathsf{M}(\Box)$ as indicated in Fig. 1. In particular, if some $(\textsc{d}_X) \in \mathsf{M}(\Box)$, then $\mathcal{R}_\Box$ is serial; and if $(\textsc{t}) \in \mathsf{M}(\Box)$, then $\mathcal{R}_\Box$ is reflexive.[6]
(3) $\mathcal{V}$ is a function assigning a bivaluation $\mathcal{V}_w$ to every $w \in W$, such that $\mathcal{V}(w)(\Box\psi) = \min\{\mathcal{V}(w')(\psi) \mid w' \in \mathcal{R}_\Box[w]\}$ whenever $\Box\psi \in dom(\mathcal{V}(w))$ and $\psi \in dom(\mathcal{V}(w'))$ for every $w' \in \mathcal{R}_\Box[w]$.[7]

If $dom(\mathcal{V}_w) = \mathcal{F}$ for every $w \in W$, we call $\mathcal{W}$ an $\mathcal{F}$*-biframe* for M.

Notation 6.7. Let $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V} \rangle$ be a biframe for M. For a set $W' \subseteq W$, we write $\mathcal{V}_{W'}(\psi)$ to denote $\min\{\mathcal{V}_{w'}(\psi) \mid w' \in W'\}$. This notation is extended to sequents and sets of sequents in the natural way (e.g., $\mathcal{V}_{W'}(S) = \min\{\mathcal{V}_{w'}(s) \mid s \in S, w' \in W'\}$). In addition, we denote by $dom(\mathcal{W})$ the intersection of all sets $dom(\mathcal{V}_w)$ for every $w \in W$.

In particular, we have $\mathcal{V}_w(\Box\psi) = \mathcal{V}_{\mathcal{R}_\Box[w]}(\psi)$ for every $w \in \mathcal{W}$ and $\psi, \Box\psi \in \mathcal{L}_{\boxdot}$ such that $\Box\psi \in dom(\mathcal{V}_w)$ and $\psi \in dom(\mathcal{V}_{w'})$ for every $w' \in \mathcal{R}_\Box[w]$.

Next, we adopt the semantic viewpoint of pure rules in order to retain the connection between sequent calculi and their semantics.

Definition 6.8. A biframe $\langle W, \mathcal{R}, \mathcal{V} \rangle$ for M is called **G**-*legal* for an $\mathcal{L}$-calculus **G** if $\mathcal{V}_w$ is **G**-legal for every $w \in W$ (see Def. 2.9).

[6] An accessibility relation R is called *transitive* if wRu and uRv imply wRv; *symmetric* if wRu implies uRw; *functional* if wRu and wRv imply $u = v$; *euclidian* if wRu and wRv imply uRv; *reflexive* if wRw for every $w \in \mathcal{W}$; and *serial* if for all $w \in W$, we have wRu for some u.

[7] Recall that $\min\ \emptyset = 1$.

We turn to proving soundness and completeness.

We note that the rule (4) and its two variants (45) and (B4) are not sound for every possible set $\mathcal{F}$ of formulas. For example, the sequent $\Box\varphi \Rightarrow \Box\Box\varphi$ is derivable using (4) and (ID), using only formulas from $\{\Box\varphi, \Box\Box\varphi\}$. However, this sequent is not valid in the $\{\Box\varphi, \Box\Box\varphi\}$-biframe $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V}\rangle$, in which $\mathcal{R}_\Box$ is transitive, given by $W = \{w_1, w_2\}$, $\mathcal{R}_\Box = \{\langle w_1, w_2\rangle\}$, $\mathcal{V}_{w_1}(\Box\varphi) = 1$, $\mathcal{V}_{w_1}(\Box\Box\varphi) = 0$, $\mathcal{V}_{w_2}(\Box\varphi) = 0$, $\mathcal{V}_{w_2}(\Box\Box\varphi) = 1$. The reason $\mathcal{W}$ is indeed a biframe is the fact that φ is missing from the domains of the bivaluations. Thus, in the presence of any of the rules (4), (45) and (B4), we require that $\mathcal{F}$ is "closed" with respect to $\boxdot$, that is, $\varphi \in \mathcal{F}$ whenever $\Box\varphi \in \mathcal{F}$ for some $\Box \in \boxdot$.

THEOREM 6.9 (SOUNDNESS). *Let* G *be an $\mathcal{L}$-calculus, $\mathcal{F}$ a set of $\mathcal{L}_\boxdot$-formulas, S a set of $\mathcal{F}$-sequents and s an $\mathcal{F}$-sequent. Suppose that for every $\Box \in \boxdot$, if $\{(4), (45), (B4)\} \cap M(\Box) \neq \emptyset$, then $\psi \in \mathcal{F}$ whenever $\Box\psi \in \mathcal{F}$. If $S \vdash^{\mathcal{F}}_{G_M} s$, then $\mathcal{V}_W(S) \leq \mathcal{V}_W(s)$ for every* G*-legal $\mathcal{F}$-biframe $\langle W, \mathcal{R}, \mathcal{V}\rangle$ for M.*

PROOF. Let $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V}\rangle$ be a G-legal $\mathcal{F}$-biframe for M. Suppose that $\mathcal{V}_W(S) = 1$. We prove that $\mathcal{V}_W(s) = 1$ by induction on the length of the derivation of s from S in G_M (that consists only of $\mathcal{F}$-sequents). If $s \in S$, or s is the conclusion of an application of a non-modal rule, then this is shown like in the proof of Thm. 2.11. If s is the conclusion of an application of some rule in $M(\Box)$, then the proof carries on according to the identity of this rule. We explicitly handle the cases of (K), (4) and (T), leaving the other cases for the reader.

(1) If s is the conclusion of an application of (K) for some $\Box \in \boxdot$, then s has the form $\Box\Gamma \Rightarrow \Box\varphi$ for some $\Gamma \subseteq \mathcal{F}$ and $\varphi \in \mathcal{F}$, and $S \vdash^{\mathcal{F}}_{G_M} \Gamma \Rightarrow \varphi$. Suppose for contradiction that $\mathcal{V}_w(\Box\Gamma \Rightarrow \Box\varphi) = 0$ for some $w \in W$. Then, $\mathcal{V}_w(\Box\varphi) = 0$, and $\mathcal{V}_w(\Box\psi) = 1$ for every $\psi \in \Gamma$. In particular, there exists a world $w' \in R[w]$ such that $\mathcal{V}_{w'}(\varphi) = 0$, and $\mathcal{V}_{w'}(\psi) = 1$ for every $\psi \in \Gamma$, which contradicts the induction hypothesis, according to which $\mathcal{V}_{w'}(\Gamma \Rightarrow \varphi) = 1$.

(2) If s is the conclusion of an application of (4) for some $\Box \in \boxdot$, then s has the form $\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box\varphi$ for some $\Gamma_2 \subseteq \mathcal{F}$, $\varphi \in \mathcal{F}$ and Γ_1 such that $\Box\Gamma_1 \subseteq \mathcal{F}$, and $S \vdash^{\mathcal{F}}_{G_M} \Box\Gamma_1, \Gamma_2 \Rightarrow \varphi$. In particular, $\Gamma_2 \subseteq \mathcal{F}$ and $\varphi \in \mathcal{F}$. In addition, since $(4) \in M(\Box)$, we have $\Gamma_1 \subseteq \mathcal{F}$ as well. Suppose for contradiction that $\mathcal{V}_w(\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box\varphi) = 0$ for some $w \in W$. Then, $\mathcal{V}_w(\Box\varphi) = 0$, and $\mathcal{V}_w(\Box\psi) = 1$ for every $\psi \in \Gamma_1 \cup \Gamma_2$. In particular, there exists a world $w' \in R[w]$ such that $\mathcal{V}_{w'}(\varphi) = 0$, and $\mathcal{V}_{w'}(\psi) = 1$ for every $\psi \in \Gamma_2$. Now, let $\psi \in \Gamma_1$ and $w'' \in R[w']$. Since $(4) \in M(\Box)$, we have that $\mathcal{R}_\Box$ is transitive, which means that $w'' \in R[w]$. Therefore, $\mathcal{V}_{w''}(\psi) = 1$ for every such w'', and hence $\mathcal{V}_{w'}(\Box\psi) = 1$ for every $\psi \in \Gamma_1$. We therefore have $\mathcal{V}_{w'}(\Box\Gamma_1, \Gamma_2 \Rightarrow \varphi) = 0$, contradicting the induction hypothesis.

(3) If s is the conclusion of an application of (T) for some $\Box \in \boxdot$, then s has the form $\Gamma, \Box\varphi \Rightarrow \Delta$ for some $\Gamma, \Delta \subseteq \mathcal{F}$ and $\varphi \in \mathcal{F}$, and $S \vdash^{\mathcal{F}}_{G_M} \Gamma, \varphi \Rightarrow \Delta$. Let $w \in W$. By the induction hypothesis, $\mathcal{V}_w(\Gamma, \varphi \Rightarrow \Delta) = 1$, which means that either $\mathcal{V}_w(\psi) = 0$ for some $\psi \in \Gamma$, $\mathcal{V}_w(\psi) = 1$ for some $\psi \in \Delta$, or $\mathcal{V}_w(\varphi) = 0$. In the first two cases, we have $\mathcal{V}_w(\Gamma, \Box\varphi \Rightarrow \Delta) = 1$ as well. In the third case, note that since $(T) \in M(\Box)$ we have that $\mathcal{R}_\Box$ is reflexive. This, together with the fact that $\mathcal{V}_w(\varphi) = 0$, means that $\mathcal{V}_w(\Box\varphi) = 0$, and hence $\mathcal{V}_w(\Gamma, \Box\varphi \Rightarrow \Delta) = 1$. □

We turn to completeness. Here, we follow the canonical construction of a countermodel, whose worlds are maximal unprovable sequents, but adjust it to the case where only formulas from a certain set $\mathcal{F}$ are allowed in derivations. When $\mathcal{F}$ is infinite, this requires us to use ω-sequents (defined as in the proof of Thm. 2.11).

THEOREM 6.10 (COMPLETENESS). *Let* G *be an $\mathcal{L}$-calculus, $\mathcal{F}$ a set of $\mathcal{L}_\boxdot$-formulas, S a set of $\mathcal{F}$-sequents and s an $\mathcal{F}$-sequent. If $S \nvdash^{\mathcal{F}}_{G_M} s$, then $\mathcal{V}_W(S) > \mathcal{V}_W(s)$ for some* G*-legal $\mathcal{F}$-biframe $\langle W, \mathcal{R}, \mathcal{V}\rangle$ for M.*

PROOF. We say that an ω-sequent $L \Rightarrow R$ is *M-S-$\mathcal{F}$-maximal unprovable* if the following hold:

- $L \cup R \subseteq \mathcal{F}$
- $S \nvdash^{\mathcal{F}}_{\mathsf{G_M}} L \Rightarrow R$
- $S \vdash^{\mathcal{F}}_{\mathsf{G_M}} L, \psi \Rightarrow R$ for every $\psi \in \mathcal{F} \setminus L$
- $S \vdash^{\mathcal{F}}_{\mathsf{G_M}} L \Rightarrow \psi, R$ for every $\psi \in \mathcal{F} \setminus R$

We denote the set of M-S-$\mathcal{F}$ maximal unprovable ω-sequents by $W(\mathsf{M}, S, \mathcal{F})$. Using (ID) and (CUT), it is easy to see that $L \cup R = \mathcal{F}$ and $L \cap R = \emptyset$ for every $L \Rightarrow R \in W(\mathsf{M}, S, \mathcal{F})$. In addition, it is a routine matter to show that every ω-sequent $L \Rightarrow R$ such that $L \cup R \subseteq \mathcal{F}$ and $S \nvdash^{\mathcal{F}}_{\mathsf{G_M}} L \Rightarrow R$ can be extended to an M-S-$\mathcal{F}$-maximal unprovable ω-sequent.

For every $L \Rightarrow R \in W(\mathsf{M}, S, \mathcal{F})$ and $\Box \in \mathsf{M}$, let

$$A^{\Box}_{L \Rightarrow R} = \{L' \Rightarrow R' \in W(\mathsf{M}, S, \mathcal{F}) \mid L^{\Box}_1 \cup L^{\Box}_2 \subseteq L' \wedge R^{\Box}_1 \cup R^{\Box}_2 \cup R^{\Box}_3 \subseteq R'\}$$

where $L^{\Box}_1, L^{\Box}_2, R^{\Box}_1, R^{\Box}_2$, and $R^{\Box}_3$ are given by:

$$L^{\Box}_1 = \{\varphi \in \mathcal{F} \mid \Box\varphi \in L\}$$

$$L^{\Box}_2 = \begin{cases} \Box\mathcal{F} \cap L & \{(4), (45), (\textsc{b4})\} \cap \mathsf{M}(\Box) \neq \emptyset \\ \emptyset & \textit{otherwise} \end{cases} \qquad R^{\Box}_1 = \begin{cases} \Box\mathcal{F} \cap R & \{(45), (\textsc{b4})\} \cap \mathsf{M}(\Box) \neq \emptyset \\ \emptyset & \textit{otherwise} \end{cases}$$

$$R^{\Box}_2 = \begin{cases} \mathcal{F} \cap \Box R & \{(\textsc{b}), (\textsc{b4})\} \cap \mathsf{M}(\Box) \neq \emptyset \\ \emptyset & \textit{otherwise} \end{cases} \qquad R^{\Box}_3 = \begin{cases} \{\varphi \in \mathcal{F} \mid \Box\varphi \in R\} & (\textsc{pf}) \in \mathsf{M}(\Box) \\ \emptyset & \textit{otherwise} \end{cases}$$

Using these definitions, we define the following countermodel $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V} \rangle$, where:

1. $W = W(\mathsf{M}, S, \mathcal{F})$.
2. for every $\Box \in \boxdot$, we define $\mathcal{R}_{\Box}$ by specifying the set $\mathcal{R}_{\Box}[L \Rightarrow R]$ for every $L \Rightarrow R \in W$:
   (a) if (PF) $\notin \mathsf{M}(\Box)$ then $\mathcal{R}_{\Box}[L \Rightarrow R]$ is $A^{\Box}_{L \Rightarrow R}$.
   (b) If (PF) $\in \mathsf{M}(\Box)$ then $\mathcal{R}_{\Box}[L \Rightarrow R]$ consists of a single arbitrary element from $A^{\Box}_{L \Rightarrow R}$, unless $A^{\Box}_{L \Rightarrow R}$ is empty, in which case so is $\mathcal{R}_{\Box}[L \Rightarrow R]$.
3. For every $\psi \in \mathcal{F}$ and $L \Rightarrow R \in W$, $\mathcal{V}_{L \Rightarrow R}(\psi) = 1$ if $\psi \in L$ and $\mathcal{V}_{L \Rightarrow R}(\psi) = 0$ otherwise.

We first show that $\mathcal{V}_W(S) > \mathcal{V}_W(s)$. For every $\Gamma \Rightarrow \Delta \in S$ and $L \Rightarrow R \in W$, since $S \vdash^{\mathcal{F}}_{\mathsf{G_M}} \Gamma \Rightarrow \Delta$ and $S \nvdash^{\mathcal{F}}_{\mathsf{G_M}} L \Rightarrow R$, there exist some $\psi \in \Gamma \setminus L$ (and then $\mathcal{V}_{L \Rightarrow R}(\psi) = 0$) or $\psi \in \Delta \setminus R$ (and then $\mathcal{V}_{L \Rightarrow R}(\psi) = 1$). Either way, $\mathcal{V}_{L \Rightarrow R}(\Gamma \Rightarrow \Delta) = 1$. In addition, since $s \subseteq L_s \Rightarrow R_s$ for some $L_s \Rightarrow R_s \in W(\mathsf{M}, S, \mathcal{F})$, we have $\mathcal{V}_{L_s \Rightarrow R_s}(s) = 0$.

It remains to prove that $\mathcal{W}$ is a **G**-legal $\mathcal{F}$-biframe for M.

- biframe: let $\Box \in \boxdot$ and $\psi, \Box\psi \in \mathcal{F}$. Let $L \Rightarrow R \in W$. If $\mathcal{V}_{L \Rightarrow R}(\Box\psi) = 1$ and $L' \Rightarrow R' \in \mathcal{R}_{\Box}[L \Rightarrow R]$, then we have $\Box\psi \in L$, which means that $\psi \in L'$, and hence $\mathcal{V}_{L' \Rightarrow R'}(\psi) = 1$. For the converse, suppose that $\mathcal{V}_{L \Rightarrow R}(\Box\psi) = 0$. Then, $\Box\psi \in R$. We prove that $S \nvdash^{\mathcal{F}}_{\mathsf{G_M}} L^{\Box}_1, L^{\Box}_2 \Rightarrow \psi, R^{\Box}_1, R^{\Box}_2, R^{\Box}_3$, extend this sequent to an element $L' \Rightarrow R'$ of $\mathcal{R}_{\Box}[L \Rightarrow R]$, and then obtain that $\mathcal{V}_{L' \Rightarrow R'}(\psi) = 0$ (as $\psi \in R'$). Assume for contradiction that $S \vdash^{\mathcal{F}}_{\mathsf{G_M}} L^{\Box}_1, L^{\Box}_2 \Rightarrow \psi, R^{\Box}_1, R^{\Box}_2, R^{\Box}_3$. Then there exist finite $\Gamma_1 \subseteq L^{\Box}_1$, $\Gamma_2 \subseteq L^{\Box}_2$, $\Delta_1 \subseteq R^{\Box}_1$, $\Delta_2 \subseteq R^{\Box}_2$ and $\Delta_3 \subseteq R^{\Box}_3$, such that $S \vdash^{\mathcal{F}}_{\mathsf{G_M}} \Gamma_1, \Gamma_2 \Rightarrow \psi, \Delta_1, \Delta_2, \Delta_3$. Let $\Delta'_2 = \{\varphi \in \mathcal{F} \mid \Box\varphi \in \Delta_2\}$. By applying the only rule in $\mathsf{M}(\Box) \cap \{(\textsc{k}), (4), (45), (\textsc{b}), (\textsc{b4}), (\textsc{pf})\}$, we obtain $S \vdash^{\mathcal{F}}_{\mathsf{G_M}} \Box\Gamma_1, \Gamma_2 \Rightarrow \Box\psi, \Delta_1, \Delta'_2, \Box\Delta_3$. Clearly, $\Box\Gamma_1, \Gamma_2 \Rightarrow \Box\psi, \Delta_1, \Delta'_2, \Box\Delta_3 \subseteq L \Rightarrow R$, and so $S \vdash^{\mathcal{F}}_{\mathsf{G_M}} L \Rightarrow R$, which is a contradiction. Now, $L^{\Box}_1, L^{\Box}_2 \Rightarrow \psi, R^{\Box}_1, R^{\Box}_2, R^{\Box}_3$ can be extended to some element $L' \Rightarrow R'$ of W, and every such extension is an element of $A^{\Box}_{L \Rightarrow R}$. Thus we have some $L' \Rightarrow R' \in \mathcal{R}_{\Box}[L \Rightarrow R]$ that extends $L^{\Box}_1, L^{\Box}_2 \Rightarrow R^{\Box}_1, R^{\Box}_2, R^{\Box}_3$. In particular, since $\Box\psi \in R$, we must have $\psi \in R'$, and so $\mathcal{V}_{L' \Rightarrow R'}(\psi) = 0$.

- for M: let $\Box \in \boxdot$. We show that $\mathcal{R}_\Box$ has the properties that are induced by M. We separately consider each of the cases:
  - Suppose that (DX) $\in$ M($\Box$) for some X. We show that $\mathcal{R}_\Box$ is serial. Similarly to the proof above that $S \nvdash^{\mathcal{F}}_{\mathrm{G_M}} L_1^\Box, L_2^\Box \Rightarrow \psi, R_1^\Box, R_2^\Box, R_3^\Box$, it can be shown that $S \nvdash^{\mathcal{F}}_{\mathrm{G_M}} L_1^\Box, L_2^\Box \Rightarrow R_1^\Box, R_2^\Box, R_3^\Box$, by applying (DX) rather than X, for the only (DX) $\in$ M($\Box$), and that $L_1^\Box, L_2^\Box \Rightarrow R_1^\Box, R_2^\Box, R_3^\Box$ can be extended to some element $L' \Rightarrow R'$ in W such that $(L \Rightarrow R)\mathcal{R}_\Box(L' \Rightarrow R')$.
  - Suppose that (T) $\in$ M($\Box$). We show that $\mathcal{R}_\Box$ is reflexive. Let $L \Rightarrow R \in W$. We show that $(L \Rightarrow R)\mathcal{R}_\Box(L \Rightarrow R)$, that is, $L_1^\Box, L_2^\Box \Rightarrow R_1^\Box, R_2^\Box, R_3^\Box \subseteq L \Rightarrow R$. Let $\psi \in L_1^\Box$, and assume for contradiction that $\psi \notin L$, that is, $\psi \in R$. Since $\psi \in L_1^\Box$, we have that $\Box\psi \in L$, and therefore, $\Box\psi \Rightarrow \psi \subseteq L \Rightarrow R$, which is impossible, as (T) $\in$ M($\Box$). The facts that $L_2^\Box \subseteq L$ and $R_1^\Box \subseteq R$ are trivial. Now let $\psi \in R_2^\Box$, and assume for contradiction that $\psi \notin R$, that is, $\psi \in L$. Since $\psi \in R_2^\Box$, we have that $\psi = \Box\psi'$ for some $\psi' \in R$, and that $\psi \in \mathcal{F}$. This means that $\psi \Rightarrow \psi' \subseteq L \Rightarrow R$, which is again impossible by the presence of (T) in M($\Box$). Finally, since (T) $\in$ M($\Box$), we have that (PF) $\notin$ M($\Box$), which means that $R_3^\Box = \emptyset \subseteq R$.

  In the following items, $L_a \Rightarrow R_a$, $L_b \Rightarrow R_b$ and $L_c \Rightarrow R_c$ denote arbitrary elements of W.
  - Suppose that (4) $\in$ M($\Box$). We show that $\mathcal{R}_\Box$ is transitive. Suppose that $(L_a \Rightarrow R_a)\mathcal{R}_\Box(L_b \Rightarrow R_b)$ and $(L_b \Rightarrow R_b)\mathcal{R}_\Box(L_c \Rightarrow R_c)$. We prove that $(L_a \Rightarrow R_a)\mathcal{R}_\Box(L_c \Rightarrow R_c)$, that is, $(L_a)_1^\Box, (L_a)_2^\Box \Rightarrow (R_a)_1^\Box, (R_a)_2^\Box, (R_a)_3^\Box \subseteq L_c \Rightarrow R_c$. Since (4) $\in$ M($\Box$), we have $(R_a)_1^\Box = (R_a)_2^\Box = (R_a)_3^\Box = \emptyset$. Now, let $\psi \in (L_a)_1^\Box$. Then both $\psi \in \mathcal{F}$ and $\Box\psi \in L_a$, which means that $\Box\psi \in (L_a)_2^\Box \subseteq L_b$. Together with the fact that $\psi \in \mathcal{F}$, we have $\psi \in (L_b)_1^\Box \subseteq L_c$. Next, let $\psi \in (L_a)_2^\Box$. Then $\psi = \Box\psi'$ for some $\psi' \in \mathcal{F}$, and $\psi \in L_b$. Therefore, $\psi \in (L_b)_2^\Box \subseteq L_c$.
  - Suppose that (45) $\in$ M($\Box$). We show that $\mathcal{R}_\Box$ is transitive and euclidian.
    * Transitivity: suppose that $(L_a \Rightarrow R_a)\mathcal{R}_\Box(L_b \Rightarrow R_b)$ and $(L_b \Rightarrow R_b)\mathcal{R}_\Box(L_c \Rightarrow R_c)$. We prove that $(L_a \Rightarrow R_a)\mathcal{R}_\Box(L_c \Rightarrow R_c)$, that is, $(L_a)_1^\Box, (L_a)_2^\Box \Rightarrow (R_a)_1^\Box, (R_a)_2^\Box, (R_a)_3^\Box \subseteq L_c \Rightarrow R_c$. Since (45) $\in$ M($\Box$), we have $(R_a)_2^\Box = (R_a)_3^\Box = \emptyset$. Similarly to the case of (4), $(L_a)_1^\Box, (L_a)_2^\Box \subseteq (L_c)$. Now let $\psi \in (R_a)_1^\Box$. Then $\psi = \Box\psi'$ for some $\psi' \in \mathcal{F}$ and $\psi \in R_b$. Therefore, $\psi \in (R_b)_1^\Box \subseteq R_c$.
    * Euclideaness: suppose that $(L_a \Rightarrow R_a)\mathcal{R}_\Box(L_b \Rightarrow R_b)$ and $(L_a \Rightarrow R_a)\mathcal{R}_\Box(L_c \Rightarrow R_c)$. We prove that $(L_b \Rightarrow R_b)\mathcal{R}_\Box(L_c \Rightarrow R_c)$, that is, $(L_b)_1^\Box, (L_b)_2^\Box \Rightarrow (R_b)_1^\Box, (R_b)_2^\Box, (R_b)_3^\Box \subseteq L_c \Rightarrow R_c$. Since (45) $\in$ M($\Box$), we have $(R_b)_2^\Box = (R_b)_3^\Box = \emptyset$. Let $\psi \in (L_b)_1^\Box$. Then $\Box\psi \in L_b$ and $\psi \in \mathcal{F}$. Hence $\Box\psi \notin R_b$, and therefore $\Box\psi \notin (R_a)_1^\Box$. Since we have $\psi \in \mathcal{F}$, this means that $\Box\psi \notin R_a$, and hence also $\Box\psi \in L_a$. Again, since $\psi \in \mathcal{F}$, $\Box\psi \in (L_a)_2^\Box \subseteq L_c$. Next, let $\psi \in (L_b)_2^\Box$. Then $\psi = \Box\psi'$ for some $\psi' \in \mathcal{F}$ and $\psi \in L_b$. In particular, $\psi \notin R_b$. Since $(R_a)_1^\Box \subseteq R_b$, we also have $\psi \notin (R_a)_1^\Box$. Together with the fact that $\psi' \in \mathcal{F}$, we have $\psi \notin R_a$. This, in turn, means that $\psi \in L_a$, which, together with $\psi' \in \mathcal{F}$, means that $\psi \in (L_a)_2^\Box \subseteq L_c$. The fact that $(R_b)_1^\Box \subseteq R_c$ is proven symmetrically.
  - Suppose that (B) $\in$ M($\Box$). We show that $\mathcal{R}_\Box$ is symmetric. Suppose that $(L_a \Rightarrow R_a)\mathcal{R}_\Box(L_b \Rightarrow R_b)$. We prove that $(L_b \Rightarrow R_b)\mathcal{R}_\Box(L_a \Rightarrow R_a)$, that is, $(L_b)_1^\Box, (L_b)_2^\Box \Rightarrow (R_b)_1^\Box, (R_b)_2^\Box, (R_b)_3^\Box \subseteq L_a \Rightarrow R_a$. Since (B) $\in$ M($\Box$), we have $(L_b)_2^\Box = (R_b)_1^\Box = (R_b)_3^\Box = \emptyset$. Let $\psi \in (L_b)_1^\Box$. Then $\Box\psi \in L_b \subseteq \mathcal{F}$, and hence $\Box\psi \notin R_b$, and in particular, $\Box\psi \notin (R_a)_2^\Box$. Since $\Box\psi \in \mathcal{F}$, we have also $\psi \notin R_a$, which means that $\psi \in L_a$. Next, let $\psi \in (R_b)_2^\Box$. Then $\psi = \Box\psi'$ for some $\psi' \in R_b \subseteq \mathcal{F}$. Hence $\psi' \notin L_b$, and in particular, $\psi' \notin (L_a)_1^\Box$. Since $\psi' \in \mathcal{F}$, we also have $\psi \notin L_a$, which means that $\psi \in R_a$.
  - Suppose that (B4) $\in$ M($\Box$). We show that $\mathcal{R}_\Box$ is transitive and symmetric.
    * Transitivity: suppose that $(L_a \Rightarrow R_a)\mathcal{R}_\Box(L_b \Rightarrow R_b)$ and $(L_b \Rightarrow R_b)\mathcal{R}_\Box(L_c \Rightarrow R_c)$. We prove that $(L_a \Rightarrow R_a)\mathcal{R}_\Box(L_c \Rightarrow R_c)$, that is, $(L_a)_1^\Box, (L_a)_2^\Box \Rightarrow (R_a)_1^\Box, (R_a)_2^\Box, (R_a)_3^\Box \subseteq$

$L_c \Rightarrow R_c$. First, note that $(R_a)^{\Box}_3 = \emptyset$. Second, $(L_a)^{\Box}_1, (L_a)^{\Box}_2 \subseteq L_c$ and $(R_a)^{\Box}_1 \subseteq R_c$ are shown similarly to the case of (45). Let $\psi \in (R_a)^{\Box}_2 \subseteq R_b$. Then $\psi \in \mathcal{F}$, and $\psi = \Box\psi'$ for some $\psi' \in \mathcal{F}$. Hence $\psi \in (R_b)^{\Box}_1 \subseteq R_c$.

∗ Symmetry: suppose that $(L_a \Rightarrow R_a)\mathcal{R}_{\Box}(L_b \Rightarrow R_b)$. We prove that $(L_b \Rightarrow R_b)\mathcal{R}_{\Box}(L_a \Rightarrow R_a)$, that is, $(L_b)^{\Box}_1, (L_b)^{\Box}_2 \Rightarrow (R_b)^{\Box}_1, (R_b)^{\Box}_2, (R_b)^{\Box}_3 \subseteq L_a \Rightarrow R_a$. First, note that $(R_a)^{\Box}_3 = \emptyset$. Second, $(L_b)^{\Box}_1 \subseteq L_a$ and $(R_b)^{\Box}_2 \subseteq L_a$ are shown similarly to the case of (B). Let $\psi \in (L_b)^{\Box}_2$. Then $\psi \in L_b$, and $\psi = \Box\psi'$ for some $\psi' \in \mathcal{F}$. In particular, $\psi \notin R_b$, and hence also $\psi \notin (R_a)^{\Box}_1$. Together with the fact that $\psi' \in \mathcal{F}$, we have that $\psi \notin R_a$, which means that $\psi \in L_a$. The fact that $(R_b)^{\Box}_1 \subseteq R_a$ is shown symmetrically.

– Suppose that (PF) $\in \mathsf{M}(\Box)$. By definition, $\mathcal{R}_{\Box}$ is functional.

- **G**-legal: For every $L \Rightarrow R \in W$, the bivaluation $\mathcal{V}_{L\Rightarrow R}$ is shown to be **G**-legal similarly to the proof of Thm. 2.11. □

6.3 Analyticity with Modal Operators

In this section we show that in a wide family of calculi ⊚-k-analyticity is preserved when augmenting a pure calculus with rules for the modal operators. Semantics will play a major role here, as what will actually be shown is how to use the ability to extend partial bivaluations in order to extend partial biframes.

We focus on a slightly restricted subfamily of calculi, namely *standard* calculi, thus ruling out some degenerate cases. Roughly speaking, a calculus is called standard if whenever an atomic formula occurs in one of its rules, it also occurs as a subformula in the same rule. This is formally defined as follows:

Definition 6.11. An atomic variable p is called *shared* in a rule r if it is a proper subformula of some formula in the conclusion of r. A rule is called *standard* if all atomic variables that occur in it are shared in it. A calculus is called *standard* if each of its rules is standard.

Example 6.12. All calculi considered in examples above are standard. In contrast, p_3 is not shared in the rule $\Rightarrow p_1, p_3 \ / \ \Rightarrow p_1 \lor p_2$, and so every calculus that includes this rule is not standard. Aside from such contrived examples, we are not aware of a non-standard calculus in the literature.

The main result of this section is:

THEOREM 6.13. *Let* **G** *be a standard* $\mathcal{L}$*-calculus. If* **G** *is* ⊚-k*-analytic then so is* $\mathbf{G}_{\mathsf{M}}$.

Note that if $\mathbf{G}_{\mathsf{M}}$ is ⊚-k-analytic, then **G** must also be ⊚-k-analytic: given that S and s do not include modal operators, $S \vdash_{\mathbf{G}} s$ implies $S \vdash_{\mathbf{G}_{\mathsf{M}}} s$. The ⊚-$k$-analyticity of $\mathbf{G}_{\mathsf{M}}$ then means that there is a derivation of s from S in $\mathbf{G}_{\mathsf{M}}$ that uses only of $sub^{⊚}_k(S \cup \{s\})$-formulas. This derivation cannot contain applications of the modal rules, and hence it is also a derivation in **G**.

Before turning to its proof, we present several examples of applications of Thm. 6.13.

Example 6.14. All sequent calculi for classical modal logics that are obtained from **LK** by the adding the rules of Fig. 1 are known to be analytic. Thm. 6.13 makes this fact a direct consequence of the analyticity of **LK**.

Example 6.15. The quotations employed in primal infon logic [19] are unary connectives of the form q `said`, where q ranges over a finite set of principals. If we take ⊡ to include these connectives and set $\mathsf{M}(q$ `said`$) = \{$(PF), (D$_{\text{PF}}$)$\}$ for every such q, we have $\vdash_{\mathbf{P}_{\mathsf{M}}} \Gamma \Rightarrow \psi$ (see Example 2.4) iff ψ is derivable from Γ in the Hilbert system for primal infon logic given in [19]. (This can be shown by induction on the lengths of the derivations.) Since **P** is standard and analytic, so is $\mathbf{P}_{\mathsf{M}}$. In contrast, the Hilbert system for primal infon logic in [19] admits a similar property that concerns local formulas (see Def. 7.1 below) rather than subformulas. Similarly, quotations can be added to the extension **EP** of **P** (Example 5.14), and the resulting calculus is analytic.

Example 6.16. One can add modal operators to the paraconsistent logic C1 (see Example 2.5), by augmenting the calculus $\mathbf{C_1}$ with one of the rules for modal operators. The $\{\neg\}$-1-analyticity of $\mathbf{C_1}$ will then entail the $\{\neg\}$-1-analyticity of the extended calculus.

Next, we prove Thm. 6.13. We use the soundness and completeness theorems and show how to extend partial biframes into full ones. The general notion of biframes (that allows for different domains in each world) and the predefined semantics of the connectives from $\boxdot$ make the proof more challenging than that of Thm. 3.9. The following definitions are therefore needed. First, we introduce a more delicate technical notion of closure under $\circledcirc$-k-subformulas.

Definition 6.17. A set of $\mathcal{L}_{\boxdot}$-formulas is called *$\circledcirc$-k-closed* if whenever it contains a formula of the form $\circ\varphi$ for some $\circ \in \circledcirc$, it also contains φ, and whenever it contains a formula of the form $\diamond(\varphi_1, \dots, \varphi_n)$ for some $\diamond \in \Diamond_{\mathcal{L}} \setminus \circledcirc$ it also contains $\circledcirc^{\leq k}\psi_i$ for every $1 \leq i \leq n$.

Note that every set that is closed under $\circledcirc$-k-subformulas is also $\circledcirc$-k-closed. However, since ψ is a subformula of $\Box\psi$ (i.e., $\psi \in sub_k^{\circledcirc}(\Box\psi)$ for any $\psi \in \mathcal{L}_{\boxdot}$, $\circledcirc \subseteq \Diamond^1_{\mathcal{L}}$, $k > 0$ and $\Box \in \boxdot$), the converse may not hold. For example, $\{(\Box p_1) \wedge (\Box p_1), \Box p_1\}$ is $\emptyset$-k-closed for any k, but it is not closed under $\emptyset$-k-subformulas, as p_1 is missing.

Next, we define *$\circledcirc$-k-closed biframes*:

Definition 6.18. A biframe $\langle W, \mathcal{R}, \mathcal{V}\rangle$ for M is *$\circledcirc$-k-closed* if the following hold for every $w \in W$:

- $dom(\mathcal{V}_w)$ is $\circledcirc$-k-closed and finite.
- For every $\Box \in \boxdot$, if $\Box\psi \in dom(\mathcal{V}_w)$, then $\psi \in dom(\mathcal{V}_{w'})$ for every $w' \in \mathcal{R}_{\Box}[w]$.

Similarly to the case of pure calculi, the ability to extend partial models is essential also when introducing modal operators. We thus explicitly define what it means to extend a biframe.

Definition 6.19. A biframe $\langle W, \mathcal{R}, \mathcal{V}\rangle$ for M *extends* a biframe $\langle W', \mathcal{R}', \mathcal{V}'\rangle$ for M if $W = W'$, $\mathcal{R} = \mathcal{R}'$, and $\mathcal{V}_w$ extends $\mathcal{V}'_w$ (i.e., $\mathcal{V}_w(\psi) = \mathcal{V}'_w(\psi)$ whenever $\mathcal{V}'_w(\psi)$ is defined) for every $w \in W$.

Finally, the main part of the proof of Thm. 6.13 is the following lemma, which is proven in the next section. The theorem immediately follows from this lemma using Theorems 6.9 and 6.10.

Lemma 6.20. *Let* **G** *be a standard $\circledcirc$-k-analytic $\mathcal{L}$-calculus and $\mathcal{W}$ a* **G***-legal $\circledcirc$-k-closed biframe for M. Then, $\mathcal{W}$ can be extended to a* **G***-legal $\mathcal{L}_{\boxdot}$-biframe for M.*

Before proving the lemma, we use it to prove Thm. 6.13.

Proof of Theorem 6.13. Suppose that $S \vdash_{\mathbf{G}_\mathrm{M}} s$. Let S' be a finite subset of S such that $S' \vdash_{\mathbf{G}_\mathrm{M}} s$. We prove that $S' \vdash^{\mathcal{F}}_{\mathbf{G}_\mathrm{M}} s$ for $\mathcal{F} = sub_k^{\circledcirc}(S' \cup \{s\})$ (and so $S \vdash^{\mathcal{F}}_{\mathbf{G}_\mathrm{M}} s$). Otherwise, by Thm. 6.10, there exists a **G**-legal $\mathcal{F}$-biframe $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V}\rangle$ for M such that $\mathcal{V}_W(S') > \mathcal{V}_w(s)$. $\mathcal{W}$ is $\circledcirc$-k-closed, and by Lemma 6.20, it can be extended to a **G**-legal $\mathcal{L}_{\boxdot}$-biframe $\mathcal{W}' = \langle W, \mathcal{R}, \mathcal{V}'\rangle$ for M. After this extension, we still have $\mathcal{V}'_W(S') > \mathcal{V}'_w(s)$. By Thm. 6.9, we have $S' \nvdash_{\mathbf{G}_\mathrm{M}} s$, which is a contradiction. □

Note that Lemma 6.20 will also be used in the next section, where we extend the reduction of §4.

6.4 Proof of Lemma 6.20

Lemma 6.20 allows one to extend partial biframes into full ones. For the extension method that we propose here, the following property of $\circledcirc$-k-closed sets is useful:

Proposition 6.21. *If $\mathcal{F} \subseteq \mathcal{L}_{\boxdot}$ is $\circledcirc$-k-closed and $\varphi \in \mathcal{L}$ is a $\circledcirc$-k-subformula of $\psi \in \mathcal{L}$, then $\sigma(\psi) \in \mathcal{F}$ implies $\sigma(\varphi) \in \mathcal{F}$.*

Our extension method is gradual: We add all formulas of the language to the domain of the biframe, not one by one—but many at a time. The following three lemmas establish the required ingredients for the full extension construction.

For the rest of this section, let **G** be a standard $\circledcirc$-k-analytic $\mathcal{L}$-calculus.

Lemma 6.22. *Let $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V}\rangle$ be a* **G***-legal $\circledcirc$-k-closed biframe for* M*. Given $p \in At$, $\mathcal{W}$ can be extended to a* **G***-legal $\circledcirc$-k-closed biframe $\mathcal{W}'$ for* M*, such that $p \in dom(\mathcal{W}')$.*

Proof. Let $\mathcal{W}' = \langle W, \mathcal{R}, \mathcal{V}'\rangle$, where $\mathcal{V}'$ is the function assigning to every $w \in W$, the $dom(\mathcal{V}_w) \cup \{p\}$-bivaluation $\mathcal{V}'_w$ obtained by extending $\mathcal{V}_w$ with the value 0 (say) for p whenever $p \notin dom(\mathcal{V}_w)$. Clearly, $\mathcal{W}'$ is a $\circledcirc$-k-closed biframe for M that extends $\mathcal{W}$, and $p \in dom(\mathcal{W}')$. It remains to show that $\mathcal{W}'$ is **G**-legal. Let $w \in W$, $s_1, \dots, s_n / s \in$ **G**, $s'_1 \subseteq s_1, \dots, s'_n \subseteq s_n$, and σ a substitution such that $\sigma(frm(\{s'_1, \dots, s'_n, s\})) \subseteq dom(\mathcal{V}'_w)$. We prove that $\mathcal{V}'_w(\sigma(\{s'_1, \dots, s'_n\})) \leq \mathcal{V}'_w(\sigma(s))$. If $p \notin \sigma(frm(\{s'_1, \dots, s'_n, s\}))$ or $p \in dom(\mathcal{V}_w)$, then this follows from the fact that $\mathcal{V}_w$ is **G**-legal. The fact that **G** is standard entails that these are actually the only two options for p. Indeed, if $p \in \sigma(frm(\{s'_1, \dots, s'_n, s\}))$, then $p = \sigma(p')$ for some atomic variable $p' \in frm(\{s'_1, \dots, s'_n, s\})$. Since **G** is standard, p' is a proper subformula of some $\varphi \in frm(s)$. Since $\sigma(\varphi) \in dom(\mathcal{V}'_w)$ and $\sigma(\varphi) \neq p$, we have $\sigma(\varphi) \in dom(\mathcal{V}_w)$. By Prop. 6.21, $p \in dom(\mathcal{V}_w)$. □

Lemma 6.23. *Let $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V}\rangle$ be a* **G***-legal $\circledcirc$-k-closed biframe for* M*. Then, $\mathcal{W}$ can be extended to a* **G***-legal $\circledcirc$-k-closed biframe $\mathcal{W}'$ for* M*, such that $\boxdot dom(\mathcal{W}) \subseteq dom(\mathcal{W}')$.*

Proof. For every $w \in W$, let $\mathcal{F}_w = dom(\mathcal{V}_w) \cup \boxdot dom(\mathcal{W})$. Let $\mathcal{W}' = \langle W, \mathcal{R}, \mathcal{V}'\rangle$, where $\mathcal{V}'$ is the function assigning to every $w \in W$, the $\mathcal{F}_w$-bivaluation $\mathcal{V}'_w$ defined by:

$$\mathcal{V}'_w(\psi) = \begin{cases} \mathcal{V}_w(\psi) & \psi \in dom(\mathcal{V}_w) \\ \mathcal{V}_{\mathcal{R}_\Box[w]}(\varphi) & \psi = \Box\varphi \in \mathcal{F}_w \setminus dom(\mathcal{V}_w) \end{cases}$$

We show first that $\mathcal{W}'$ is a biframe for M. Let $w \in W$. Let $\Box\psi \in dom(\mathcal{V}'_w)$ such that $\psi \in dom(\mathcal{V}'_{w'})$ for every $w' \in \mathcal{R}_\Box[w]$. If $\Box\psi \in dom(\mathcal{V}_w)$, then since $\mathcal{W}$ is $\circledcirc$-k-closed, $\psi \in dom(\mathcal{V}_{w'})$ for every $w' \in \mathcal{R}_\Box[w]$. Hence since $\mathcal{W}$ is a biframe for M, $\mathcal{V}'_w(\Box\psi) = \mathcal{V}_w(\Box\psi) = \mathcal{V}_{\mathcal{R}_\Box[w]}(\psi) = \mathcal{V}'_{\mathcal{R}_\Box[w]}(\psi)$. If $\Box\psi \notin dom(\mathcal{V}_w)$, then by the definition of $\mathcal{V}'$ in this case, $\mathcal{V}'_w(\Box\psi) = \mathcal{V}_{\mathcal{R}_\Box[w]}(\psi) = \mathcal{V}'_{\mathcal{R}_\Box[w]}(\psi)$.

Obviously, $\mathcal{W}'$ extends $\mathcal{W}$ and $\boxdot dom(\mathcal{W}) \subseteq dom(\mathcal{W}')$. It remains to show that $\mathcal{W}'$ is $\circledcirc$-k-closed and **G**-legal.

(1) $\circledcirc$-k-closed: For every $w \in W$, $dom(\mathcal{V}_w)$ is $\circledcirc$-k-closed and finite. Since we only added a finite number of formulas, all from $\boxdot\mathcal{L}_{\boxdot}$, $dom(\mathcal{V}'_w)$ is also $\circledcirc$-k-closed and finite for every $w \in W$. Now, suppose that $\Box\psi \in dom(\mathcal{V}'_w)$. If $\Box\psi \in dom(\mathcal{V}_w)$, then $\psi \in dom(\mathcal{V}'_{w'})$ for every $w' \in \mathcal{R}_\Box[w]$ since $\mathcal{W}$ is $\circledcirc$-k-closed. If $\Box\psi \notin dom(\mathcal{V}_w)$, then $\psi \in dom(\mathcal{W}) \subseteq dom(\mathcal{W}')$, and in particular $\psi \in dom(\mathcal{V}'_{w'})$ for every $w' \in \mathcal{R}_\Box[w]$.

(2) **G**-legal: Let $w \in W$, $s_1, \dots, s_n / s \in$ **G**, $s'_1 \subseteq s_1, \dots, s'_n \subseteq s_n$, and σ a substitution such that $\sigma(frm(\{s'_1, \dots, s'_n, s\})) \subseteq dom(\mathcal{V}'_w)$. We prove that $\sigma(frm(\{s'_1, \dots, s'_n, s\})) \subseteq dom(\mathcal{V}_w)$, and then $\mathcal{V}'_w(\sigma(\{s'_1, \dots, s'_n\})) \leq \mathcal{V}'_w(\sigma(s))$ follows from the fact that $\mathcal{W}$ is **G**-legal. Indeed, let $\psi \in \sigma(frm(\{s'_1, \dots, s'_n, s\}))$. If $\psi \notin \boxdot\mathcal{L}_{\boxdot}$, then $\psi \in dom(\mathcal{V}_w)$. Otherwise, $\psi = \sigma(p)$ for some atomic variable $p \in frm(\{s'_1, \dots, s'_n, s\})$. Since **G** is standard, p is a proper subformula of some compound $\mathcal{L}$-formula $\varphi \in frm(s)$. Since φ is a compound $\mathcal{L}$-formula, we have $\sigma(\varphi) \notin \boxdot\mathcal{L}_{\boxdot}$, and hence $\sigma(\varphi) \in dom(\mathcal{V}_w)$. By Prop. 6.21, since $dom(\mathcal{V}_w)$ is $\circledcirc$-k-closed, $\psi \in dom(\mathcal{V}_w)$. □

Lemma 6.24. *Let* $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V} \rangle$ *be a* **G**-*legal* $\circledcirc$-k-*closed biframe for* M. *Then,* $\mathcal{W}$ *can be extended to a* **G**-*legal* $\circledcirc$-k-*closed biframe* $\mathcal{W}'$ *for* M*, such that* $\circledcirc dom(\mathcal{W}) \subseteq dom(\mathcal{W}')$*, and for every* $\diamond \in \diamond^n_{\mathcal{L}} \setminus \circledcirc$*,* $\diamond(\varphi_1, \ldots, \varphi_n) \in dom(\mathcal{W}')$ *whenever* $\circledcirc^{\leq k}\{\varphi_1, \ldots, \varphi_n\} \subseteq dom(\mathcal{W})$*.*

Proof. We define $\mathcal{W}'$ in several steps.

Embedding $\mathcal{L}$ in $\mathcal{L}_{\boxdot}$: Let σ_0 be some bijection from At to $At \cup \boxdot\mathcal{L}_{\boxdot}$. As a substitution, σ_0 is naturally extended to apply on all $\mathcal{L}$-formulas. It is straightforward to verify that its extension is a bijection from $\mathcal{L}$ to $\mathcal{L}_{\boxdot}$.

Translating $\mathcal{V}$: For every $w \in W$, let $\mathcal{F}_w = \{\varphi \in \mathcal{L} \mid \sigma_0(\varphi) \in dom(\mathcal{V}_w)\}$. By Prop. 6.21 and the fact that $\mathcal{W}$ is $\circledcirc$-k-closed, we have that $\mathcal{F}_w$ is closed under $\circledcirc$-k-subformulas for every $w \in W$. Since σ_0 is a bijection, we also have that $\mathcal{F}_w$ is finite for every $w \in W$. Now, for every $w \in W$, let u_w be the $\mathcal{F}_w$-bivaluation given by $u_w = \lambda\varphi \in \mathcal{F}_w.\ \mathcal{V}_w(\sigma_0(\varphi))$. We show that u_w is **G**-legal for every $w \in W$. Let $w \in W$, $s_1, \ldots, s_n \,/\, s \in \mathbf{G}$, $s'_1 \subseteq s_1, \ldots, s'_n \subseteq s_n$, and σ a substitution such that $\sigma(\{s'_1, \ldots, s'_n, s\})) \subseteq \mathcal{F}_w$. We prove that $u_w(\sigma(\{s'_1, \ldots, s'_n\})) \leq u_w(\sigma(s))$. Consider the substitution $\sigma' = \sigma_0 \circ \sigma$. It is easy to see that $\sigma'(\varphi) = \sigma_0(\sigma(\varphi))$ for every formula φ. Therefore, $\sigma'(frm(\{s'_1, \ldots, s'_n, s\})) = \sigma_0(\sigma(frm(\{s'_1, \ldots, s'_n, s\}))) \subseteq \sigma_0(\mathcal{F}_w) \subseteq dom(\mathcal{V}_w)$. Since $\mathcal{W}$ is **G**-legal, we have $u_w(\sigma(\{s'_1, \ldots, s'_n\})) = \mathcal{V}_w(\sigma'(\{s'_1, \ldots, s'_n\})) \leq \mathcal{V}_w(\sigma'(s)) = u_w(\sigma(s))$.

Extending the translation: Let $w \in W$. Then, u_w is a **G**-legal bivaluation whose domain $\mathcal{F}_w$ is a finite subset of $\mathcal{L}$ closed under $\circledcirc$-k-subformulas. Since **G** is $\circledcirc$-k-analytic, by Thm. 3.9, u_w can be extended to a **G**-legal $\mathcal{L}$-bivaluation u^*_w.

Defining $\mathcal{W}'$: For every $w \in W$, let $\mathcal{F}'_w = dom(\mathcal{V}_w) \cup \circledcirc dom(\mathcal{W}) \cup \{\diamond(\varphi_1, \ldots, \varphi_n) \mid \diamond \in (\diamond^n_{\mathcal{L}} \setminus \circledcirc), \circledcirc^{\leq k}\{\varphi_1, \ldots, \varphi_n\} \subseteq dom(\mathcal{W})\}$. Let α be the inverse of the extended σ_0. α is a bijection from $\mathcal{L}_{\boxdot}$ to $\mathcal{L}$. Let $\mathcal{W}' = \langle W, \mathcal{R}, \mathcal{V}' \rangle$, where $\mathcal{V}'$ is the function assigning to every $w \in W$, the $\mathcal{F}'_w$-bivaluation $\mathcal{V}'_w$ defined by:

$$\mathcal{V}'_w(\psi) = \begin{cases} \mathcal{V}_w(\psi) & \psi \in dom(\mathcal{V}_w) \\ u^*_w(\alpha(\psi)) & \psi \in \mathcal{F}'_w \setminus dom(\mathcal{V}_w) \end{cases}$$

First, we prove that $\mathcal{W}'$ is a biframe for M. Let $w \in W$ and $\psi, \Box\psi \in \mathcal{L}_{\boxdot}$. Suppose that $\Box\psi \in dom(\mathcal{V}'_w)$ and $\psi \in dom(\mathcal{V}'_{w'})$ for every $w' \in \mathcal{R}_\Box[w]$. Then, since $\Box\psi \in \boxdot\mathcal{L}_{\boxdot}$, we have $\Box\psi \in dom(\mathcal{V}_w)$. Since $\mathcal{W}$ is $\circledcirc$-k-closed, $\psi \in dom(\mathcal{V}_{w'})$ for every $w' \in \mathcal{R}_\Box[w]$. Since $\mathcal{W}$ is a biframe, $\mathcal{V}'_w(\Box\psi) = \mathcal{V}_w(\Box\psi) = \mathcal{V}_{\mathcal{R}_\Box[w]}(\psi) = \mathcal{V}'_{\mathcal{R}_\Box[w]}(\psi)$.

Clearly, $\mathcal{W}'$ extends $\mathcal{W}$, $\circledcirc dom(\mathcal{W}) \subseteq dom(\mathcal{W}')$, and for every $\diamond \in \diamond^n_{\mathcal{L}} \setminus \circledcirc$, $\diamond(\varphi_1, \ldots, \varphi_n) \in dom(\mathcal{W}')$ whenever $\circledcirc^{\leq k}\{\varphi_1, \ldots, \varphi_n\} \subseteq dom(\mathcal{W})$.

It remains to show that $\mathcal{W}'$ is $\circledcirc$-k-closed and **G**-legal.

(1) $\circledcirc$-k-closed: Let $w \in W$. First, $dom(\mathcal{V}'_w)$ is finite since $dom(\mathcal{W})$ and $\diamond_{\mathcal{L}}$ are finite. Second, let $\circ\varphi \in dom(\mathcal{V}'_w)$ for some $\circ \in \circledcirc$. If $\circ\varphi \in dom(\mathcal{V}_w)$, then since $\mathcal{W}$ is $\circledcirc$-k-closed, $\varphi \in dom(\mathcal{V}_w) \subseteq dom(\mathcal{V}'_w)$. Otherwise, $\circ\varphi \in \mathcal{F}'_w \setminus dom(\mathcal{V}_w)$, which means that $\varphi \in dom(\mathcal{W}) \subseteq dom(\mathcal{V}_w) \subseteq dom(\mathcal{V}'_w)$. Third, let $\diamond(\psi_1, \ldots, \psi_n) \in dom(\mathcal{V}'_w)$. We show that $\circledcirc^{\leq k}\psi_i \subseteq dom(\mathcal{V}'_w)$ for every $1 \leq i \leq n$. If $\diamond(\psi_1, \ldots, \psi_n) \in dom(\mathcal{V}_w)$ then this holds since $dom(\mathcal{V}_w)$ is $\circledcirc$-k-closed. Otherwise, $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{F}'_w \setminus dom(\mathcal{V}_w)$, which means that $\circledcirc^{\leq k}\psi_i \subseteq dom(\mathcal{W}) \subseteq dom(\mathcal{V}_w) \subseteq dom(\mathcal{V}'_w)$ for every $1 \leq i \leq n$. Finally, let $\Box\psi \in dom(\mathcal{V}'_w)$. Then, since $\Box\psi \in \boxdot\mathcal{L}_{\boxdot}$, $\Box\psi \in dom(\mathcal{V}_w)$. Since $\mathcal{W}$ is $\circledcirc$-k-closed, $\psi \in dom(\mathcal{V}_{w'}) \subseteq dom(\mathcal{V}'_{w'})$ for every $w' \in \mathcal{R}_\Box[w]$.

(2) **G**-legal: Let $w \in W$, $s_1, \ldots, s_n \,/\, s \in \mathbf{G}$, $s'_1 \subseteq s_1, \ldots, s'_n \subseteq s_n$, and σ a substitution such that $\sigma(frm(\{s'_1, \ldots, s'_n, s\})) \subseteq dom(\mathcal{V}'_w)$. We prove that $\mathcal{V}'_w(\sigma(\{s'_1, \ldots, s'_n\})) \leq \mathcal{V}'_w(\sigma(s))$. For that, we first prove that $\mathcal{V}'_w(\psi) = u^*_w(\alpha(\psi))$ for every $\psi \in dom(\mathcal{V}'_w)$. If $\psi \notin dom(\mathcal{V}_w)$, then this holds by definition. Suppose that $\psi \in dom(\mathcal{V}_w)$. Since $\sigma_0(\alpha(\psi)) = \psi$, $\alpha(\psi) \in \mathcal{F}_w$.

Hence $u^*_w(\alpha(\psi)) = u_w(\alpha(\psi))$. By definition, $u_w(\alpha(\psi)) = \mathcal{V}_w(\sigma_0(\alpha(\psi))) = \mathcal{V}_w(\psi)$. Since $\psi \in dom(\mathcal{V}_w)$, $u^*_w(\alpha(\psi)) = \mathcal{V}'_w(\psi)$. Now, consider the substitution $\sigma' = \alpha \circ \sigma$. It is easy to see that $\sigma'(\psi) = \alpha(\sigma(\psi))$ for every $\psi \in frm(\{s'_1, \dots, s'_n, s\})$. Clearly, $\sigma'(frm(\{s'_1, \dots, s'_n, s\})) \subseteq \mathcal{L}$. Since u^*_w is **G**-legal, we have $\mathcal{V}'_w(\sigma(\{s'_1, \dots, s'_n\})) = u^*_w(\alpha(\sigma(\{s'_1, \dots, s'_n\}))) = u^*_w(\sigma'(\{s'_1, \dots, s'_n\})) \leq u^*_w(\sigma'(s)) = u^*_w(\alpha(\sigma(s))) = \mathcal{V}'_w(\sigma(s))$. □

To complete the proof of Lemma 6.20, we use Lemmas 6.22 to 6.24 repeatedly, and construct a full biframe from a partial one. First, recursively construct an infinite sequence $\mathcal{W}^0 = \langle W, \mathcal{R}, \mathcal{V}^0 \rangle$, $\mathcal{W}^1 = \langle W, \mathcal{R}, \mathcal{V}^1 \rangle$, ... such that:

- $\mathcal{W}^0 = \mathcal{W}$.
- For every i, $\mathcal{W}^i$ is a **G**-legal $\circledcirc$-k-closed biframe for M.
- Each $\mathcal{W}^{i+1}$ extends $\mathcal{W}^i$.
- For every $\psi \in \mathcal{L}_{\boxdot}$, $\psi \in dom(\mathcal{W}^i)$ for some $i \geq 0$.

We begin with $\mathcal{W}^0 = \mathcal{W}$. Given $\mathcal{W}^i$, $\mathcal{W}^{i+1}$ is obtained as follows. By Lemma 6.22, $\mathcal{W}^i$ can be extended to a **G**-legal $\circledcirc$-k-closed biframe $\mathcal{W}^i_1$ for M such that $p_i \in dom(\mathcal{W}^i_1)$. In turn, Lemma 6.23 gives us that $\mathcal{W}^i_1$ can be extended to a **G**-legal $\circledcirc$-k-closed biframe $\mathcal{W}^i_2$ for M such that $\boxdot dom(\mathcal{W}^i_1) \subseteq dom(\mathcal{W}^i_2)$. Finally, by Lemma 6.24, $\mathcal{W}^i_2$ can be extended to a **G**-legal $\circledcirc$-k-closed biframe $\mathcal{W}^i_3$ for M such that $\circledcirc dom(\mathcal{W}^i_2) \subseteq dom(\mathcal{W}^i_3)$, and for every $\diamond \in \diamond^n_{\mathcal{L}} \setminus \circledcirc$, $\diamond(\varphi_1, \dots, \varphi_n) \in dom(\mathcal{W}^i_3)$ whenever $\circledcirc^{\leq k}\{\varphi_1, \dots, \varphi_n\} \subseteq dom(\mathcal{W}^i_2)$. We take $\mathcal{W}^{i+1} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V}^{i+1} \rangle$ to be $\mathcal{W}^i_3$.

Clearly, for every $i \geq 0$, $\mathcal{W}^{i+1}$ is a **G**-legal $\circledcirc$-k-closed biframe for M that extends $\mathcal{W}^i$. We prove that for every $\psi \in \mathcal{L}_{\boxdot}$ there exists some $i \geq 0$ such that $\psi \in dom(\mathcal{W}^i)$, by induction on the complexity of ψ:

(1) If $\psi \in At$ then $\psi = p_i$ for some $i \geq 1$. By our construction, $p_i \in dom(\mathcal{W}^i_1)$ and hence $p_i \in dom(\mathcal{W}^{i+1})$.
(2) If $\psi = \Box\varphi$ then by the induction hypothesis, $\varphi \in dom(\mathcal{W}^i)$ for some $i \geq 0$. By our construction, $\Box\varphi \in dom(\mathcal{W}^i_2)$ and hence $\psi \in dom(\mathcal{W}^{i+1})$.
(3) If $\psi = \circ\varphi$, then by the induction hypothesis, there exists i such that $\varphi \in dom(\mathcal{W}^i)$. By our construction, $\circ\varphi \in dom(\mathcal{W}^i_3)$, and hence $\circ\varphi \in dom(\mathcal{W}^{i+1})$.
(4) If $\psi = \diamond(\psi_1, \dots, \psi_n)$ then by the induction hypothesis, there exist $i_1, \dots, i_n$ such that $\psi_j \in dom(\mathcal{W}^{i_j})$ for every $1 \leq j \leq n$. Let $i = \max\{i_1, \dots, i_n\}$. By our construction, there exists $i_0 > i$ such that $\circledcirc^{\leq k}\psi_j \subseteq dom(\mathcal{W}^{i_0})$ for every $1 \leq j \leq n$ (in each step we add $\circ\varphi$ for every $\varphi \in dom(\mathcal{W}^i)$ and $\circ \in \circledcirc$. Since $\circledcirc^{\leq k}$ is finite, we exhaust it at some point). Hence $\diamond(\psi_1, \dots, \psi_n) \in dom(\mathcal{W}^{i_0}_3)$, which means that $\diamond(\psi_1, \dots, \psi_n) \in dom(\mathcal{W}^{i_0+1})$.

We now define $\mathcal{W}' = \langle W, \mathcal{R}, \mathcal{V}' \rangle$, a **G**-legal $\mathcal{L}_{\boxdot}$-biframe for M that extends $\mathcal{W}$. For every $\psi \in \mathcal{L}_{\boxdot}$, let i_ψ denote the first i such that $\psi \in dom(\mathcal{W}^i)$. For every $w \in W$, $\mathcal{V}'_w$ is defined by $\mathcal{V}'_w(\psi) = \mathcal{V}^{i_\psi}_w(\psi)$.

We prove that $\mathcal{W}'$ is a **G**-legal $\mathcal{L}_{\boxdot}$-biframe for M that extends $\mathcal{W}$. Clearly, $dom(\mathcal{W}') = \mathcal{L}_{\boxdot}$ and $\mathcal{W}'$ extends $\mathcal{W}$. We prove that $\mathcal{W}'$ is a biframe: Let $w \in W$ and $\psi, \Box\psi \in \mathcal{L}_{\boxdot}$. Let $k = \max\{i_\psi, i_{\Box\psi}\}$. Since $\mathcal{W}^i$ extends $\mathcal{W}^{i-1}$ for every i, we have $\mathcal{V}'_{w'}(\psi) = \mathcal{V}^k_{w'}(\psi)$ and $\mathcal{V}'_{w'}(\Box\psi) = \mathcal{V}^k_{w'}(\Box\psi)$ for every $w' \in W$. Since $\mathcal{W}^k$ is a biframe, $\mathcal{V}'_w(\Box\psi) = \mathcal{V}^k_w(\Box\psi) = \mathcal{V}^k_{\mathcal{R}_\Box[w]}(\psi) = \mathcal{V}'_{\mathcal{R}_\Box[w]}(\psi)$. It remains to show that $\mathcal{W}'$ is **G**-legal. Let $w \in W$, $s_1, \dots, s_n \,/\, s \in \mathbf{G}$, $s'_1 \subseteq s_1, \dots, s'_n \subseteq s_n$, and σ a substitution. We prove that $\mathcal{V}'_w(\sigma(\{s'_1, \dots, s'_n\})) \leq \mathcal{V}'_w(\sigma(s))$. Let $k = \max\{i_\psi \mid \psi \in \sigma(frm(\{s'_1, \dots, s'_n, s\}))\}$. Since $\mathcal{W}^i$ extends W^{i-1} for every i, we have $\mathcal{V}'_w(\psi) = \mathcal{V}^k_w(\psi)$ for every $\psi \in \sigma(frm(\{s'_1, \dots, s'_n, s\}))$. Since $\mathcal{V}^k_w$ is **G**-legal, $\mathcal{V}'_w(\sigma(\{s'_1, \dots, s'_n\})) = \mathcal{V}^k_w(\sigma(\{s'_1, \dots, s'_n\})) \leq \mathcal{V}^k_w(\sigma(s)) = \mathcal{V}'_w(\sigma(s))$. □

7 DECISION PROCEDURE FOR PURE CALCULI WITH NEXT OPERATORS

In this section we extend the reduction from §4 to standard pure calculi with modal operators that are defined by (PF) and (D_{PF}).[8] The implementation described in §4.2 includes this extension.

Semantically, such calculi are characterized by biframes in which the accessibility relations are total functions. We call such operators *Next operators*, as they are often employed in temporal logics to express the "next state". Recall, that the quotations employed in primal infon logic [19] are also governed by these rules, and hence quotations are *Next* operators (Example 6.15).

We start by defining a useful variant of the $\circledcirc$-k-subformula relation in §7.1. This relation is used in §7.2 in order to extend the reduction of §4, and to prove the correctness of the extended reduction. In what follows, we denote the specification function M that assigns $\{(\text{PF}), (D_{\text{PF}})\}$ to every $\Box \in \boxdot$ by *Next*. In turn, biframes for *Next* are called *totally functional* biframes.

7.1 Local Formulas

To generalize the reduction in §4, we replace $\circledcirc$-k-subformulas by $\circledcirc$-k-*local formulas*. This notion generalizes the *local formulas relation* from [28]. A sequence $\bar{\Box} = \Box_1 \dots \Box_m$ ($m \geq 0$) of elements of $\boxdot$ is called a $\boxdot$-*prefix*. We say that $\bar{\Box}$ is a $\boxdot$-*prefix of a formula* φ if φ has the form $\bar{\Box}\psi$ for some $\psi \in \mathcal{L}_{\boxdot}$. The notation $\Box\mathcal{F}$ is naturally extended to prefixes $\bar{\Box}$.

Definition 7.1. Denote by $\bar{\Box}_\psi$ the longest (possibly, empty) $\boxdot$-prefix of ψ, and by b_ψ the formula for which $\psi = \bar{\Box}_\psi b_\psi$. A formula φ is *immediately* $\circledcirc$-k-*local* to a formula ψ if $\varphi = \bar{\Box}_\psi \varphi'$ for some immediate $\circledcirc$-k-subformula φ' of b_ψ. The $\circledcirc$-k-local formula relation is the reflexive transitive closure of the immediate $\circledcirc$-k-local formula relation. We denote the set of $\circledcirc$-k-local formulas of a formula ψ by $loc_k^{\circledcirc,\boxdot}(\psi)$. This notation is naturally extended to sequents, sets of sequents etc. When $\circledcirc = \emptyset$, we call φ a *local* formula of ψ.

Note that for $\boxdot = \emptyset$, we have $loc_k^{\circledcirc,\boxdot}(\psi) = sub_k^{\circledcirc}(\psi)$ for every formula ψ.

Example 7.2. $loc_1^{\{\neg\},\{\Box,\boxtimes\}}(\Box(\boxtimes p_1 \supset p_2)) = \{\Box \boxtimes p_1, \Box\neg \boxtimes p_1, \Box p_2, \Box\neg p_2, \Box(\boxtimes p_1 \supset p_2)\}$.

Similarly to $\circledcirc$-k-subformulas, since every formula has finitely many immediate $\circledcirc$-k-local formulas, we have that $loc_k^{\circledcirc,\boxdot}(\psi)$ is finite for every $\psi \in \mathcal{L}$. The following lemma provides an alternative inductive definition of $loc_k^{\circledcirc,\boxdot}(\psi)$:

LEMMA 7.3. (1) $loc_k^{\circledcirc,\boxdot}(p) = \{p\}$ *for every* $p \in At$.
(2) $loc_k^{\circledcirc,\boxdot}(\circ\psi) = \{\circ\psi\} \cup loc_k^{\circledcirc,\boxdot}(\psi)$ *for every* $\circ \in \circledcirc$.
(3) $loc_k^{\circledcirc,\boxdot}(\diamond(\psi_1, \dots, \psi_n)) = \{\diamond(\psi_1, \dots, \psi_n)\} \cup \bigcup_{1 \leq i \leq n} \circledcirc^{\leq k}\psi_i \cup loc_k^{\circledcirc,\boxdot}(\psi_i)$ *for every* $\diamond \in \diamond_{\mathcal{L}} \setminus \circledcirc$.
(4) $loc_k^{\circledcirc,\boxdot}(\Box\psi) = \Box loc_k^{\circledcirc,\boxdot}(\psi)$.

7.2 Extending The Reduction

For the case that the set of assumptions is empty, we extend the reduction from §4 to sequent calculi with *Next* operators. As before, we assume **G** is axiomatic.

Definition 7.4. The SAT instance associated with a given axiomatic $\mathcal{L}$-calculus **G** and an $\mathcal{L}_{\boxdot}$-sequent s, denoted $\mathsf{SAT}_k^{\circledcirc,\boxdot}(\mathbf{G}, s)$, consists of the following clauses:

(1) $\mathsf{SAT}^-(s)$
(2) $\mathsf{SAT}^+(\bar{\Box}\sigma(s'))$ for every rule $\emptyset \,/\, s'$ of **G**, substitution σ and $\boxdot$-prefix $\bar{\Box}$ such that $\bar{\Box}\sigma(frm(s')) \subseteq loc_k^{\circledcirc,\boxdot}(s)$.

[8] Note that we cannot expect a similar reduction for all modalities studied here, as for example, $\mathbf{LK}_M$ with $M(\Box) = \{(\text{K})\}$ is known [32] to be PSPACE-complete.

The following theorem states that the reduction is correct.

Theorem 7.5. *Let* **G** *be a standard* $\circledcirc$*-*k*-analytic* $\mathcal{L}$*-calculus and* s *an* $\mathcal{L}_{\boxdot}$*-sequent. Then* $\vdash_{\mathbf{G}_{Next}} s$ *iff* $\mathsf{SAT}_k^{\circledcirc,\boxdot}(\mathbf{G}, s)$ *is unsatisfiable.*

Proof. For a totally functional biframe $\langle W, \mathcal{R}, \mathcal{V}\rangle$ and a world $w \in W$, we denote by $\mathcal{R}_{\Box}(w)$ the (unique) world w' such that $\langle w, w'\rangle \in \mathcal{R}_{\Box}$. Then, we have $\mathcal{V}_w(\Box\psi) = \mathcal{V}_{\mathcal{R}_{\Box}(w)}(\psi)$ whenever $\Box\psi \in dom(\mathcal{V}_w)$ and $\psi \in dom(\mathcal{V}_{\mathcal{R}_{\Box}(w)})$.

($\Rightarrow$): Suppose that $\nvdash_{\mathbf{G}_{Next}} s$. By Thm. 6.10, we have $\mathcal{V}_w(s) = 0$ for some **G**-legal $\mathcal{L}_{\boxdot}$-biframe $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V}\rangle$ for *Next* and $w \in W$. Consider the classical assignment u that assigns true to x_ψ iff $\mathcal{V}_w(\psi) = 1$. Since $\mathcal{V}_w(s) = 0$, u satisfies $\mathsf{SAT}^-(s)$. It remains to prove that $\mathcal{V}_w(\bar{\Box}\sigma(s')) = 1$ for every $\emptyset / s' \in \mathbf{G}$, substitution σ and $\boxdot$-prefix $\bar{\Box}$ such that $\bar{\Box}\sigma(frm(s')) \subseteq loc_k^{\circledcirc,\boxdot}(s)$. Suppose that $\bar{\Box} = \Box_1 \dots \Box_n$, and let $w = w_0, w_1, \dots, w_n$ be a sequence of worlds of $\mathcal{W}$ such that $\mathcal{R}_{\Box_i}(w_{i-1}) = w_i$ for every $1 \le i \le n$. Then $\mathcal{V}_{w_0}(\Box_1 \dots \Box_n\psi) = \mathcal{V}_{w_1}(\Box_2 \dots \Box_n\psi) = \dots = \mathcal{V}_{w_n}(\psi)$ for every $\psi \in \mathcal{L}_{\boxdot}$. Since $\mathcal{W}$ is **G**-legal, the bivaluation $\mathcal{V}_{w_n}$ is **G**-legal, and therefore, $\mathcal{V}_w(\bar{\Box}\sigma(s')) = \mathcal{V}_{w_n}(\sigma(s')) = 1$.

($\Leftarrow$): Let u be an assignment that satisfies $\mathsf{SAT}_k^{\circledcirc,\boxdot}(\mathbf{G}, s)$. Define the following biframe $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V}\rangle$:

(1) W is the set of all $\boxdot$-prefixes.
(2) For every $\Box \in \boxdot$ and $\bar{\Box} \in W$, $\mathcal{R}_{\Box}(\bar{\Box}) = \bar{\Box}\Box$.
(3) $\mathcal{V}_{\bar{\Box}}$ is defined by induction on the length of $\bar{\Box}$: $dom(\mathcal{V}_\epsilon) = loc_k^{\circledcirc,\boxdot}(s)$ and $\mathcal{V}_\epsilon(\psi) = 1$ iff u satisfies x_ψ; $dom(\mathcal{V}_{\Box_1\dots\Box_n}) = \{\varphi \mid \Box_n\varphi \in dom(\mathcal{V}_{\Box_1\dots\Box_{n-1}})\}$ and $\mathcal{V}_{\Box_1\dots\Box_n}(\psi) = \mathcal{V}_{\Box_1\dots\Box_{n-1}}(\Box_n\psi)$.

Clearly, $\mathcal{R}_{\Box}$ is a total function for every $\Box \in \boxdot$. Since u satisfies $\mathsf{SAT}^-(s)$, $\mathcal{V}_\epsilon(s) = 0$. We prove that $\mathcal{W}$ is a **G**-legal $\circledcirc$-k-closed biframe for *Next* (see Def. 6.18).

(1) biframe for *Next*: By the definition of $\mathcal{V}$.
(2) **G**-legal: We prove that $\mathcal{V}_{\Box_1\dots\Box_n}$ is **G**-legal for every $\Box_1 \dots \Box_n \in W$. Let $\emptyset / s' \in \mathbf{G}$ and σ be a substitution such that $\sigma(frm(s')) \subseteq dom(\mathcal{V}_{\Box_1\dots\Box_n})$. We prove that $\mathcal{V}_{\Box_1\dots\Box_n}(\sigma(s')) = 1$. We actually prove a stronger claim, namely that $\mathcal{V}_{\Box_1\dots\Box_n}(\bar{\Box}\sigma(s')) = 1$ for every $\boxdot$-prefix $\bar{\Box}$ (including ϵ) such that $\bar{\Box}\sigma(frm(s')) \subseteq dom(\mathcal{V}_{\Box_1\dots\Box_n})$. We do so by induction on n. For $n = 0$ we have $\mathcal{V}_\epsilon(\bar{\Box}\sigma(s')) = 1$ because u satisfies $\mathsf{SAT}^+(\bar{\Box}\sigma(s'))$. Now, let $n \ge 1$. Since $\bar{\Box}\sigma(frm(s')) \subseteq dom(\mathcal{V}_{\Box_1\dots\Box_n})$, we have $\Box_n\bar{\Box}\sigma(frm(s')) \subseteq dom(\mathcal{V}_{\Box_1\dots\Box_{n-1}})$. By the induction hypothesis, $\mathcal{V}_{\Box_1\dots\Box_{n-1}}(\Box_n\bar{\Box}\sigma(s')) = 1$. By $\mathcal{V}$'s definition, $\mathcal{V}_{\Box_1\dots\Box_n}(\bar{\Box}\sigma(s')) = 1$.
(3) $\circledcirc$-k-closed: $dom(\mathcal{V}_{\bar{\Box}})$ is finite for every $\bar{\Box}$ since $dom(\mathcal{V}_\epsilon) = loc_k^{\circledcirc,\boxdot}(s)$ is finite. In addition, if $\Box\psi \in dom(\mathcal{V}_{\bar{\Box}})$ then by our construction, $\psi \in dom(\mathcal{V}_{\bar{\Box}\Box})$. It remains to prove that for every $\bar{\Box} \in W$, $dom(\mathcal{V}_{\bar{\Box}})$ is $\circledcirc$-k-closed. First, note that every set which is closed under $\circledcirc$-k-local formulas is also $\circledcirc$-k-closed. This holds since ψ is $\circledcirc$-k-local to $\circ\psi$ for every $\circ \in \circledcirc$, and $\bar{\circ}\psi_i$ is $\circledcirc$-k-local to $\diamond(\psi_1, \dots, \psi_n)$ for every $1 \le i \le n$ and $\bar{\circ} \in \circledcirc^{\le k}$. Therefore, it suffices to prove that $dom(\mathcal{V}_{\bar{\Box}})$ is closed under $\circledcirc$-k-local formulas for every $\bar{\Box} \in W$. We do so by induction on the length of $\bar{\Box}$. First, we have that $dom(\mathcal{V}_\epsilon) = loc_k^{\circledcirc,\boxdot}(s)$ is closed under $\circledcirc$-k-local formulas. Now, let $\Box_1 \dots \Box_n \in W$ ($n \ge 1$). We prove that $loc_k^{\circledcirc,\boxdot}(\psi) \subseteq dom(\mathcal{V}_{\Box_1\dots\Box_n})$ for every $\psi \in dom(\mathcal{V}_{\Box_1\dots\Box_n})$. Let $\psi \in dom(\mathcal{V}_{\Box_1\dots\Box_n})$. Then, $\Box_n\psi \in dom(\mathcal{V}_{\Box_1\dots\Box_{n-1}})$. By the induction hypothesis, $dom(\mathcal{V}_{\Box_1\dots\Box_{n-1}})$ is closed under $\circledcirc$-k-local formulas. Therefore, $loc_k^{\circledcirc,\boxdot}(\Box_n\psi) \subseteq dom(\mathcal{V}_{\Box_1\dots\Box_{n-1}})$. Now, let $\varphi \in loc_k^{\circledcirc,\boxdot}(\psi)$. Then, by Lemma 7.3, $\Box_n\varphi \in \Box_n loc_k^{\circledcirc,\boxdot}(\psi) = loc_k^{\circledcirc,\boxdot}(\Box_n\psi)$. Hence $\Box_n\varphi \in dom(\mathcal{V}_{\Box_1\dots\Box_{n-1}})$. By $\mathcal{V}$'s definition, $\varphi \in dom(\mathcal{V}_{\Box_1\dots\Box_n})$.

Now, since **G** is $\circledcirc$-k-analytic, By Lemma 6.20, $\mathcal{W}$ can be extended to a **G**-legal $\mathcal{L}_{\boxdot}$-biframe $\langle W, \mathcal{R}, \mathcal{V}'\rangle$ for *Next*. By Thm. 6.9, since $\mathcal{V}'_\epsilon(s) = 0$, we have $\nvdash_{\mathbf{G}_{Next}} s$. ◻

Note that Thm. 7.5 is restricted to derivability problems with an empty set of assumptions. The main difficulty with encoding a countermodel for the derivability of s from S is that every element of S must hold in every world of the countermodel. This is in contrast to the rules of **G**, which are required to hold only in worlds whose domains include the instances of the rules. We leave the handling of non-empty sets of assumptions for future work.

For the case that $\circledcirc = \emptyset$ (and $S = \emptyset$), the polynomial time algorithm from Thm. 4.6 can be modified to accommodate *Next* operators. In particular, the derivability problem for such calculi is also in co-NP.

THEOREM 7.6. *Let* **G** *be an axiomatic* $\mathcal{L}$*-calculus, such that* $c_k^{\circledcirc}(\mathbf{G}) = m$*. Given an* $\mathcal{L}_{\boxdot}$*-sequent* s*, it is possible to compute* $\mathsf{SAT}_k^{\emptyset,\boxdot}(\mathbf{G}, s)$ *in* $O(n^m)$ *time, where* n *is the length of the string representing the input sequent.*

PROOF. The algorithm in the proof of Thm. 4.6 is reused with several modifications. As in [19], an auxiliary trie (an ordered tree data structure commonly used for string processing) for $\boxdot$-prefixes is constructed in linear time, and every node in the input parse tree has a pointer to a node in this trie. Now, each node in the parse tree corresponds to an occurrence of a formula that is local to s. The tree is then compressed to a dag as in the proof of Thm. 4.6. The nodes of the dag one-to-one correspond to the local formulas of s. The rest of the algorithm is exactly as in the proof of Thm. 4.6. □

Example 7.7. Following Example 4.11, our general reduction provides linear time algorithms for the extensions of **P** and **EP** with any finite set of *Next* operators. (A different linear time algorithm was developed in [19] for **P**.)

To conclude, we note that in some cases, Propositions 6.2 and 6.5 allow for modal operators other than *Next* operators to be applicable for the reduction to SAT. Indeed, for Horn calculi (Def. 4.8), *Next* is equivalent to a specification M in which $\mathsf{M}(\Box) = \{(\textsc{k}), (\textsc{d}_\textsc{k})\}$; and if the calculus is also definite (Def. 6.3), $(\textsc{d}_\textsc{k})$ can be eliminated, leaving just $\{(\textsc{k})\}$.

8 CONCLUSIONS

We have studied the family of pure sequent calculi focusing on (generalized) analyticity (rather than the more traditional cut-elimination property). The key tool in this general study is a modular and uniform *semantic* interpretation of pure sequent calculi. The semantics was used to characterize analyticity, provide useful sufficient criteria for it, as well as to obtain an effective SAT-based decision procedure for derivability in analytic pure calculi. We then further considered the extension of pure calculi with various rules for modal operators, and showed that such extension always preserves analyticity. This result, together with the criteria for analyticity in pure calculi, provides simple approach to develop analytic-by-construction calculi for non-classical logics with modal operators. Finally, the SAT-based decision procedure was extended for a restricted type of modal operators that correspond to *Next* operators in temporal logics.

Further research is required for extending the methods of this paper to provide analyticity conditions and decision procedures for many-sided sequent calculi, that are more expressive than ordinary two-sided calculi, as well as for richer languages, which employ, e.g., diamond-like modalities, negative modalities, and quantifiers.

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