Basic Constructive Connectives, Determinism and Matrix-based Semantics

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Main Contribution

- A semantic characterization of syntactic properties of single-conclusion canonical sequent systems.
- A link between invertibility, axiom-expansion and determinism of Kripke-style semantics for such systems.
- A matrix-based presentation of non-deterministic Kripke-style semantics, allowing for a decision procedure for checking determinism.

Invertibility

A rule is *invertible* in a calculus **G** if each of its premises is derivable in **G** from its conclusion.

$$\frac{\Gamma \Rightarrow \varphi \qquad \Gamma, \psi \Rightarrow E}{\Gamma, \varphi \supset \psi \Rightarrow E} \qquad\qquad \frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \supset \psi}$$

The left rule is not invertible, while the right rule is:

$$\frac{\Gamma \Rightarrow \varphi \supset \psi \qquad \dfrac{\dfrac{\varphi \Rightarrow \varphi}{\Gamma, \varphi \Rightarrow \varphi} \qquad \dfrac{\psi \Rightarrow \psi}{\Gamma, \varphi, \psi \Rightarrow \psi}}{\Gamma, \varphi, \varphi \supset \psi \Rightarrow \psi}}{\Gamma, \varphi \Rightarrow \psi}$$

Axiom-Expansion

An n-ary connective $\diamond$ admits *axiom-expansion* in **G**, if $\diamond(p_1, \ldots, p_n) \Rightarrow \diamond(p_1, \ldots, p_n)$ has a cut-free proof in **G** that does not contain non-atomic axioms.

$$\frac{\Gamma \Rightarrow \varphi \quad \Gamma, \psi \Rightarrow E}{\Gamma, \varphi \supset \psi \Rightarrow E} \qquad \frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \supset \psi} \qquad \dfrac{\dfrac{p_1 \Rightarrow p_1 \quad p_2 \Rightarrow p_2}{p_1, p_1 \supset p_2 \Rightarrow p_2}}{p_1 \supset p_2 \Rightarrow p_1 \supset p_2}$$

$$\frac{\Gamma \Rightarrow \psi \quad \Gamma, \varphi \Rightarrow E}{\Gamma, \psi \rightsquigarrow \varphi \Rightarrow E} \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \psi \rightsquigarrow \varphi} \qquad \begin{array}{c} p_1 \Rightarrow p_1, p_2 \Rightarrow p_2 \not\vdash \\ p_1 \rightsquigarrow p_2 \Rightarrow p_1 \rightsquigarrow p_2 \end{array}$$

What is a Canonical Rule?

- An *"ideal" logical rule*: an introduction rule for *exactly one connective*, on *exactly one side of a sequent*.
- In its formulation: *exactly one occurrence* of the introduced connective, no other occurrences of other connectives.
- Its active formulas: *immediate subformulas* of its principal formula.

Multiple-Conclusion Canonical Rules

Stage 1:

$$\frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \wedge \varphi}$$

Stage 2:

$$\frac{\psi, \varphi \Rightarrow}{\psi \wedge \varphi \Rightarrow} \qquad \frac{\Rightarrow \psi \quad \Rightarrow \varphi}{\Rightarrow \psi \wedge \varphi}$$

Stage 3:

$$\{p_1, p_2 \Rightarrow\} \;/\; p_1 \wedge p_2 \Rightarrow \qquad \{\Rightarrow p_1 \;;\; \Rightarrow p_2\} \;/\; \Rightarrow p_1 \wedge p_2$$

Multiple-Conclusion Canonical Systems ([Avron,Lev 2001])

- Multiple-conclusion sequent calculi consist of identity axioms, cut, weakening and multiple-conclusion canonical rules.
- Have a semantic characterization using non-deterministic two-valued matrices (2Nmatrices).
- Remarkable correspondence: Invertibility of rules - Axiom-expansion - Determinism of the corresponding 2Nmatrix

Intuitition for Introducing Non-determinism

Standard rules for classical negation and disjunction:

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg\psi \Rightarrow \Delta} \qquad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\psi}$$

$$\frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \vee \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi, \varphi}{\Gamma \Rightarrow \Delta, \psi \vee \varphi}$$

Intuition for Introducing Non-determinism

Written in the canonical notation:

$$\frac{\Rightarrow p}{\neg p \Rightarrow} \qquad \frac{p \Rightarrow}{\Rightarrow \neg p}$$

$$\frac{p_1 \Rightarrow \quad p_2 \Rightarrow}{p_1 \vee p_2 \Rightarrow} \qquad \frac{\Rightarrow p_1, p_2}{\Rightarrow p_1 \vee p_2}$$

Intuition for Introducing Non-determinism

Correponding to the classical semantics:

$$\frac{\Rightarrow p}{\neg p \Rightarrow} \qquad \frac{p \Rightarrow}{\Rightarrow \neg p}$$

$$\frac{p_1 \Rightarrow \quad p_2 \Rightarrow}{p_1 \vee p_2 \Rightarrow} \qquad \frac{\Rightarrow p_1, p_2}{\Rightarrow p_1 \vee p_2}$$

	$\neg$
t	**f**
f	**t**

		$\vee$
t	**t**	**t**
t	**f**	**t**
f	**t**	**t**
f	**f**	**f**

Intuition for Introducing Non-determinism

Correponding to the classical semantics:

$$\frac{\Rightarrow p}{\neg p \Rightarrow} \qquad \frac{\qquad}{\qquad}$$

$$\frac{\qquad\qquad}{\qquad\qquad} \qquad \frac{\Rightarrow p_1, p_2}{\Rightarrow p_1 \vee p_2}$$

	$\neg$
t	**f**
f	???

		$\vee$
t	**t**	**t**
t	**f**	**t**
f	**t**	**t**
f	**f**	???

Intuition for Introducing Non-determinism

Correponding to the classical semantics:

$$\frac{\Rightarrow p}{\neg p \Rightarrow} \qquad \frac{}{}$$

$$\frac{}{} \qquad \frac{\Rightarrow p_1, p_2}{\Rightarrow p_1 \vee p_2}$$

	$\neg$
t	{**f**}
f	{**t**,**f**}

		$\vee$
t	**t**	{**t**}
t	**f**	{**t**}
f	**t**	{**t**}
f	**f**	{**t**, **f**}

Single-Conclusion Canonical Systems

- Single-conclusion canonical systems were defined in [Avron,Lahav 2010], and used to proof-theoretically characterize basic constructive connectives.
- Have a semantic characterization using non-deterministic Kripke-style semantics.
- A gap to be filled: Deterministic semantics? Invertibility? Axiom-expansion?

Single-conclusion Right Canonical Rules

- A canonical right rule:

$$\{\Pi_i \Rightarrow E_i\}_{1\leq i\leq m} / \ \Rightarrow \diamond(p_1, \ldots, p_n)$$

- An application of the rule:

$$\frac{\{\Gamma, \sigma(\Pi_i) \Rightarrow \sigma(E_i)\}_{1\leq i\leq m}}{\Gamma \Rightarrow \sigma(\diamond(p_1, \ldots, p_n))}$$

Implication:

$$\{p_1 \Rightarrow p_2\} \ / \ \ \Rightarrow p_1 \supset p_2$$

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \supset \psi}$$

Semi-implication:

$$\{\Rightarrow p_2\} \ / \ \ \Rightarrow p_1 \leadsto p_2$$

$$\frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \leadsto \psi}$$

Single-conclusion Left Canonical Rules

- A canonical left rule:

$$\langle \{\Pi_i \Rightarrow E_i\}_{1 \le i \le m}, \{\Sigma_i \Rightarrow\}_{1 \le i \le k} \rangle / \ \diamond(p_1, \ldots, p_n) \Rightarrow$$

- An application of the rule:

$$\frac{\{\Gamma, \sigma(\Pi_i) \Rightarrow \sigma(E_i)\}_{1 \le i \le m} \quad \{\Gamma, \sigma(\Sigma_i) \Rightarrow E\}_{1 \le i \le k}}{\Gamma, \sigma(\diamond(p_1, \ldots, p_n)) \Rightarrow E}$$

Implication:

$$\langle \{\Rightarrow p_1\}, \{p_2 \Rightarrow\} \rangle \ / \ p_1 \supset p_2 \Rightarrow$$

$$\frac{\Gamma \Rightarrow \varphi \quad \Gamma, \psi \Rightarrow E}{\Gamma, \varphi \supset \psi \Rightarrow E}$$

Weak affirmation:

$$\langle \{p_1 \Rightarrow\}, \emptyset \rangle \ / \ \blacktriangleright p_1 \Rightarrow$$

$$\frac{\Gamma, \varphi \Rightarrow}{\Gamma, \blacktriangleright \varphi \Rightarrow E}$$

Single-Conclusion Canonical Systems

- Identity axioms (the sequents of the form $\psi \Rightarrow \psi$)
- Cut rule
- Weakening
- Single-conclusion canonical rules:
  - Right rules
  - Left rules

Semantics for Single-conclusion Canonical Systems

- Let $\mathcal{F}$ be a set of formulas closed under subformulas. An *$\mathcal{F}$-semiframe* is a triple $\mathcal{W} = \langle W, \leq, v \rangle$ such that:
  1. $\langle W, \leq \rangle$ is a nonempty partially ordered set.
  2. v is a persistent function from $W \times \mathcal{F}$ to $\{t, f\}$: if $v(a, \psi) = t$, then for all $b \geq a$, $v(b, \psi) = t$.
- When $\mathcal{F}$ is the set of all wffs of the language, we call $\mathcal{W}$ a (full) frame.
- Each canonical rule imposes a semantic condition on v. Combining the conditions imposed by all rules of a canonical system **G**, we obtain the set of **G**-*legal* frames, for which **G** is sound and complete.

Example 1: Implication

$$\frac{\Gamma \Rightarrow \varphi \qquad \Gamma, \psi \Rightarrow E}{\Gamma, \varphi \supset \psi \Rightarrow E} \qquad \frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \supset \psi}$$

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The right rule imposes the condition $v(a, \varphi \supset \psi) = t$ whenever for every $b \geq a$, either $v(b, \varphi) = f$ or $v(b, \psi) = t$.

Example 1: Implication

$$\frac{\Gamma \Rightarrow \varphi \quad \Gamma, \psi \Rightarrow E}{\Gamma, \varphi \supset \psi \Rightarrow E} \qquad \frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \supset \psi}$$

The right rule imposes the condition $v(a, \varphi \supset \psi) = t$ whenever for every $b \geq a$, either $v(b, \varphi) = f$ or $v(b, \psi) = t$.

The left rule imposes the condition $v(a, \varphi \supset \psi) = f$ whenever $v(b, \varphi) = t$ for every $b \geq a$ and $v(a, \psi) = f$.

Example 2: Semi-Implication ([Gurevich, Neeman 2009])

$$\frac{\Gamma \Rightarrow \varphi \qquad \Gamma, \psi \Rightarrow E}{\Gamma, \varphi \rightsquigarrow \psi \Rightarrow E} \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \rightsquigarrow \psi}$$

Example 2: Semi-Implication ([Gurevich, Neeman 2009])

$$\frac{\Gamma \Rightarrow \varphi \quad \Gamma, \psi \Rightarrow E}{\Gamma, \varphi \rightsquigarrow \psi \Rightarrow E} \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \rightsquigarrow \psi}$$

The right rule imposes the condition $v(a, \varphi \rightsquigarrow \psi) = t$ whenever for every $b \geq a$, $v(b, \psi) = t$.
The left rule imposes the condition $v(a, \varphi \rightsquigarrow \psi) = f$ whenever $v(b, \varphi) = t$ for every $b \geq a$ and $v(a, \psi) = f$.

If $v(a, \psi) = f$ and there is no $b \geq a$ such that $v(b, \varphi) = t$ and $v(b, \psi) = f$, then $v(a, \varphi \rightsquigarrow \psi)$ is not restricted — non-determinism!

A Note on the Importance of Analycity

- Analycity: to determine whether a sequent s follows from a set $\mathcal{S}$ of sequents, it should be sufficient to consider only partial valuations, related to the relevant set of subformulas of $\mathcal{S} \cup \{s\}$.
- The semantics of **G**-legal non-deterministic frames is analytic in this sense: each **G**-legal $\mathcal{F}$-semi-frame can be extended to a full **G**-legal frame.

What is a Deterministic Connective?

- $\diamond$ is *deterministic* in **G** if every **G**-legal $SF(\psi_1, ..., \psi_n)$-semiframe has a unique extension to a **G**-legal $SF(\diamond(\psi_1, \ldots, \psi_n))$-semiframe.
- Implication is deterministic (in **G** with standard implication rules).
- Semi-Implication is non-deterministic (in **G** with the two semi-implication rules). For instance, define a simple $\{p_1, p_2\}$-semiframe $\mathcal{W}$ with one world w, in which $v(w, p_1) = v(w, p_2) = f$. Then there are two different **G**-legal $\{p_1, p_2, p_1 \rightsquigarrow p_2\}$-semiframes extending $\mathcal{W}$!

Semantic Characterization of Axiom-Expansion

Theorem

A connective $\diamond$ admits axiom-expansion in a canonical system **G** *iff* $\diamond$ *is deterministic in* **G**.

Semantic Characterization of Invertibility

Theorem

If **G** *contains exactly one right rule for* $\diamond$*, then this rule is invertible in* **G** *iff* $\diamond$ *is deterministic in* **G**.

Deciding Determinism

- The current formulation of Kripke-style semantics does not induce a straightforward algorithm for checking determinism of connectives.
- We have nice non-deterministic matrix-based semantics for multiple-conclusioned canonical systems, in which non-determinism is immediately detectable:

		$\diamond$
t	**t**	$\{\mathbf{t}, \mathbf{f}\}$
t	**f**	$\{\mathbf{t}\}$
f	**t**	$\{\mathbf{t}\}$
f	**f**	$\{\mathbf{t}, \mathbf{f}\}$

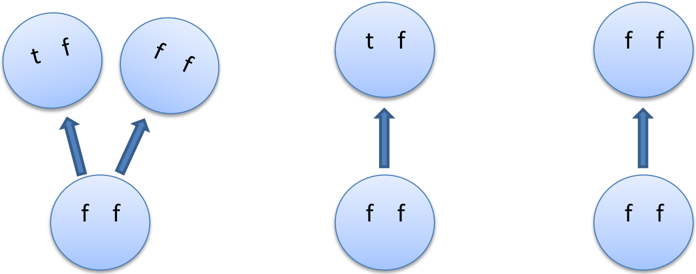
		$\vee$
t	**t**	$\{\mathbf{t}\}$
t	**f**	$\{\mathbf{t}\}$
f	**t**	$\{\mathbf{t}\}$
f	**f**	$\{\mathbf{f}\}$

Matrix-based Approach to Kripke-style Semantics: Intuition

- In standard Nmatrices the truth-value of $\diamond(\psi_1, \ldots, \psi_n)$ depends on (although is not necessarily uniquely determined by) the truth-values assigned to $\psi_1, \ldots, \psi_n$.
- In Kripke-style semantics the interpretation is more complex: the truth-value assigned to $\diamond(\psi_1, \ldots, \psi_n)$ in a world a depends, in addition to the truth-values assigned to $\psi_1, \ldots, \psi_n$ in a, also on the truth-values assigned to these formulas in all worlds $b \geq a$.
- However, which truth-values are assigned to $\psi_1, \ldots, \psi_n$ in which world is immaterial, what matters is their distribution:

$$D_a = \{\langle v(b, \psi_1), \ldots, v(b, \psi_n)\rangle \mid b \geq a\}$$

Distribution Examples



<f,f> {<f,f>,<t,f>} <f,f> {<f,f>,<t,f>} <f,f> {<f,f>}

"Reading Off" the Semantics from Canonical Rules

The simplest canonical calculus with no canonical rules for $\supset$:

	D	$\tilde{\supset}$
$\langle t, t\rangle$	$\{\langle t, t\rangle\}$	$\{t, f\}$
$\langle t, f\rangle$	$\{\langle t, f\rangle\}$	$\{t, f\}$
$\langle t, f\rangle$	$\{\langle t, f\rangle, \langle t, t\rangle\}$	$\{t, f\}$
$\langle f, t\rangle$	$\{\langle f, t\rangle\}$	$\{t, f\}$
$\langle f, t\rangle$	$\{\langle f, t\rangle, \langle t, t\rangle\}$	$\{t, f\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle\}$	$\{t, f\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle t, t\rangle\}$	$\{t, f\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle f, t\rangle\}$	$\{t, f\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle t, f\rangle\}$	$\{t, f\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle t, t\rangle, \langle f, t\rangle\}$	$\{t, f\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle t, f\rangle, \langle f, f\rangle\}$	$\{t, f\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle t, f\rangle, \langle f, t\rangle\}$	$\{t, f\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle t, f\rangle, \langle f, t\rangle, \langle t, t\rangle\}$	$\{t, f\}$

"Reading Off" the Semantics from Canonical Rules

Add the rule $\{p_1 \Rightarrow p_2\}/ \Rightarrow p_1 \supset p_2$:

	D	$\tilde{\supset}$
$\langle t, t\rangle$	$\{\langle t, t\rangle\}$	$\{\mathbf{t}\}$
$\langle t, f\rangle$	$\{\langle t, f\rangle\}$	$\{t, f\}$
$\langle t, f\rangle$	$\{\langle t, f\rangle, \langle t, t\rangle\}$	$\{t, f\}$
$\langle f, t\rangle$	$\{\langle f, t\rangle\}$	$\{\mathbf{t}\}$
$\langle f, t\rangle$	$\{\langle f, t\rangle, \langle t, t\rangle\}$	$\{\mathbf{t}\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle\}$	$\{\mathbf{t}\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle t, t\rangle\}$	$\{\mathbf{t}\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle f, t\rangle\}$	$\{\mathbf{t}\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle t, f\rangle\}$	$\{t, f\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle t, t\rangle, \langle f, t\rangle\}$	$\{\mathbf{t}\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle t, f\rangle, \langle f, f\rangle\}$	$\{t, f\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle t, f\rangle, \langle f, t\rangle\}$	$\{t, f\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle t, f\rangle, \langle f, t\rangle, \langle t, t\rangle\}$	$\{t, f\}$

"Reading Off" the Semantics from Canonical Rules

Add the rule $\langle\{\Rightarrow p_1\},\{p_2 \Rightarrow\}\rangle / p_1 \supset p_2 \Rightarrow$:

	D	$\tilde{\supset}$
$\langle t,t\rangle$	$\{\langle t,t\rangle\}$	$\{\mathbf{t}\}$
$\langle t,f\rangle$	$\{\langle t,f\rangle\}$	$\{\mathbf{f}\}$
$\langle t,f\rangle$	$\{\langle t,f\rangle,\langle t,t\rangle\}$	$\{\mathbf{f}\}$
$\langle f,t\rangle$	$\{\langle f,t\rangle\}$	$\{\mathbf{t}\}$
$\langle f,t\rangle$	$\{\langle f,t\rangle,\langle t,t\rangle\}$	$\{\mathbf{t}\}$
$\langle f,f\rangle$	$\{\langle f,f\rangle\}$	$\{\mathbf{t}\}$
$\langle f,f\rangle$	$\{\langle f,f\rangle,\langle t,t\rangle\}$	$\{\mathbf{t}\}$
$\langle f,f\rangle$	$\{\langle f,f\rangle,\langle f,t\rangle\}$	$\{\mathbf{t}\}$
$\langle f,f\rangle$	$\{\langle f,f\rangle,\langle t,f\rangle\}$	$\{t,f\}$
$\langle f,f\rangle$	$\{\langle f,f\rangle,\langle t,t\rangle,\langle f,t\rangle\}$	$\{\mathbf{t}\}$
$\langle f,f\rangle$	$\{\langle f,f\rangle,\langle t,f\rangle,\langle f,f\rangle\}$	$\{t,f\}$
$\langle f,f\rangle$	$\{\langle f,f\rangle,\langle t,f\rangle,\langle f,t\rangle\}$	$\{t,f\}$
$\langle f,f\rangle$	$\{\langle f,f\rangle,\langle t,f\rangle,\langle f,t\rangle,\langle t,t\rangle\}$	$\{t,f\}$

"Reading Off" the Semantics from Canonical Rules

	D	$\tilde{\supset}$
$\langle t, t\rangle$	$\{\langle t, t\rangle\}$	$\{\mathbf{t}\}$
$\langle \mathbf{t}, \mathbf{f}\rangle$	$\{\langle \mathbf{t}, \mathbf{f}\rangle\}$	$\{\mathbf{f}\}$
$\langle \mathbf{t}, \mathbf{f}\rangle$	$\{\langle \mathbf{t}, \mathbf{f}\rangle, \langle \mathbf{t}, \mathbf{t}\rangle\}$	$\{\mathbf{f}\}$
$\langle f, t\rangle$	$\{\langle f, t\rangle\}$	$\{\mathbf{t}\}$
$\langle f, t\rangle$	$\{\langle f, t\rangle, \langle t, t\rangle\}$	$\{\mathbf{t}\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle\}$	$\{\mathbf{t}\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle t, t\rangle\}$	$\{\mathbf{t}\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle f, t\rangle\}$	$\{\mathbf{t}\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle t, f\rangle\}$	$\{t, f\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle t, t\rangle, \langle f, t\rangle\}$	$\{\mathbf{t}\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle t, f\rangle, \langle f, f\rangle\}$	$\{t, f\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle t, f\rangle, \langle f, t\rangle\}$	$\{t, f\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle t, f\rangle, \langle f, t\rangle, \langle t, t\rangle\}$	$\{t, f\}$

"Reading Off" the Semantics from Canonical Rules

	D	$\tilde{\supset}$
$\langle t, t\rangle$	$\{\langle t, t\rangle\}$	$\{\mathbf{t}\}$
$\langle \mathbf{t}, \mathbf{f}\rangle$	$\{\langle \mathbf{t}, \mathbf{f}\rangle\}$	$\{\mathbf{f}\}$
$\langle \mathbf{t}, \mathbf{f}\rangle$	$\{\langle \mathbf{t}, \mathbf{f}\rangle, \langle \mathbf{t}, \mathbf{t}\rangle\}$	$\{\mathbf{f}\}$
$\langle f, t\rangle$	$\{\langle f, t\rangle\}$	$\{\mathbf{t}\}$
$\langle f, t\rangle$	$\{\langle f, t\rangle, \langle t, t\rangle\}$	$\{\mathbf{t}\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle\}$	$\{\mathbf{t}\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle t, t\rangle\}$	$\{\mathbf{t}\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle f, t\rangle\}$	$\{\mathbf{t}\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle t, f\rangle\}$	$\{\mathbf{f}\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle t, t\rangle, \langle f, t\rangle\}$	$\{\mathbf{t}\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle t, f\rangle, \langle f, f\rangle\}$	$\{\mathbf{f}\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle t, f\rangle, \langle f, t\rangle\}$	$\{\mathbf{f}\}$
$\langle f, f\rangle$	$\{\langle f, f\rangle, \langle t, f\rangle, \langle f, t\rangle, \langle t, t\rangle\}$	$\{\mathbf{f}\}$

An Algorithm for Removing Illegal Options

Let $\tilde{\diamond} : \mathbf{V}_n \to P^+(\{t, f\})$ be an interpretation of an n-ary connective $\diamond$. The reduced interpretation $\mathrm{R}(\tilde{\diamond})$ is obtained by the following algorithm:

- $L_0 \leftarrow \tilde{\diamond}$ and $i \leftarrow 0$.

Repeat

- $i \leftarrow i + 1$ and $L_i \leftarrow L_{i-1}$.
- Let $V = \langle \overline{x}, D \rangle$, such that $L_{i-1}(V) = \{t, f\}$. If there is some $\overline{y} \in D$, such that for every $D' \subseteq D$, such that $\langle \overline{y}, D' \rangle \in \mathbf{V}_n$: $L_{i-1}(\langle \overline{y}, D' \rangle) = \{f\}$, then $L_i(V) \leftarrow \{f\}$.

Until $L_i = L_{i-1}$

Determinism in Canonical Calculi is Decidable

Theorem

A connective $\diamond$ *is deterministic in* **G** *iff its truth-table read off the canonical rules of* **G** *for* $\diamond$ *and updated by the algorithm above, has no non-deterministic lines.*

Conclusions and Future Work

- Defined in precise terms determinism of Kripke-style semantics for canonical single-conclusion systems.
- Used it to semantically characterize (right) invertibility and axiom-expansion in these systems.
- Introduced a matrix-based presentation of Kripke-style non-deterministic semantics, which allows to decide determinism of connectives.
- Future work:
  - General theory of matrix-based Kripke-style semantics.
  - Characterization of further properties of canonical calculi.
  - Extending the results to more general systems.