Semantic Investigation of Proof Systems for Non-classical Logics

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Background

- Sequent calculi and their natural extensions (as many-sided sequents and hypersequents) are a prominent proof-theoretic framework.
- Suitable for a variety of logics:
  - Classical logic, intuitionistic logic
  - Modal logics, intermediate logics, bi-intuitionistic logic
  - Many-valued logics, fuzzy logics
  - Paraconsistent logics
  - Substructural logics, relevance logics
- Provide an "algorithmic presentation" of a logic, useful for:
  - working with the logic
  - studying its properties (decidability, interpolation, Herbrand theorem)

Two-Sided Sequent Calculi

- Proof systems that manipulate *sequents* rather than formulas
- Sequents are objects of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sequences/multisets/*sets* of formulas
- Semantic intuition:

$$A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m \quad \leftrightsquigarrow \quad A_1 \wedge \ldots \wedge A_n \supset B_1 \vee \ldots \vee B_m$$

- Logics can be obtained by:

$$A \text{ follows from } \mathcal{T} \quad \text{iff} \quad \{ \Rightarrow B \mid B \in \mathcal{T}\} \vdash \Rightarrow A$$

The calculus **LK** [Gentzen 1934]

Identity Axiom and Cut:

$$(id)\quad \frac{}{\Gamma, A \Rightarrow A, \Delta} \qquad (cut)\quad \frac{\Gamma, A \Rightarrow \Delta \quad \Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \Delta}$$

Structural Rules:

$$(W \Rightarrow)\quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad (\Rightarrow W)\quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}$$

Logical Rules:

$$(\neg \Rightarrow)\quad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta} \qquad (\Rightarrow \neg)\quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}$$

$$(\wedge \Rightarrow)\quad \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad (\Rightarrow \wedge)\quad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$$

$$(\supset \Rightarrow)\quad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta} \qquad (\Rightarrow \supset)\quad \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}$$

Maehara's Multiple-Conclusion **LJ**

Identity Axiom and Cut:

$$(id)\quad \frac{}{\Gamma, A \Rightarrow A, \Delta} \qquad (cut)\quad \frac{\Gamma, A \Rightarrow \Delta \quad \Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \Delta}$$

Structural Rules:

$$(W \Rightarrow)\quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad (\Rightarrow W)\quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}$$

Logical Rules:

$$(\neg \Rightarrow)\quad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta} \qquad (\Rightarrow \neg)\quad \frac{\Gamma, A \Rightarrow}{\Gamma \Rightarrow \neg A}$$

$$(\wedge \Rightarrow)\quad \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad (\Rightarrow \wedge)\quad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$$

$$(\supset \Rightarrow)\quad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta} \qquad (\Rightarrow \supset)\quad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B}$$

Some Useful Properties of Sequent Calculi

Cut-admissibility

If $\vdash \Gamma \Rightarrow \Delta$ then there is a cut-free proof of $\Gamma \Rightarrow \Delta$

Analyticity

If $\Omega \vdash \Gamma \Rightarrow \Delta$ then there is a proof of $\Gamma \Rightarrow \Delta$ from Ω using only the "syntactic material" inside Ω and $\Gamma \Rightarrow \Delta$

Semantics

A set of models for which the sequent calculus is sound and complete

Decidability

Given $\Omega, \Gamma \Rightarrow \Delta$ it is decidable whether $\Omega \vdash \Gamma \Rightarrow \Delta$ or not

Motivation For This Work

- Gentzen calculi for non-classical logics arise very often.
- A variety of works is devoted to study particular calculi for specific logics, proving:
  - Soundness and completeness with respect to the intended semantics.
  - Cut-elimination and/or analyticity.
- Traditional syntactic methods to prove cut-elimination are tedious and error prone.
  - They sometimes fail, e.g. for second-order logics.
- The intimate relations between proof-theory and semantics are still mysterious.

Main Contributions of the Thesis

- We study several general families of Gentzen-type calculi, and obtain:
  - General and modular methods to extract semantics from calculi
  - Criteria for the effectiveness of the semantics and general decidability results
- Semantic characterizations of proof-theoretic properties.
- This is done on the propositional level.
- In addition:
  - New semantic proof of cut-admissibility in the hypersequent calculus for first-order Gödel logic.
  - Completeness and cut-admissibility in the hypersequent calculus for second-order Gödel logic.

The Rest of this Talk

High-level overview of

- the different families of calculi included in this investigation
- the main contributions for each of them

- Pure many-sided sequent calculi
- Canonical many-sided sequent calculi
- Quasi-canonical many-sided sequent calculi
- Basic sequent calculi
- Canonical hypersequent calculi

Pure Calculi

Pure logical rules are rules that allow any context.

$$\frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta} \text{ but not } \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B}$$

Three-valued Lukasiewicz's Implication

$$\frac{\Gamma \Rightarrow \Delta \Rightarrow A, \Theta \quad B, \Gamma \Rightarrow \Delta \Rightarrow \Theta}{A \supset B, \Gamma \Rightarrow \Delta \Rightarrow \Theta}$$

$$\frac{\Gamma \Rightarrow A, \Delta \Rightarrow A, \Theta \quad \Gamma \Rightarrow A, B, \Delta \Rightarrow \Theta \quad B, \Gamma \Rightarrow \Delta \Rightarrow A, \Theta}{\Gamma \Rightarrow A \supset B, \Delta \Rightarrow \Theta}$$

$$\frac{A, \Gamma \Rightarrow A, \Delta \Rightarrow B, \Theta \quad A, \Gamma \Rightarrow B, \Delta \Rightarrow B, \Theta}{\Gamma \Rightarrow \Delta \Rightarrow A \supset B, \Theta}$$

Pure Calculi

Labelled presentation of many-sided sequents

- A finite set of labels $\pounds = \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}, {\color{green}\blacksquare}, \dots\}$
- Labelled formula:= $\square : A$
- Sequent:= a finite set of labelled formulas

Two-sided sequents $\pounds = \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$

$$p_1, p_1 \supset p_2 \Rightarrow p_2 \quad \leftrightsquigarrow \quad \{{\color{red}\blacksquare} : p_1, {\color{red}\blacksquare} : p_1 \supset p_2, {\color{blue}\blacksquare} : p_2\}$$

$$\frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \Delta} \quad \leftrightsquigarrow \quad \frac{\{{\color{red}\blacksquare} : A\} \cup s \qquad \{{\color{blue}\blacksquare} : A\} \cup s}{s}$$

$$\frac{}{\Gamma, A \Rightarrow A, \Delta} \quad \leftrightsquigarrow \quad \frac{}{\{{\color{red}\blacksquare} : A, {\color{blue}\blacksquare} : A\} \cup s}$$

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta} \quad \leftrightsquigarrow \quad \frac{\{{\color{red}\blacksquare} : A\} \cup s}{\{{\color{blue}\blacksquare} : \neg A\} \cup s}$$

Pure Calculi

1. All standard structural rules
   (exchange, contraction, weakening)
2. A finite set of primitive rules
3. A finite set of pure logical rules

Primitive Rules

All the premises and the conclusion have the form $\{\Box_1 : A, \ldots, \Box_n : A\} \cup s$ for some $\Box_1, \ldots, \Box_n \in \pounds$.

E.g.

$$\frac{\{{\color{blue}\blacksquare} : A\} \cup s \quad \{{\color{red}\blacksquare} : A\} \cup s}{\{{\color{orange}\blacksquare} : A, {\color{green}\blacksquare} : A\} \cup s}$$

Pure Calculi

Pure Logical Rules

$$\frac{c_1 \cup s \qquad \ldots \qquad c_n \cup s}{c \cup s}$$

for some sequent schemes $c_1, \ldots, c_n, c$ involving at least one connective

Example (Sequent Calculus for C_1 [Avron, Konikowska, Zamansky '12])

Several rules involve negation:

$$\frac{\{{\color{red}\blacksquare} : A\} \cup s}{\{{\color{blue}\blacksquare} : \neg A\} \cup s} \qquad \frac{\{{\color{red}\blacksquare} : A\} \cup s}{\{{\color{red}\blacksquare} : \neg\neg A\} \cup s}$$

$$\frac{\{{\color{blue}\blacksquare} : A\} \cup s \qquad \{{\color{blue}\blacksquare} : \neg A\} \cup s}{\{{\color{red}\blacksquare} : \neg(A \wedge \neg A)\} \cup s} \qquad \frac{\{{\color{red}\blacksquare} : \neg A\} \cup s \qquad \{{\color{red}\blacksquare} : \neg B\} \cup s}{\{{\color{red}\blacksquare} : \neg(A \wedge B)\} \cup s}$$

Semantics for Pure Calculi

Many-Valued System

Set of truth values + semantic conditions on valuations

Example (A Two-valued System)

$\mathcal{V} = \{0, 1\}$

1. If $v(A) = 0$ then $v(\neg\neg A) = 0$
2. If $v(A) = 1$ then $v(\neg\neg A) = 1$

Semantics for Pure Calculi

- Truth values are sets of labels: $\mathcal{V} \subseteq P(\pounds)$
- The set $\mathcal{V}$ is determined according to the primitive rules
- A valuation $v : Frm_{\mathcal{L}} \to \mathcal{V}$ is a model of $\square : A$ if $\square \in v(A)$
- v is a model of a sequent s if it is a model of some $\square : A$ in s

Example (The Case of Two-sided Calculi)

$$(id) \quad \frac{}{\{{\color{red}\blacksquare} : A, {\color{blue}\blacksquare} : A\} \cup s} \qquad (cut) \quad \frac{\{{\color{red}\blacksquare} : A\} \cup s \quad \{{\color{blue}\blacksquare} : A\} \cup s}{s}$$

	with (cut)	without (cut)
with (id)	$\mathcal{V} = \{\{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}\}$	$\mathcal{V} = \{\{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}\}$
without (id)	$\mathcal{V} = \{\emptyset, \{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}\}$	$\mathcal{V} = \{\emptyset, \{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}\}$

Semantics for Pure Calculi

Example (Sequent Calculus for C_1)

$$\frac{\{{\color{red}\blacksquare} : A\} \cup s}{\{{\color{blue}\blacksquare} : \neg A\} \cup s} \qquad \frac{\{{\color{red}\blacksquare} : A\} \cup s}{\{{\color{red}\blacksquare} : \neg\neg A\} \cup s}$$

$$\frac{\{{\color{blue}\blacksquare} : A\} \cup s \quad \{{\color{blue}\blacksquare} : \neg A\} \cup s}{\{{\color{red}\blacksquare} : \neg(A \wedge \neg A)\} \cup s} \qquad \frac{\{{\color{red}\blacksquare} : \neg A\} \cup s \quad \{{\color{red}\blacksquare} : \neg B\} \cup s}{\{{\color{red}\blacksquare} : \neg(A \wedge B)\} \cup s}$$

Corresponding semantic conditions:

1. If ${\color{red}\blacksquare} \in v(A)$ then ${\color{blue}\blacksquare} \in v(\neg A)$.
2. If ${\color{red}\blacksquare} \in v(A)$ then ${\color{red}\blacksquare} \in v(\neg\neg A)$.
3. If ${\color{blue}\blacksquare} \in v(A)$ and ${\color{blue}\blacksquare} \in v(\neg A)$ then ${\color{red}\blacksquare} \in v(\neg(A \wedge \neg A))$.
4. If ${\color{red}\blacksquare} \in v(\neg A)$ and ${\color{red}\blacksquare} \in v(\neg B)$ then ${\color{red}\blacksquare} \in v(\neg(A \wedge B)))$.

This semantics is non-deterministic.

Semantics for Pure Calculi

Theorem

Every pure calculus can be characterized by a many-valued system (with at most $2^{|\mathcal{L}|}$ truth values).

Analyticity

Notation: $\Omega \vdash^{\mathcal{F}} s$ iff there exists an $\mathcal{F}$-derivation of s from Ω (i.e. a derivation that consists only of formulas from $\mathcal{F}$).

Analyticity:= $\Omega \vdash s \quad \Longrightarrow \quad \Omega \vdash^{sub[\Omega,s]} s$

Observation

$\mathcal{F}$-derivations correspond to partial valuations whose domain is $\mathcal{F}$.

Theorem

A calculus is analytic iff every legal partial valuation whose domain is closed under subformulas can be extended to a full valuation.

This property is called *semantic analyticity*.

Cut-admissibility

- Cuts are special primitive rules.
- They forbid some truth values.

Example (Semantic Effect of Cuts)

$$\frac{\{\textcolor{red}{\blacksquare} : A\} \cup s \quad \{\textcolor{blue}{\blacksquare} : A\} \cup s \quad \{\textcolor{orange}{\blacksquare} : A\} \cup s}{s}$$

$\{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\} \not\subseteq X$ for every truth value $X \in \mathcal{V}$.

Semantic Equivalent of Cut-Admissibility

If there is a counter model then there is a counter model without the "forbidden truth values".

Effectiveness?

Drawback: the price to pay for the high generality is the fact that semantics is not always effective.

Semantic Decision Procedure

To decide whether s is valid, check one-by-one all legal partial valuations defined on the subformulas of s, and look for one which is not a model of s.

Hidden assumption: All legal partial valuations can be extended to full ones (semantic analyticity).

Corollary

Semantic analyticity $\Longrightarrow$ *Effective semantics*

Example: Generalized Semantic and Syntactic Analyticity

Example (Sequent Calculus for C_1)

- The many-valued system for C_1 does not enjoy semantic analyticity.
  - $\mathcal{V} = \{\{\color{red}\blacksquare\}, \{\color{blue}\blacksquare\}\}$
  - If $\color{red}\blacksquare$ $\in v(A)$ then $\color{blue}\blacksquare$ $\in v(\neg A)$.
  - If $\color{red}\blacksquare$ $\in v(A)$ then $\color{red}\blacksquare$ $\in v(\neg\neg A)$.
  - If $\color{blue}\blacksquare$ $\in v(A)$ and $\color{blue}\blacksquare$ $\in v(\neg A)$ then $\color{red}\blacksquare$ $\in v(\neg(A \wedge \neg A))$.
  - If $\color{red}\blacksquare$ $\in v(\neg A)$ and $\color{red}\blacksquare$ $\in v(\neg B)$ then $\color{red}\blacksquare$ $\in v(\neg(A \wedge B)))$.

  There is no way to assign a value to $\neg\neg p_2$ when:
  - $v(p_1) = v(p_2) = v(\neg\neg p_1) = \{\color{red}\blacksquare\}$
  - $v(\neg p_1) = v(\neg p_2) = v(\neg p_1 \wedge \neg p_2) = v(\neg(\neg p_1 \wedge \neg p_2)) = \{\color{blue}\blacksquare\}$
- However it enjoys *nsub*-analyticity:

  $$nsub = (sub \cup \{\langle \neg A_i, \neg(A_1 \diamond A_2)\rangle \mid A_1, A_2 \in Frm_{\mathcal{L}}, \diamond \in \{\wedge, \vee, \supset\}, i = 1, 2\})^*$$

  - Every partial valuation whose domain is closed under *nsub* can be extended to a full one.
  - Effective semantics + Syntactic *nsub*-analyticity

Many-Sided Canonical Calculi

- Pure calculi whose logical rules are canonical.

Example (Canonical Logical Rules [Avron, Lev '05] [Avron, Zamansky '09])

$$\frac{\{\color{red}{\blacksquare} : A\} \cup s}{\{\color{blue}{\blacksquare} : \neg A\} \cup s} \qquad \frac{\{\color{red}{\blacksquare} : A, \color{blue}{\blacksquare} : B\} \cup s \qquad \{\color{orange}{\blacksquare} : C\} \cup s}{\{\color{blue}{\blacksquare} : \heartsuit(A, B, C), \color{red}{\blacksquare} : \heartsuit(A, B, C)\} \cup s}$$

but not: $$\frac{\{\color{blue}{\blacksquare} : A\} \cup s \qquad \{\color{blue}{\blacksquare} : \neg A\} \cup s}{\{\color{red}{\blacksquare} : \neg(A \wedge \neg A)\} \cup s} \qquad \frac{\{\color{red}{\blacksquare} : \neg A\} \cup s \qquad \{\color{red}{\blacksquare} : \neg B\} \cup s}{\{\color{red}{\blacksquare} : \neg(A \wedge B)\} \cup s}$$

- This notion serves an old tradition in proof theory.

Our Contribution [Baaz, L, Zamansky IJCAR'12]

- Many-sided canonical calculi can be characterized by partial non-deterministic truth tables.
- This semantics is always effective.
- Analyticity and cut-admissibility are easily decided using the semantics.

Semantics Many-Sided Canonical Calculi

- We represent the semantic conditions by *non-deterministic truth tables*.
- The many-valued systems for canonical calculi correspond to the Nmatrices of [Avron, Lev '05].

Example (Primal Implication)

$$\frac{\{{\color{blue}\blacksquare} : A\} \cup s \quad \{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \rightsquigarrow B\} \cup s}$$

$$\frac{\{{\color{blue}\blacksquare} : B\} \cup s}{\{{\color{blue}\blacksquare} : A \rightsquigarrow B\} \cup s}$$

$$\mathcal{V} = \{\{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}\}$$

- If ${\color{blue}\blacksquare} \in v(A)$ and ${\color{red}\blacksquare} \in v(B)$ then ${\color{red}\blacksquare} \in v(A \rightsquigarrow B)$.
- If ${\color{blue}\blacksquare} \in v(B)$ then ${\color{blue}\blacksquare} \in v(A \rightsquigarrow B)$.

$\widetilde{\rightsquigarrow}$	$\{{\color{red}\blacksquare}\}$	$\{{\color{blue}\blacksquare}\}$
$\{{\color{red}\blacksquare}\}$	$\{\{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}\}$	$\{\{{\color{blue}\blacksquare}\}\}$
$\{{\color{blue}\blacksquare}\}$	$\{\{{\color{red}\blacksquare}\}\}$	$\{\{{\color{blue}\blacksquare}\}\}$

Semantics Many-Sided Canonical Calculi

- Non-determinism is a result of syntactic under-specification, or of non-standard primitive rules.

Example (Ordinary Implication without (*cut*))

$\mathcal{V} = \{\{\textcolor{red}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}\}, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}\}$

$\widetilde{\supset}$	$\{\textcolor{red}{\blacksquare}\}$	$\{\textcolor{blue}{\blacksquare}\}$	$\{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}$
$\{\textcolor{red}{\blacksquare}\}$	$\{\{\textcolor{blue}{\blacksquare}\}, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}\}$	$\{\{\textcolor{blue}{\blacksquare}\}, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}\}$	$\{\{\textcolor{blue}{\blacksquare}\}, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}\}$
$\{\textcolor{blue}{\blacksquare}\}$	$\{\{\textcolor{red}{\blacksquare}\}, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}\}$	$\{\{\textcolor{blue}{\blacksquare}\}, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}\}$	$\{\{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}\}$
$\{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}$	$\{\{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}\}$	$\{\{\textcolor{blue}{\blacksquare}\}, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}\}$	$\{\{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}\}$

New formulation of Girard's three-valued logic and Schütte valuations.

"The Empty Set Problem" – From Nmatrices to PNmatrices

$$\mathcal{V} = \{\{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}\}$$

$$\frac{\{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \diamond B\} \cup s} \qquad \frac{\{{\color{blue}\blacksquare} : A\} \cup s}{\{{\color{blue}\blacksquare} : A \diamond B\} \cup s}$$

$\widetilde{\diamond}$	$\{{\color{red}\blacksquare}\}$	$\{{\color{blue}\blacksquare}\}$
$\{{\color{red}\blacksquare}\}$	$\{\{{\color{red}\blacksquare}\}\}$	$\{\{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}\}$
$\{{\color{blue}\blacksquare}\}$	$\emptyset$	$\{\{{\color{blue}\blacksquare}\}\}$

$\{{\color{red}\blacksquare}\}$ and $\{{\color{blue}\blacksquare}\}$ cannot be used by the same valuation.

- We lose semantic analyticity.
- Effectiveness is recovered using:

Theorem

Given a PNmatrix, it is decidable whether a legal partial valuation can be extended to a full one or not.

Analyticity and Cut-admissibility

Theorem

The following are equivalent for every many-sided canonical calculus **G***:*

- *There are no empty sets in the PNmatrix constructed for* **G***.*
- **G** *is analytic.*
- **G** *enjoys a strong form of cut-admissibility.*

- Extends the result of [Avron,Lev '05].

Quasi-Canonical Calculi

Example (BK - fundamental logic of formal inconsistency)

$$(\wedge\Rightarrow)\quad \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad (\Rightarrow\wedge)\quad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$(\supset\Rightarrow)\quad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta} \qquad (\Rightarrow\supset)\quad \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}$$

$$(\Rightarrow\neg)\quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}$$

$$(\circ\Rightarrow)\quad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow \neg A, \Delta}{\Gamma, \circ A \Rightarrow \Delta} \qquad (\Rightarrow\circ)\quad \frac{\Gamma, A, \neg A \Rightarrow \Delta}{\Gamma \Rightarrow \circ A, \Delta}$$

$$(cut)\quad \frac{\Gamma, A \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta} \qquad (id)\quad \frac{}{\Gamma, A \Rightarrow \Delta, A} \qquad (weak)\quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Quasi-Canonical Calculi

Example (*BK* - fundamental logic of formal inconsistency)

$$({\color{red}\blacksquare} : \wedge) \quad \frac{\{{\color{red}\blacksquare} : A, {\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \wedge B\} \cup s} \qquad ({\color{blue}\blacksquare} : \wedge) \quad \frac{\{{\color{blue}\blacksquare} : A\} \cup s \quad \{{\color{blue}\blacksquare} : B\} \cup s}{\{{\color{blue}\blacksquare} : A \wedge B\} \cup s}$$

$$({\color{red}\blacksquare} : \supset) \quad \frac{\{{\color{red}\blacksquare} : A\} \cup s \quad \{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \supset B\} \cup s} \qquad ({\color{blue}\blacksquare} : \supset) \quad \frac{\{{\color{red}\blacksquare} : A, {\color{blue}\blacksquare} : B\} \cup s}{\{{\color{blue}\blacksquare} : A \supset B\} \cup s}$$

$$({\color{blue}\blacksquare} : \neg) \quad \frac{\{{\color{red}\blacksquare} : A\} \cup s}{\{{\color{blue}\blacksquare} : \neg A\} \cup s}$$

$$({\color{red}\blacksquare} : \circ) \quad \frac{\{{\color{blue}\blacksquare} : A\} \cup s \quad \{{\color{blue}\blacksquare} : \neg A\} \cup s}{\{{\color{red}\blacksquare} : \circ A\} \cup s} \qquad ({\color{blue}\blacksquare} : \circ) \quad \frac{\{{\color{red}\blacksquare} : A, {\color{red}\blacksquare} : \neg A\} \cup s}{\{{\color{blue}\blacksquare} : \circ A\} \cup s}$$

$$(cut) \quad \frac{\{{\color{red}\blacksquare} : A\} \cup s \quad \{{\color{blue}\blacksquare} : A\} \cup s}{s} \qquad (id) \quad \frac{}{\{{\color{red}\blacksquare} : A, {\color{blue}\blacksquare} : A\} \cup s} \qquad (weak) \quad \frac{s}{s \cup s'}$$

Quasi-Canonical Calculus $\to$ Canonical Calculus

[Baaz, L, Zamansky JAR'13]

Example (BK - fundamental logic of formal inconsistency)

- Add two labels: $\textcolor{red}{\blacksquare}_{\textcolor{red}{\neg}}$ and $\textcolor{blue}{\blacksquare}_{\textcolor{blue}{\neg}}$.
-

$$\frac{\{\textcolor{blue}{\blacksquare} : A\} \cup s \quad \{\textcolor{blue}{\blacksquare} : \neg A\} \cup s}{\{\textcolor{red}{\blacksquare} : \circ A\} \cup s} \qquad \frac{\{\textcolor{red}{\blacksquare} : A, \textcolor{red}{\blacksquare} : \neg A\} \cup s}{\{\textcolor{blue}{\blacksquare} : \circ A\} \cup s}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\frac{\{\textcolor{blue}{\blacksquare} : A\} \cup s \quad \{\textcolor{blue}{\blacksquare}_{\textcolor{blue}{\neg}} : A\} \cup s}{\{\textcolor{red}{\blacksquare} : \circ A\} \cup s} \qquad \frac{\{\textcolor{red}{\blacksquare} : A, \textcolor{red}{\blacksquare}_{\textcolor{red}{\neg}} : A\} \cup s}{\{\textcolor{blue}{\blacksquare} : \circ A\} \cup s}$$

- Add cut and axiom for the new labels:

$$\frac{\{\textcolor{red}{\blacksquare}_{\textcolor{red}{\neg}} : A\} \cup s \quad \{\textcolor{blue}{\blacksquare}_{\textcolor{blue}{\neg}} : A\} \cup s}{s} \qquad \frac{}{\{\textcolor{red}{\blacksquare}_{\textcolor{red}{\neg}} : A, \textcolor{blue}{\blacksquare}_{\textcolor{blue}{\neg}} : A\} \cup s}$$

- Add extra logical rules:

$$\frac{\{\textcolor{red}{\blacksquare}_{\textcolor{red}{\neg}} : A\} \cup s}{\{\textcolor{red}{\blacksquare} : \neg A\} \cup s} \qquad \frac{\{\textcolor{blue}{\blacksquare}_{\textcolor{blue}{\neg}} : A\} \cup s}{\{\textcolor{blue}{\blacksquare} : \neg A\} \cup s}$$

Quasi-Canonical Calculi

- Now, we can use the previous method to obtain a PNmatrix for this calculus, and use it in a decision procedure.

This translation is possible for calculi with logical rules of the form:

$$\frac{\textit{prem}_1 \quad \ldots \quad \textit{prem}_m}{\textit{conc}}$$

where:

- *conc* has one of the following forms:
  - $\Gamma, \diamond(A_1, \ldots, A_n) \Rightarrow \Delta$
  - $\Gamma \Rightarrow \diamond(A_1, \ldots, A_n), \Delta$
  - $\Gamma, \star\diamond(A_1, \ldots, A_n) \Rightarrow \Delta$ for some unary connective $\star$
  - $\Gamma \Rightarrow \star\diamond(A_1, \ldots, A_n), \Delta$ for some unary connective $\star$.
- Each $\textit{prem}_i$ has the form: $\Gamma, \Pi \Rightarrow \Sigma, \Delta$ where Π and Σ consist of A_i's and formulas of the form $\star A_i$ for some unary connective $\star$.

Non-pure Sequent Calculi
[L, Avron TOCL'13]

Various well-known calculi for important logics employ non-pure rules

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B} \qquad \frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A} \qquad \frac{\Gamma_1, \Box\Gamma_2 \Rightarrow A}{\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box A}$$

- We introduce and study *basic sequent systems* – a general family of (two-sided) fully-structural propositional sequent calculi, that allow rules of this kind.
- This family includes analytic calculi for:
  - Intuitionistic logic, its dual, and bi-intuitionistic logic
  - Many important modal logic
  - Primal logic with quotations [Gurevich, Neeman '11]

Main Results

- A correspondence between basic calculi and Kripke semantics.
  - General soundness and completeness
  - In many cases, we can easily derive the usual semantics
  - Modularity: each ingredient of the calculus corresponds to a semantic restriction
  - We may obtain non-deterministic semantics
- Semantic characterizations of analyticity and cut-admissibility
  - Analyticity corresponds to extensions of partial Kripke models to full ones
  - Cut-admissibility is characterized using three-valued non-deterministic Kripke models

Example: Bi-intuitionistic Logic

$$\frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, B \Rightarrow, \Delta}{\Gamma, A \supset B \Rightarrow \Delta} \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B}$$

$$\frac{A \Rightarrow B, \Delta}{A \prec B \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma \Rightarrow A \prec B, \Delta}$$

Accessibility relations $R, T \subseteq W \times W$ such that $R = T^{-1}$ and:

- If $v(w_1, A) = \mathrm{T}$ then $v(w_2, A) = \mathrm{T}$ for every w_2 such that $w_1 R w_2$.
- If $v(w_1, A) = \mathrm{F}$ then $v(w_2, A) = \mathrm{F}$ for every w_2 such that $w_1 T w_2$.

A Kripke valuation $v : W \times Frm_{\mathcal{L}} \to \{\mathrm{F}, \mathrm{T}\}$ satisfying:

- $v(w_1, A \supset B) = \mathrm{T}$ if $v(w_2, A) = \mathrm{F}$ or $v(w_2, B) = \mathrm{T}$ for every w_2 such that $w_1 R w_2$.
- $v(w, A \supset B) = \mathrm{F}$ if $v(w, A) = \mathrm{T}$ and $v(w, B) = \mathrm{F}$.
- $v(w_1, A \prec B) = \mathrm{F}$ if $v(w_2, A) = \mathrm{F}$ or $v(w_2, B) = \mathrm{T}$ for every w_2 such that $w_1 T w_2$.
- $v(w, A \prec B) = \mathrm{T}$ if $v(w, A) = \mathrm{T}$ and $v(w, B) = \mathrm{F}$.

Example: Bi-intuitionistic Logic

Corollary

The above calculus for bi-intuitionistic logic is analytic.

Proof.

Proving that this calculus is analytic reduces to proving that any partial Kripke model can be extended to a full one.
This can be easily done by structural induction. □

Note that this calculus does not enjoy cut-admissibility.

Hypersequent Calculi

Gödel Logic

- The truth values are $[0,1]$, where 1 is the only designated value.
- $\widetilde{\bot} = 1$, $\widetilde{\top} = 1$.
- $\widetilde{\wedge} = \min$, $\widetilde{\vee} = \max$.
- $\widetilde{\supset}$ is Gödel implication: $v(A \supset B) = \begin{cases} 1 & v(A) \leq v(B) \\ v(B) & otherwise \end{cases}$

"Syntactically" Gödel logic is obtained by adding (*Linearity*) to an axiomatization of intuitionistic logic.

$$(Linearity) \qquad (A \supset B) \vee (B \supset A)$$

The Proof-Theory of Gödel Logic

- Various sequent systems with ad-hoc logical rules of a nonstandard form (e.g., [Sonobe '75], [Corsi '86], [Avellone et al. '99], [Dyckhoff '99], [Avron, Konikowska '01], [Dyckhoff, Negri '06]).

- **HG** [Avron '91] employs standard logical rules, obtained by lifting **LJ** to the hypersequent level and adding the communication rule.

A *hypersequent* is a finite set of sequents denoted by:

$$\Gamma_1 \Rightarrow \Sigma_1 \mid \Gamma_2 \Rightarrow \Sigma_2 \mid \dots \mid \Gamma_n \Rightarrow \Sigma_n$$

The Calculus **HG**

Structural Rules:

$$(IW\Rightarrow)\quad \frac{H \mid \Gamma \Rightarrow \Sigma}{H \mid \Gamma, A \Rightarrow \Sigma} \qquad (\Rightarrow IW)\quad \frac{H \mid \Gamma \Rightarrow}{H \mid \Gamma \Rightarrow A} \qquad (EW)\quad \frac{H}{H \mid \Gamma \Rightarrow \Sigma}$$

$$(com)\quad \frac{H \mid \Gamma, \Delta \Rightarrow \Sigma_1 \qquad H \mid \Gamma, \Delta \Rightarrow \Sigma_2}{H \mid \Gamma \Rightarrow \Sigma_1 \mid \Delta \Rightarrow \Sigma_2}$$

Identity Rules:

$$(id)\quad \frac{}{A \Rightarrow A} \qquad (cut)\quad \frac{H \mid \Gamma \Rightarrow A \qquad H \mid \Gamma, A \Rightarrow \Sigma}{H \mid \Gamma \Rightarrow \Sigma}$$

Logical Rules:

$$(\Rightarrow\supset)\quad \frac{H \mid \Gamma, A \Rightarrow B}{H \mid \Gamma \Rightarrow A \supset B} \qquad (\supset\Rightarrow)\quad \frac{H \mid \Gamma \Rightarrow A \qquad H \mid \Gamma, B \Rightarrow \Sigma}{H \mid \Gamma, A \supset B \Rightarrow \Sigma}$$

$$(\Rightarrow\wedge)\quad \frac{H \mid \Gamma \Rightarrow A \qquad H \mid \Gamma \Rightarrow B}{H \mid \Gamma \Rightarrow A \wedge B} \qquad (\wedge\Rightarrow)\quad \frac{H \mid \Gamma, A, B \Rightarrow \Sigma}{H \mid \Gamma, A \wedge B \Rightarrow \Sigma}$$

Semantic Investigation of Hypersequent Gödel calculi

We studied hypersequent calculi with (*com*) and arbitrary canonical rules.

Example (And/Or)

$$\frac{H \mid \Gamma \Rightarrow A \quad H \mid \Gamma \Rightarrow B}{H \mid \Gamma \Rightarrow A \mathbin{\wedge\!\!\!\vee} B} \qquad \frac{H \mid \Gamma, A \Rightarrow E \quad H \mid \Gamma, B \Rightarrow E}{H \mid \Gamma, A \mathbin{\wedge\!\!\!\vee} B \Rightarrow E}$$

$$v(A \mathbin{\wedge\!\!\!\vee} B) \in [\min(v(A), v(B)), \max(v(A), v(B))]$$

Main idea for Characterizing Cut-Admissibility

When (*cut*) is not available, two truth values are assigned for each formula: one for its “left occurrences” and one for its “right occurrences”.

$$(id) \qquad v^{left}(A) \leq v^{right}(A)$$

$$(cut) \qquad v^{left}(A) \geq v^{right}(A)$$

First-Order and Second-Order Gödel Logic

- $\forall$ and $\exists$ are interpreted as inf and sup.
- The rules for the quantifiers are the usual hypersequential versions of the classical rules.

Our contribution

- A semantic proof of cut-admissibility in **HIF**, the extension of **HG** with rules for first-order quantifiers
  - Proved syntactically in [Baaz,Zach '00]
- Cut-admissibility for $\mathbf{HIF}^2$, the extension of **HIF** with rules for second-order quantifiers
  - Usual syntactic arguments fail

Conclusions

- We obtained new insights into the relations between semantics and proof-theory, in particular:
  - Semantic understanding of crucial proof-theoretic properties
  - Many-valued non-deterministic semantics is essential for characterizing abstract calculi
- We developed semantic toolbox for Gentzen-type calculi
  - Useful for studying particular calculi
  - Intended to complement the usual proof-theoretic methods

Thank you!