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Canonical Constructive Systems

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by

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Abstract

We define the notions of a canonical inference rule and a canonical system in the framework of single-conclusion sequential systems, and give a constructive condition for non-triviality of a canonical system. We develop a general non-deterministic Kripke-style semantics for such systems, and show that every constructive canonical system (i.e. coherent canonical single-conclusion system) induces a class of non-deterministic Kripke-style frames for which it is strongly sound and complete. We use this non-deterministic semantics to show that all constructive canonical systems admit a strong form of the cut-elimination theorem, and to provide a decision procedure for such systems. These results identify a large family of basic constructive connectives, each having both a proof-theoretical characterization in terms of a coherent set of canonical rules, as well as a semantic characterization using non-deterministic frames. The family includes the standard intuitionistic connectives, together with many other independent connectives.

Contents

Introduction

The Problem

The standard intuitionistic connectives ($\supset, \wedge, \vee$ and $\perp$ or $\supset, \wedge, \vee$ and $\neg$) are of great importance in theoretical computer science, especially in type theory, where they correspond to basic operations on types (via the formulas-as-types principle and Curry-Howard isomorphism).

Now a natural question is: what is so special about these connectives? The standard answer is that they are all constructive connectives. But then what exactly is a constructive connective, and can we define other basic constructive connectives beyond the standard intuitionistic ones? And what does the last question mean anyway: how do we "define" new (or old) connectives? Two main approaches can be taken to answer the last question:

1. **The proof theoretic approach.** According to this approach, a connective is defined by a set of axioms and derivation rules in some appropriate axiomatic proof system. These axioms and rules determine the way in which the connective is used in proofs. In particular, constructive connectives are defined by axioms and rules in axiomatic systems, which allow only constructive derivations. The best example for such a system is LJ, Gentzen's single-conclusion sequential system for intuitionistic logic.

2. **The model theoretic approach.** According to this approach, a connective is characterized by formal restrictions on the semantic values that models assign to formulas containing this connective. These restrictions determine which models are relevant when the connective under discussion is used. They might lead to deterministic semantics, in which there is only one way to set the value of the formula given the values of its subformulas, or to non-deterministic one, in which there is more than one way of doing so. In particular, constructive connectives are defined within the framework of some denotational semantics suitable for constructive reasoning. The best example is the semantics of Kripke frames for intuitionistic logic.

Our Approach

Our goal in this work is to combine the two approaches outlined above. Initially we take the first approach, and show a general way to define constructive connectives by providing a set of inference rules for it. We do this by introducing the notion of canonical constructive systems, which generalizes (as we show) Gentzen's original LJ. Then we generalize the semantics of Kripke frames for intuitionistic logic in order to provide semantics for every connective which can be defined in some canonical system. The key for doing this is the use of non-deterministic semantics. This makes it possible to provide simple semantics for many connectives, which lacked one up to now. We prove soundness and completeness of our semantics, and use it to prove some important properties of our proof systems and of the consequence relations they induce.

The main inspiration for this work is previous works about classical logic ([3, 4]). In these papers the notion of multiple-conclusion canonical rule was introduced. These rules are "well-behaved" inference rules in a multiple-conclusion Gentzen system: each rule is associated with exactly one connective; it includes exactly one occurrence of the introduced connective and no occurrences of other connectives; it is context-independent (pure in the sense of [2]) and there are no side conditions limiting its application. This gives canonical rules the *subformula property*, i.e. the formulas in the premises of the rule are immediate subformulas of its conclusion. Equipped with this notion, "multiple-conclusion canonical propositional Gentzen systems" were defined as sequential systems which have only canonical rules as logical rules. Then, a simple *coherence* property was suggested, and proved to be a necessary and sufficient condition for non-triviality and cut-elimination in multiple-conclusion canonical systems. This coherence criterion plays also an important role in our work about *single-conclusion* canonical systems (see definition 21).

These previous works also provide semantics for such systems in the form of two-valued *non-deterministic matrices* (two-valued *Nmatrices*). Nmatrices are a natural generalization of the classical truth-tables. They are obtained by discarding the principle of truth-functionality, and permitting truth tables to leave the value of the compound formula undetermined in some entries. In this way, a large family of connectives (which includes all the classical connectives) is defined. Each connective in this family has both a proof-theoretical characterization in terms of a coherent set of canonical rules, and a semantic characterization using two-valued Nmatrices. The present work can be seen as the constructive counterpart of [3, 4].

The central property that we prove for our canonical constructive systems is a strong form of *cut-elimination*. This ensures that the definition of a connective by set of inference rules in some canonical constructive system is independent of the system (which may include rules for other connectives). This kind of modularity is characteristic for systems with cut-elimination, in particular: canonical constructive systems.

Since cut-elimination is the most important technique in proof theory, a lot of effort has been put into characterizing systems which admit it (see [8]). Our work contributes in this field too. First, we simultaneously prove a strong form of cut-elimination for a large family of single-conclusion systems (the *canonical constructive systems*). Second, we offer a *coherence* criterion which is necessary and sufficient for cut-elimination in canonical systems. This criterion is very simple. Accordingly, checking whether a canonical system is *coherent* (and so admits cut-elimination) is an easy task. Finally, we investigate the connections between our strong form of cut-elimination and the usual one.

Thesis Organization

The rest of this thesis consists of 4 chapters:

1. Chapter 1 provides some necessary background about Gentzen's single-conclusion calculus and Kripke semantics for intuitionistic logic.
2. Chapter 2 describes some related work and discusses some of its advantages and shortcomings.
3. Chapter 3 is the main chapter of this thesis. It is divided into two sections: one about strict constructive systems and the other about non-strict ones (the difference between these two versions is explained in Section 1.1).
4. Chapter 4 provides our solution to the problem described above (i.e. "what is a constructive connective?"). It also describes the connections between our results and those that are presented in Chapter 2. Finally, it presents some directions for further research.

Finally, note that some results from this thesis appear in [5].

Chapter 1

Preliminaries

1.1 Gentzen's Single-Conclusion Calculus

Since a finitary consequence relation, $\vdash$, is determined by the set of pairs $\langle \Gamma, \varphi \rangle$ such that $\Gamma \vdash \varphi$, it is natural to base proof systems for logics on the use of such pairs. Originally, this was done by Gentzen in [9], when he introduced systems which manipulate *sequents*, instead of formulas. Gentzen suggested two different variants of a sequent:

1. A *multiple-conclusion* sequent which is constructed from two finite (possibly empty) sequences of formulas, separated by a new symbol (e.g. $\Rightarrow$).

2. A *single-conclusion* sequent which is a multiple-conclusion sequent of the form $\Gamma \Rightarrow \Delta$, such that Δ contains at most one formula.

The use of sequents made it possible to obtain "ideal" inference rules for the basic connectives. Each of these rules deals with a single connective and contains this connective only, and exactly once. These rules are divided to two types: left introduction rules, which introduce the connective on the left side of the sequent, and right introduction rules, which do it on the right side. By specifying the premises needed to introduce a connective on each side, this "ideal" kind of rules reflects the independent meaning of the connective, and thus these rules are traditionally considered as definitions of connectives.

Two sequential calculi were introduced in [9]: LK and LJ. LK was proved to be sound and complete with respect to classical logic, and LJ was proved to be sound and complete with respect to intuitionistic logic. While the inference rules of these two systems are the same, the difference between them is the type of sequents that are used in their derivations: while derivations in LK use multiple-conclusion sequents, LJ limits its sequents to be single-conclusion.

Although Gentzen used finite *sequences* of formulas, on both sides of a sequent, it is often more convenient to use *sets*, which eliminate the need for structural rules to deal with permutations and repetitions of formulas on both sides of a sequent. In the present work we assume the standard structural rules of minimal logic, i.e. we always assume the existence of the interchange rules and contraction rules. Thus, we will define a sequent using *sets* on both sides of a sequent.

We give here a set-version of the propositional fragment of LJ, which is the starting point of our work. The following axioms and rules are used in derivations by replacing φ and ψ by concrete formulas, replacing Γ by a finite set of concrete formulas, and either omitting E, or replacing it by a single concrete formula.

- Axioms

$$\varphi \Rightarrow \varphi$$

- Structural Rules
  - Weakening

$$\frac{\Gamma \Rightarrow E}{\Gamma, \Delta \Rightarrow E} \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \varphi}$$

  - Cut

$$\frac{\Gamma \Rightarrow \varphi \quad \Delta, \varphi \Rightarrow E}{\Gamma, \Delta \Rightarrow E}$$

- Logical Rules
  - Conjunction rules

$$\frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi} \qquad \frac{\Gamma, \varphi \Rightarrow E}{\Gamma, \varphi \wedge \psi \Rightarrow E} \quad \frac{\Gamma, \psi \Rightarrow E}{\Gamma, \varphi \wedge \psi \Rightarrow E}$$

  - Disjunction rules

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi} \quad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi} \qquad \frac{\Gamma, \varphi \Rightarrow E \quad \Gamma, \psi \Rightarrow E}{\Gamma, \varphi \vee \psi \Rightarrow E}$$

  - Implication rules

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \supset \psi} \qquad \frac{\Gamma \Rightarrow \varphi \quad \Gamma, \psi \Rightarrow E}{\Gamma, \varphi \supset \psi \Rightarrow E}$$

  - Negation rules

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma, \neg\varphi \Rightarrow} \qquad \frac{\Gamma, \varphi \Rightarrow}{\Gamma \Rightarrow \neg\varphi}$$

Gentzen proved the cut-elimination theorem ("Hauplatz") for LJ (and also for LK), which says that whenever there exists a proof (from no assumptions) of some sequent in LJ, then there exists a proof of it without any application of the cut rule. This property has many corollaries and applications, including decidability of the intuitionistic logic. The original proof of the cut-elimination theorem was done by complex syntactic arguments, and involved many case distinctions in order to go through all possible combinations of rules. This kind of proofs often leaves many details to the reader, and tends to contain inaccuracies or mistakes. In the present work, we will not follow this approach, and thus we do not describe it here.

In the same paper Gentzen also introduced natural deduction systems. These systems are another type of sequential systems, in which instead of using left introduction rules, one uses elimination rules. A single-conclusion sequent in these systems is defined as a multiple-conclusion sequent whose right side includes *exactly* one formula. In this framework it is impossible to have "ideal" rules for negation. To solve this, $\supset, \wedge, \vee$ and $\bot$ are used as basic connectives, and $\neg\varphi$ is an abbreviation for $\varphi \supset \bot$. [1]

Some later presentations of LJ also use single-conclusion sequents with exactly one formula on the right side, and again $\neg\varphi$ is defined as an abbreviation for $\varphi \supset \bot$. This version is more natural for deriving a consequence relation, since provability of a sequent of the form $\Gamma \Rightarrow \varphi$ is naturally identified with $\Gamma \vdash \varphi$. We will refer to this version of LJ as the *strict* version, while the original version will be called the *non-strict* one. In the present work we initially work within the strict framework and present a generalization of this version of LJ. Afterwards we present a generalization of the original non-strict version.

Our generalization of the strict version of LJ can be done in the natural deduction framework as well. This is briefly described in Subsection 3.2.6. Except for this subsection, henceforth by "sequential system", we mean Gentzen-type systems like LK and LJ. However, we will use the natural deduction terms "introduction rule" and "elimination rule", also for Gentzen-type systems, instead of "left introduction rule" and "right introduction rule".

[1] This is also natural (or even unavoidable) for the usual semantic point of view, in which the various connectives are characterized in terms of their proofs. A proof of $\neg\varphi$ is defined as a procedure that transforms any proof of φ to a proof of an absurd.

1.2 Kripke Semantics for the Intuitionistic Logic

The most useful semantics for propositional intuitionistic logic is that of Kripke frames, introduced in [12].

Let $\mathcal{F}$ is the set of wffs in the language of $\supset, \wedge, \vee$ and $\bot$.

Definition. A *Kripke frame* is a triple $\mathcal{W} = \langle W, \leq, v \rangle$ such that:

1. $\langle W, \leq \rangle$ is a nonempty partially ordered set. The elements of W will be referred as *worlds*.
2. v is a function from $W \times \mathcal{F}$ to $\{t, f\}$ satisfying the following conditions for every $a \in W$, and every two formulas φ and ψ:
   (a) persistence condition: $v(a, \varphi) = t$ implies $v(b, \varphi) = t$ for every $b \geq a$.
   (b) $v(a, \varphi \wedge \psi) = t$ iff $v(a, \varphi) = t$ and $v(a, \psi) = t$.
   (c) $v(a, \varphi \vee \psi) = t$ iff $v(a, \varphi) = t$ or $v(a, \psi) = t$.
   (d) $v(a, \varphi \supset \psi) = t$ iff $v(b, \varphi) = t$ implies $v(b, \psi) = t$ for every $b \geq a$.
   (e) $v(a, \bot) = f$.

Definition. A frame $\mathcal{W} = \langle W, \leq, v \rangle$ is a model of a sequent $\Gamma \Rightarrow E$ if for every $a \in W$, either $v(a, \psi) = f$ for some $\psi \in \Gamma$, or $E = \{\psi\}$ and $v(a, \psi) = t$.

Definition. $\mathcal{S} \vDash^{seq}_{Kripke} s$ (where $\mathcal{S}$ is a set of sequents and s is a sequent) iff every frame which is a model of $\mathcal{S}$ is also a model of s.

Kripke proved that this semantics is sound and complete for intuitionistic logic. It was later strengthened to the following strong soundness and completeness theorem:

Theorem. A sequent s is provable in LJ from a set of sequents $\mathcal{S}$ iff $\mathcal{S} \vDash^{seq}_{Kripke} s$.

Chapter 2

Related Previous Work

In this chapter we give an overview of some previous papers that are most relevant to this thesis. In the last chapter, after presenting our work, we return to compare our results to these works.

2.1 Logical Connectives for Intuitionistic Propositional Logic, Dean P. McCullough

In [13] McCullough examined the question of adding new constructive connectives to intuitionistic propositional logic. He showed that the set of four basic intuitionistic connectives ($\supset, \wedge, \vee$ and $\neg$ or $\supset, \wedge, \vee$ and $\bot$) is a functionally complete set for intuitionistic propositional logic. For this, he first had to characterize a general intuitionistic connective. His approach to this question was purely semantic, based on a generalization of Kripke frame semantics.

According to his definition, a constructive connective is defined by a "metalogical" formula. We reformulate his definitions, in a way suitable for our purposes. For us, McCullough's "metalogical" formulas are second order formulas, or more precisely: monadic logic of order (MLO) formulas, without quantification on second order variables. We use $X_1, X_2, ...$ as second order variables, and $x_1, x_2, ...$ as first order variables. The signature only includes a binary relation $\leq$ which is used between first order variables. A definition of an n-ary connective is a formula in this language with one first order free variable x_1, and n second order free variables $X_1, \ldots, X_n$. Using a set of such formulas, $\{\Phi_{\diamond_1}, \ldots, \Phi_{\diamond_m}\}$, the notion of Kripke frame is generalized as follows:

Definition. A $\{\Phi_{\diamond_1}, \ldots, \Phi_{\diamond_m}\}$-*frame* is a triple $\mathcal{W} = \langle W, \leq, v\rangle$ such that:

1. $\langle W, \leq\rangle$ is a nonempty partially ordered set.
2. v is a persistent function from $W \times \mathcal{F}$ (where $\mathcal{F}$ is the set of wffs in the language) to $\{t, f\}$ such that for every $1 \leq i \leq m$, $v(a, \diamond_i(\psi_1, \ldots, \psi_n)) = t$

iff $\mathcal{M}_{\mathcal{W}} \models \Phi_{\diamond_i}(a, \{b \in W \mid v(b,\psi_1) = t\}, \ldots, \{b \in W \mid v(b,\psi_n)) = t\})$ [1], where $\mathcal{M}_{\mathcal{W}}$ is the structure which is naturally induced by the frame: its domain is W and $\leq$ is interpreted as the $\leq$ relation of $\mathcal{W}$.

Example (Conjunction)**.** $\Phi_\wedge = X_1(x_1) \wedge X_2(x_1)$ defines the intuitionistic conjunction. In a $\{\Phi_\wedge\}$-frame, $v(a, \psi \wedge \varphi) = t$ iff $a \in \{b \in W \mid v(b,\psi) = t\}$ and $a \in \{b \in W \mid v(b,\varphi) = t\}$, i.e. iff $v(a,\psi) = t$ and $v(a,\varphi) = t$ as in a Kripke frame.

Example (Implication)**.** $\Phi_\supset = \forall x_2 \geq x_1 X_1(x_2) \supset X_2(x_2)$ defines the intuitionistic implication. In a $\{\Phi_\supset\}$-frame, $v(a, \psi \supset \varphi) = t$ iff for every $b \geq a$, $b \in \{c \in W \mid v(c,\psi) = f\}$ or $b \in \{c \in W \mid v(c,\varphi) = t\}$, i.e. iff for every $b \geq a$, $v(b,\psi) = f$ or $v(b,\varphi) = t$ as in a Kripke frame.

Example (Composition)**.** We can obtain other constructive connectives by composing defining formulas. For example, a ternary constructive connective can be obtained by composing $\Phi_\supset$ and $\Phi_\wedge$: $\Phi = X_1(x_1) \wedge (\forall x_2 \geq x_1 X_2(x_2) \supset X_3(x_2))$.

McCullough asserted that not every formula in this second order language can be used as a defining formula. For this he gave two more conditions:

1. $\Phi_\diamond$ is *monotone*:
   For every MLO-structure, $M = \langle D, I\rangle$ (where $I[\leq]$ is an order relation on D), $a \in D$ and upwards closed $P_1, \ldots, P_n \subseteq D$ ($b \in P_i$ and $c \geq b$ implies $c \in P_i$): if $M \models \Phi_\diamond(a, P_1, \ldots, P_n)$ and $b \geq a$ then $M \models \Phi_\diamond(b, P_1, \ldots, P_n)$.

2. All quantifiers in $\Phi_\diamond$ are bounded below, i.e all universal quantifiers are of the form $\forall b(a \leq b \supset \Psi)$, and all existential quantifiers are of the form $\exists b(a \leq b \wedge \Psi)$.

The first condition is necessary, since it reflects the persistence requirement of Kripke frames. This condition ensures the existence of $\{\Phi_\diamond\}$-frame. On the other hand, McCullough did not justify his syntactic second condition [2].

McCullough showed that every such defining formula is equivalent (over monotone valuations) to a composition of the four basic defining formulas ($\Phi_\supset$, $\Phi_\wedge, \Phi_\vee$, and $\Phi_\neg$ or $\Phi_\perp$). He concluded that every constructive connective (a connective which is characterized by such a formula) is equivalent to a composition of the four basic intuitionistic connectives. Hence, no *new* constructive connectives can be defined.

One of the main weak points in McCullough's work is the absence of any justification for the second criterion above. Some works (see [16] for example) replaced this condition with other conditions that are more natural. For our purposes it is important to note that McCullough's semantics is purely deterministic, following the principle of truth-functionality, since a $\mathcal{M}_{\mathcal{W}}$ is either a model of $\Phi_\diamond$ or not, and thus the values assigned to the subformulas of φ in all worlds b such that $b \geq a$ uniquely determine the value of φ in a.

[1] This way of writing is not totally formal, but the intention should be clear.

[2] The second condition is obviously related to the idea that the truth value of some formula should only be effected by the values of its subformulas in the *accessible* worlds.

2.2 An Extension of the Intuitionistic Propositional Calculus, K. A. Bowen

Unlike McCullough, Bowen (in [7]) followed a purely syntactic approach to define constructive connectives. In [7], Bowen offered an extension of LJ with two new intuitionistic connectives[3]. The new connectives were defined by the following introduction and elimination rules in a single-conclusion sequential system:

1. Converse Non-Implication, $\not\subset$:

$$\frac{\Gamma, \psi \Rightarrow \varphi}{\Gamma, \varphi \not\subset \psi \Rightarrow} \qquad \frac{\Gamma, \varphi \Rightarrow \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \not\subset \psi}$$

2. Not Both, $|$:

$$\frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma, \varphi \mid \psi \Rightarrow} \qquad \frac{\Gamma, \varphi \Rightarrow}{\Gamma \Rightarrow \varphi \mid \psi} \quad \frac{\Gamma, \psi \Rightarrow}{\Gamma \Rightarrow \varphi \mid \psi}$$

Bowen extended Gentzen's cut-elimination proof to the extension of LJ by $\not\subset$. He only gave the modifications that are needed in Gentzen's original proof, to prove that when we add the rules for $\not\subset$, the cut-elimination theorem is preserved. Using cut-elimination and a some syntactic arguments, he showed that $\not\subset$ cannot be expressed by the 4 basic connectives of LJ. He concluded that $\not\subset$ has no Kripke-style characterization in sense of McCullough. He left to the reader to do the same with $|$. Bowen did not provide semantic interpretation for his new connectives. He ended with an assertion that the same can be done in order to define similar n-ary connectives, which maintain the cut-elimination theorem.

His work motivated ours in the following points:

Generality We wished to offer more general theory about syntactic definitions of constructive connectives. We generalize LJ extended with Bowen's two new connectives, by showing a general type of logical rules that define constructive connectives, and generalize some of Bowen's results (especially cut-elimination) for these general systems. As a result, we do not need to repeat the proof for every connective, as Bowen did. Moreover, our work turns Bowen's final observation into precise notions.

Semantics A Kripke-style semantics helps to reveal the nature of connectives. As Bowen proved, McCullough's semantic approach for defining constructive connective does not apply to $\not\subset$ and $|$. We wanted to point the exact reason for this, and find appropriate semantics for Bowen's style connectives.

[3] He also presented "neither-nor" connective, which we do not describe here, since this connective can be expressed by the four basic intuitionistic connectives.

A simple corollary of cut-elimination in single conclusion systems is that in order to prove $\Rightarrow \diamond(\varphi_1, \ldots, \varphi_n)$ in such system, we do not have any other option but to prove the premises of one of its introduction rules, and then use that introduction rule. Bowen proved this for $\not\subset$, and again leaves $|$ for the reader. He saw this as the most characteristic property of *constructive* connectives. For example, concerning $\not\subset$ he writes: "For if we are to have evidence that A is not implied by B, what better evidence can we ask than evidence for B and evidence that A is absurd?". This argument may give another reason to claim that a connective that is defined in some single-conclusion system which admits cut-elimination is a constructive connective.

2.3 Nonstandard Connectives for Intuitionistic Propositional Logic, M. Kaminski

Kaminski (see [11]) followed the syntactic approach for defining connectives. He provided a general template of logical rules in a single-conclusion sequential system that may be used to define connectives. He defined (his definition 9) this template in a slightly opaque way. We give here our interpretation of his definition:

- The introduction template is:

$$\frac{\{\Gamma, A_i \Rightarrow B_i\}_{1 \leq i \leq m}}{\Gamma \Rightarrow \diamond(\psi_1, \ldots, \psi_n)}$$

  where:

  - m is the number of premises of the rule.
  - Each A_i should be a sequence (possibly empty) of formulas from the ψ_j's $(1 \leq j \leq n)$.
  - Each B_i should be one formula from the ψ_j's $(1 \leq j \leq n)$ or empty.

- The elimination template is:

$$\frac{\{\Gamma, A_i \Rightarrow B_i\}_{1 \leq i \leq m}}{\Gamma, \diamond(\psi_1, \ldots, \psi_n) \Rightarrow E}$$

  where:

  - m and A_i are as above.
  - Each B_i should be E, one formula from the ψ_j's $(1 \leq j \leq n)$ or empty.

The resulting rules (actually, schemes) are used in derivations as usual: by replacing the meta-variables ψ_i's by concrete formulas, replacing Γ by a finite set of concrete formulas, and either omitting E or replacing it by one concrete formula. Notice that LJ's logical rules, as well as Bowen's new rules can be formed of these templates. For example:

- Using the template of introduction rules, choosing $m = 1$, replacing A_1 by ψ_1 and B_1 by ψ_2 will give LJ's introduction rule for implication.
- Using the template of elimination rules, choosing $m = 2$, omitting A_1, replacing B_1 by ψ_1, A_2 by ψ_2 and B_2 by E will give LJ's elimination rule for implication.

Kaminski proved a cut-elimination theorem for every system which is obtained by adding one new connective with this kind of rules to LJ (his Theorem 9). However, he assumed that the resulting system is "consistent". He did not explicitly explain what he means by this term. Following his proof (his Lemma 5), it seems that he required that every sequent in the language of LK which is provable in the resulting system, is also provable in LK. This is a very strong requirement, and it is not clear how one verifies that some set of inference rules added to LJ creates a "consistent" extension.

Kaminski's cut-elimination proof is syntactic, and he only described the required additions to Gentzen's original cut-elimination proof. This makes the proof even more complex (triple induction instead of double), and harder to follow.

Kaminski did not give semantics for his general LJ extensions. Moreover, he claimed that no reasonable semantics can be used in the general case. The reason for this is that, in his opinion, a semantics that assigns the same value to φ_i and ψ_i ($1 \leq i \leq n$) should assign the same value to $\diamond(\varphi_1, \ldots, \varphi_n)$ and $\diamond(\psi_1, \ldots, \psi_n)$. This kind of semantics cannot be suitable for any such extension of LJ since there might happen (he gives one of Bowen's connective as an example) that $\Rightarrow \varphi_1 \equiv \psi_1, \ldots, \Rightarrow \varphi_n \equiv \psi_n$ are provable in a system, but $\Rightarrow \diamond(\varphi_1, \ldots, \varphi_n) \equiv \diamond(\psi_1, \ldots, \psi_n)$ is not provable (where $\equiv$ is an abbreviation for two implications). In the present work, we explicitly reject this criterion, as we offer non-deterministic semantics.

It should be noted that most of Kaminski's work is devoted to two other templates of logical rules, inspired by modal logic calculi. These templates are used to define *nonstandard* connectives. For them, he gave a (deterministic) Kripke-style semantics, and proved cut-elimination. This kind of rules is beyond the scope of this thesis.

2.4 Towards a Semantic Characterization of Cut-Elimination, A. Ciabattoni and K. Terui

In [8] Ciabattoni and Terui tried to identify exact conditions for cut-elimination in single-conclusion systems. They defined the notion of *simple calculi*, which are single-conclusion sequential systems which have $\varphi \Rightarrow \varphi$ as axioms, the cut rule, and any collection of other structural rules and logical rules from certain pre-defined families of rules. They dealt with a broad range of systems with various types of structural rules. Since in the present work we assume the standard structural rules of minimal logic, we give here an elaborated definition of logical rules in simple calculi, adapted for our framework.

Definition.

1. An introduction template[4] is an expression of the form:

$$\frac{\Gamma, \Upsilon_1 \Rightarrow \Psi_1 \quad \dots \quad \Gamma, \Upsilon_m \Rightarrow \Psi_m}{\Gamma \Rightarrow \diamond(\psi_1, \dots, \psi_n)}$$

   where Υ_i is a sequence of meta-variables from $\psi_1, \dots, \psi_n$, and Ψ_i is any meta-variable from $\psi_1, \dots, \psi_n$ or empty.

2. An elimination template is an expression of the form:

$$\frac{\Gamma, \Upsilon_1 \Rightarrow \Psi_1 \quad \dots \quad \Gamma, \Upsilon_m \Rightarrow \Psi_m}{\Gamma, \diamond(\psi_1, \dots, \psi_n) \Rightarrow E}$$

   where Υ_i is a sequence of meta-variables from $\psi_1, \dots, \psi_n$, and Ψ_i is E or some meta-variable from $\psi_1, \dots, \psi_n$.

3. Infinitely many introduction rules (or more precisely, schemes) are obtained from an introduction template by replacing Γ with a sequence of meta-variables of formulas.

4. Infinitely many elimination rules (or more precisely, schemes) are obtained from an elimination template by replacing Γ with a sequence of meta-variables of formulas, and omitting E or replacing it with a meta-variable of formula.

5. The generated rules are used in the derivations by replacing the meta-variables of formulas with concrete formulas. The formula which matches $\diamond(\psi_1, \dots, \psi_n)$ in an application of a rule as above is called the *principal formula* of the application.

Example (Conjunction)**.** One can define the intuitionistic conjunction in a simple calculus. Using the introduction template, an introduction rule is constructed for every finite Γ:

$$\frac{\Gamma \Rightarrow \psi_1 \quad \Gamma \Rightarrow \psi_2}{\Gamma \Rightarrow \psi_1 \wedge \psi_2}$$

Using the elimination template, an elimination rule is constructed for every finite Γ, and singleton or empty E:

$$\frac{\Gamma, \psi_1, \psi_2 \Rightarrow E}{\Gamma, \psi_1 \wedge \psi_2 \Rightarrow E}$$

Note that formally $\wedge$ has infinitely many rules in this simple calculus.

[4] A template is actually a "scheme of schemes".

Ciabattoni and Terui defined a syntactic property of a set of rules for a connective which they call *reductivity.* They proved that this property exactly characterize simple calculi which admits *reductive cut-elimination.* Both terms are defined in the following. Again, we adapt the definitions to our framework.

Definition. A simple calculus is called *reductive* iff whenever it includes two rules of the form:

$$\frac{\Gamma, \Upsilon_1 \Rightarrow \Psi_1 \ \dots \ \Gamma, \Upsilon_m \Rightarrow \Psi_m}{\Gamma \Rightarrow \diamond(\psi_1, \dots, \psi_n)} \quad \frac{\Gamma', \Upsilon'_1 \Rightarrow \Psi'_1 \ \dots \ \Gamma', \Upsilon'_k \Rightarrow \Psi'_k}{\Gamma', \diamond(\psi_1, \dots, \psi_n) \Rightarrow E}$$

then $\Gamma, \Gamma' \Rightarrow E$ is derivable from the assumptions $\Upsilon_1 \Rightarrow \Psi_1, \dots, \Upsilon_m \Rightarrow \Psi_m$ and $\Upsilon'_1 \Rightarrow \Psi'_1, \dots, \Upsilon'_k \Rightarrow \Psi'_k$ using only axioms, cut and weakening.

Example (Reductivity of Conjunction)**.** The set of logical rules for conjunction from the previous example is *reductive*, since the next derivation is possible for any Γ, Γ' and E:

$$\cfrac{\cfrac{\Rightarrow \psi_2 \quad \cfrac{\Rightarrow \psi_1 \quad \psi_1, \psi_2 \Rightarrow E}{\psi_2 \Rightarrow E}\,cut}{\Rightarrow E}\,cut}{\Gamma, \Gamma' \Rightarrow E}\,weak$$

Notice that formally we have to show infinitely many possible derivations, in order to prove reductivity of a set of rules of a connective.

Definition. A *reductive cut* is a application of the cut rule of the form:

$$\frac{\Gamma \Rightarrow \varphi \qquad \Pi, \varphi \Rightarrow E}{\Gamma, \Pi \Rightarrow E}$$

which satisfies at least one of the following conditions:

- $\Gamma \Rightarrow \varphi$ and $\Pi, \varphi \Rightarrow E$ are both derived by applications of a logical rule in which φ serves as the principal formula.
- $\Gamma \Rightarrow \varphi$ or $\Pi, \varphi \Rightarrow E$ is derived by application of a logical rule, in which φ does not serve as a principal formula, or by an application of weakening.
- $\Gamma \Rightarrow \varphi$ or $\Pi, \varphi \Rightarrow E$ is an axiom.

A simple calculus admits *reductive cut-elimination* iff whenever a sequent s is derivable from a set $\mathcal{S}$ of sequents, s has a derivation from $\mathcal{S}$ without any reducible cuts. The usual cut-elimination is implied by reductive cut-elimination, since the first cut in a derivation with no assumptions is always reducible.

Ciabattoni's and Terui's proof of the exact correspondence between reductivity and reductive cut-elimination is done semantically, using variants of phase semantics. This semantics is significantly more abstract and complicated than Kripke frame semantics. It is crucial to notice that non-determinism is implicitly used in this semantics.

Chapter 3

Canonical Constructive Systems

3.1 Basic Definitions and Notations

In what follows $\mathcal{L}$ is a propositional language, $\mathcal{F}$ is its set of wffs, p, q denote atomic formulas, ψ, φ, θ denote arbitrary formulas (of $\mathcal{L}$), T denotes subsets of $\mathcal{F}$, $\Gamma, \Delta, \Sigma, \Pi$ denote *finite* subsets of $\mathcal{F}$, and E, F denote subsets of $\mathcal{F}$, which are either singletons or empty. We assume that the atomic formulas of $\mathcal{L}$ are $p_1, p_2, \ldots$ (in particular: $\{p_1, \ldots, p_n\}$ are the first n atomic formulas of $\mathcal{L}$).

Definition 1. An $\mathcal{L}$*-substitution* is a function $\sigma : \mathcal{F} \to \mathcal{F}$, such that for every n-ary connective of $\mathcal{L}$, $\diamond$, we have: $\sigma(\diamond(\psi_1, \ldots, \psi_n)) = \diamond(\sigma(\psi_1), \ldots, \sigma(\psi_n))$. Obviously a substitution is determined by its values on the atomic formulas. A substitution is extended to sets of formulas in the obvious way.

Definition 2. A *sequent* is an expression of the form $\Gamma \Rightarrow E$ where Γ and E are finite sets of formulas, and E is either a singleton or empty. A sequent of the form $\Gamma \Rightarrow \{\varphi\}$ is called *definite*. A sequent of the form $\Gamma \Rightarrow \{\}$ is called *negative*. A Horn *clause* is a sequent which consists of atomic formulas only.

Notation. We mainly use s to denote a sequent and $\mathcal{S}$ to denote a set of sequents. For convenience, we identify φ and $\{\varphi\}$, and we sometimes do not write anything instead of $\{\}$. For example, we shall denote a sequent of the form $\Gamma \Rightarrow \{\varphi\}$ by $\Gamma \Rightarrow \varphi$, and a sequent of the form $\Gamma \Rightarrow \{\}$ by $\Gamma \Rightarrow$. We also employ the standard abbreviations, e.g. Γ, Δ instead of $\Gamma \cup \Delta$.

3.2 Strict Sequential Systems

This section deals with strict single-conclusion sequential systems, i.e. single-conclusion sequential systems which allow only definite sequents in their derivations.

3.2.1 Strict Canonical Constructive Systems

The following definitions formulate in exact terms the idea of an "ideal rule" which was described in the introduction and in Section 1.1.

Definition 3.

1. A *strict canonical introduction rule* is an expression of the form:

$$\{\Pi_i \Rightarrow q_i\}_{1\leq i\leq m} / \ \Rightarrow \diamond(p_1,\ldots,p_n)$$

   where $m \geq 0$, $\diamond$ is a connective of arity n, and for every $1 \leq i \leq m$, $\Pi_i \Rightarrow q_i$ is a *definite* Horn clause such that $\Pi_i \cup q_i \subseteq \{p_1,\ldots,p_n\}$.
   $\Pi_i \Rightarrow q_i$ $(1 \leq i \leq m)$ are called the *premises* of the rule.
   $\Rightarrow \diamond(p_1,\ldots,p_n)$ is called the *conclusion* of the rule.

2. A *strict canonical elimination rule* is an expression of the form

$$\{\Pi_i \Rightarrow E_i\}_{1\leq i\leq m} / \ \diamond(p_1,\ldots,p_n) \Rightarrow$$

   where $m, \diamond$ are as above, and for every $1 \leq i \leq m$, $\Pi_i \Rightarrow E_i$ is a Horn clause (either definite or negative) such that $\Pi_i \cup E_i \subseteq \{p_1,\ldots,p_n\}$.
   $\Pi_i \Rightarrow E_i$ $(1 \leq i \leq m)$ are called the *premises* of the rule.
   $\diamond(p_1,\ldots,p_n) \Rightarrow$ is called the *conclusion* of the rule.

3. An *application* of the rule $\{\Pi_i \Rightarrow q_i\}_{1\leq i\leq m} / \ \Rightarrow \diamond(p_1,\ldots,p_n)$ is any inference step of the form:

$$\frac{\{\Gamma, \sigma(\Pi_i) \Rightarrow \sigma(q_i)\}_{1\leq i\leq m}}{\Gamma \Rightarrow \sigma(\diamond(p_1,\ldots,p_n))}$$

   where Γ is a finite set of formulas and σ is a substitution in $\mathcal{L}$.

4. An *application* of the rule $\{\Pi_i \Rightarrow E_i\}_{1\leq i\leq m} / \ \diamond(p_1,\ldots,p_n) \Rightarrow$ is any inference step of the form:

$$\frac{\{\Gamma, \sigma(\Pi_i) \Rightarrow \sigma(E_i), F_i\}_{1\leq i\leq m}}{\Gamma, \sigma(\diamond(p_1,\ldots,p_n)) \Rightarrow \theta}$$

   where Γ and σ are as above, θ is a formula, and for every $1 \leq i \leq m$: $F_i = \theta$ in case E_i is empty, and F_i is empty otherwise.

Remark 4. While only definite sequents are used in the derivations of a strict system, negative sequents may appear in the formulations of elimination rules in the form of negative premises.

Example 5 (Conjunction)**.** The two usual rules for conjunction are:

$$\{p_1, p_2 \Rightarrow\ \} \ /\ p_1 \wedge p_2 \Rightarrow \quad \text{and} \quad \{\ \Rightarrow p_1\ ,\ \ \Rightarrow p_2\} \ /\ \ \Rightarrow p_1 \wedge p_2$$

Applications of these rules have the form:

$$\frac{\Gamma, \psi, \varphi \Rightarrow \theta}{\Gamma, \psi \wedge \varphi \Rightarrow \theta} \qquad \frac{\Gamma \Rightarrow \psi \quad \Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \psi \wedge \varphi}$$

The above elimination rule can easily be shown to be equivalent to the combination of the two more usual elimination rules for conjunction.

Example 6 (Disjunction)**.** The two usual introduction rules for disjunction are:

$$\{\ \Rightarrow p_1\} \ /\ \ \Rightarrow p_1 \vee p_2 \quad \text{and} \quad \{\ \Rightarrow p_2\} \ /\ \ \Rightarrow p_1 \vee p_2$$

Applications of these rules have then the form:

$$\frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \psi \vee \varphi} \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \psi \vee \varphi}$$

The usual elimination rule for disjunction is:

$$\{p_1 \Rightarrow\ ,\ \ p_2 \Rightarrow\} \ /\ p_1 \vee p_2 \Rightarrow$$

Its applications have the form:

$$\frac{\Gamma, \psi \Rightarrow \theta \quad \Gamma, \varphi \Rightarrow \theta}{\Gamma, \psi \vee \varphi \Rightarrow \theta}$$

Example 7 (Implication)**.** The two usual rules for implication are:

$$\{\Rightarrow p_1\ ,\ \ p_2 \Rightarrow\} \ /\ p_1 \supset p_2 \Rightarrow \quad \text{and} \quad \{p_1 \Rightarrow p_2\} \ /\ \ \Rightarrow p_1 \supset p_2$$

Applications of these rules have the form:

$$\frac{\Gamma \Rightarrow \psi \quad \Gamma, \varphi \Rightarrow \theta}{\Gamma, \psi \supset \varphi \Rightarrow \theta} \qquad \frac{\Gamma, \psi \Rightarrow \varphi}{\Gamma \Rightarrow \psi \supset \varphi}$$

Example 8 (Absurdity)**.** In intuitionistic logic there is no introduction rule for the absurdity constant $\bot$, and there is exactly one elimination rule for it:

$$\{\ \} \ /\ \ \bot \Rightarrow$$

Applications of this rule provide new *axioms*:

$$\Gamma, \bot \Rightarrow \theta$$

Example 9 (Semi-implication)**.** Suppose we introduce a "semi-implication" $\rightsquigarrow$ with the following two rules: [1]

$$\{\Rightarrow p_1 \ , \quad p_2 \Rightarrow\} \ / \ p_1 \rightsquigarrow p_2 \Rightarrow \quad \text{and} \quad \{\Rightarrow p_2\} \ / \quad \Rightarrow p_1 \rightsquigarrow p_2$$

Applications of these rules have the form:

$$\frac{\Gamma \Rightarrow \psi \qquad \Gamma, \varphi \Rightarrow \theta}{\Gamma, \psi \rightsquigarrow \varphi \Rightarrow \theta} \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \psi \rightsquigarrow \varphi}$$

Example 10 (Tonk)**.** In [14] Prior introduced a "connective" T (which he called "Tonk"). Its introduction and elimination rules can be formulated as canonical rules:

$$\{p_1 \Rightarrow \ \} \ / \ p_1 T p_2 \Rightarrow \qquad \text{and} \qquad \{ \ \Rightarrow p_2\} \ / \ \Rightarrow p_1 T p_2$$

Applications of these rules have the form:

$$\frac{\Gamma, \psi \Rightarrow \theta}{\Gamma, \psi T \varphi \Rightarrow \theta} \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \psi T \varphi}$$

Prior used "Tonk" to obtain a trivial system in order to show that rules alone cannot define a connective. In the following we deal with the problem that is raised by this connective.

Example 11 (Affirmation)**.** Suppose we introduce an "affirmation" connective $\triangleright$ with the following rules:

$$\{p_1 \Rightarrow \ \} \ / \ \ \triangleright p_1 \Rightarrow \quad \text{and} \quad \{\Rightarrow p_1\} \ / \ \Rightarrow \triangleright p_1$$

Applications of these rules have the form:

$$\frac{\Gamma, \varphi \Rightarrow \theta}{\Gamma, \triangleright \varphi \Rightarrow \theta} \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \triangleright \varphi}$$

Definition 12. A strict canonical system is a strict single-conclusion sequential system in which the axioms are the sequents of the form $\varphi \Rightarrow \varphi$, cut and weakening (in their strict version) are among its rules, and each of its other rules is either a strict canonical introduction rule or a strict canonical elimination rule.

Remark 13. The weakening and cut rules in their strict version are:

$$\frac{\Gamma \Rightarrow \psi}{\Gamma, \Delta \Rightarrow \psi} \qquad \frac{\Gamma \Rightarrow \varphi \qquad \Delta, \varphi \Rightarrow \psi}{\Gamma, \Delta \Rightarrow \psi}$$

Definition 14. Let **G** be a strict canonical system, and let $\mathcal{S} \cup \{s\}$ be a set of definite sequents. $\mathcal{S} \vdash^{seq}_{\mathbf{G}} s$ iff there exists a derivation in **G** of s from $\mathcal{S}$. The sequents of $\mathcal{S}$ are called *assumptions* (or *non-logical axioms*).

A crucial property of canonical systems is their consistency (on non-triviality). Intuitively, it means that "something" is not provable. It is defined as follows:

[1] The same connective was independently introduced in [10] for different purposes.

Definition 15. A strict canonical system **G** is called *consistent* iff $\nvdash^{seq}_{\mathbf{G}} p_1 \Rightarrow p_2$.

Remark 16. This property ensures the (usual) consistency of the induced consequence relation as described in Subsection 3.2.5.

It happens that in order to be consistent, the system must satisfy one more condition: [2]

Definition 17. A set $\mathcal{R}$ of strict canonical rules for an n-ary connective $\diamond$ is called *coherent* if $S_1 \cup S_2$ is classically inconsistent whenever $\mathcal{R}$ contains both $S_1 / \diamond(p_1, \ldots, p_n) \Rightarrow$ and $S_2 / \Rightarrow \diamond(p_1, \ldots, p_n)$.

Remark 18. It is known that a set of clauses is classically inconsistent iff the empty clause can be derived from it using only cuts.

Example 19. The sets of rules for the connectives $\wedge, \vee, \supset, \perp, \leadsto$ and $\rhd$, which were introduced in the examples above are coherent. For example, for the two rules for conjunction we have $S_1 = \{p_1, p_2 \Rightarrow \}$, $S_2 = \{ \Rightarrow p_1 \;,\; \Rightarrow p_2\}$, and $S_1 \cup S_2$ is the classically inconsistent set $\{p_1, p_2 \Rightarrow \;,\; \Rightarrow p_1 \;,\; \Rightarrow p_2\}$ (from which the empty sequent can be derived using two cuts).

Example 20 (Tonk)**.** The rules for Tonk were $\{p_1 \Rightarrow \} \;/\; p_1 T p_2 \Rightarrow$ and $\{ \Rightarrow p_2\} \;/\; \Rightarrow p_1 T p_2$. Now the union of the sets of premises of these two rules is $\{p_1 \Rightarrow \;,\; \Rightarrow p_2\}$, and this is a classically consistent set of clauses. It follows that Prior's set of rules for Tonk is incoherent.

Definition 21. A strict canonical system, **G**, is called *coherent* if every connective of the language of **G** has a coherent set of rules in **G**.

Theorem 22. *Every consistent strict canonical system is coherent.*

Proof. Let **G** be an incoherent strict canonical system. This means that **G** includes two rules $S_1 / \diamond(p_1, \ldots, p_n) \Rightarrow$ and $S_2 / \Rightarrow \diamond(p_1, \ldots, p_n)$, such that the set of clauses $S_1 \cup S_2$ is classically satisfiable. Let v be an assignment in $\{t, f\}$ that satisfies all the clauses in $S_1 \cup S_2$. Define a substitution σ by:

$$\sigma(p) = \begin{cases} p_{n+1} & v(p) = f \\ p & v(p) = t \end{cases}$$

Let $\Pi \Rightarrow q \in S_1 \cup S_2$. Then $\vdash^{seq}_{\mathbf{G}} p_1, \ldots, p_n, \sigma(\Pi) \Rightarrow \sigma(q)$. This is trivial in case $v(q) = t$, since in this case $\sigma(q) = q \in \{p_1, \ldots, p_n\}$. On the other hand, if $v(q) = f$ then $v(p) = f$ for some $p \in \Pi$ (since v satisfies the clause $\Pi \Rightarrow q$). Therefore in this case $\sigma(p) = \sigma(q) = p_{n+1}$, and so again $p_1, \ldots, p_n, \sigma(\Pi) \Rightarrow \sigma(q)$ is obtained by weakening of the axiom $p_{n+1} \Rightarrow p_{n+1}$. We can similarly prove that $\vdash^{seq}_{\mathbf{G}} p_1, \ldots, p_n, \sigma(\Pi) \Rightarrow p_{n+1}$ in case $\Pi \Rightarrow \; \in S_1$. Now by applying $S_1 / \diamond(p_1, \ldots, p_n) \Rightarrow$ and $S_2 / \Rightarrow \diamond(p_1, \ldots, p_n)$ to these provable sequents we get proofs in **G** of $p_1, \ldots, p_n \Rightarrow \diamond(p_1, \ldots, p_n)$ and of $p_1, \ldots, p_n, \diamond(p_1, \ldots, p_n) \Rightarrow p_{n+1}$. That $\vdash^{seq}_{\mathbf{G}} p_1, \ldots, p_n \Rightarrow p_{n+1}$ then follows using a cut. This easily entails that $\vdash^{seq}_{\mathbf{G}} p_1 \Rightarrow p_2$, and hence **G** is not consistent. □

[2] This is exactly the same condition suggested in [3, 4] for the multiple-conclusion case.

The last theorem implies that coherence is a necessary demand from any acceptable canonical system **G**. It follows that not every set of such rules is legitimate for defining constructive connectives - only coherent ones do (and this is what is wrong with "Tonk"). In the sequel (Corollary 44) we show that coherence is also sufficient to ensure the system's consistency[3]. Accordingly we define:

Definition 23. A *strict canonical constructive system* is a coherent strict canonical system.

The following definition will be needed in the sequel:

Definition 24. Let s be a sequent, $\mathcal{S}, \mathcal{S}'$ be sets of sequents, and **G** be a strict canonical system.

1. A cut is called an $\mathcal{S}$-cut if the cut formula occurs in $\mathcal{S}$.

2. A proof in **G** of s from $\mathcal{S}'$ is called an $\mathcal{S}$-proof if every cut in it is an $\mathcal{S}$-cut.

3. **G** admits cut-elimination iff whenever $\vdash^{seq}_{\mathbf{G}} s$, there exists a proof of s without cuts (i.e. there exists a $\emptyset$-proof).

4. ([2]) **G** admits strong cut-elimination iff whenever $\mathcal{S} \vdash^{seq}_{\mathbf{G}} s$, there exists an $\mathcal{S}$-proof of s from $\mathcal{S}$.

Notice that cut-elimination is a special case of strong cut-elimination with an empty $\mathcal{S}$. Also notice that by cut-elimination we mean here just the existence of proofs without (certain forms of) cuts, rather than an algorithm to transform a given proof to a cut-free one (for the assumption-free case the term *cut-admissibility* is sometimes used).

3.2.2 Semantics for Strict Canonical Systems

In this section we generalize Kripke semantics to arbitrary strict canonical constructive systems. For this we introduce *non-deterministic* Kripke frames and semiframes.

Definition 25. A *generalized $\mathcal{L}$-semiframe* is a triple $\mathcal{W} = \langle W, \leq, v\rangle$ such that:

1. $\langle W, \leq\rangle$ is a nonempty partially ordered set.

2. v is a persistent function from $W \times \mathcal{F}'$ to $\{t, f\}$, where $\mathcal{F}' \subseteq \mathcal{F}$ is closed under subformulas.

When v is defined on $W \times \mathcal{F}$ then the generalized $\mathcal{L}$-semiframe is also called *generalized $\mathcal{L}$-frame.*

Remark 26. Recall that a function $v : W \times \mathcal{F}' \to \{t, f\}$ is *persistent* iff for every $a \in W$ and $\varphi \in \mathcal{F}'$, $v(a, \varphi) = t$ implies that $v(b, \varphi) = t$ for every $b \geq a$.

[3] It is also implied by [3, 4]. It is shown there that the coherence of **G** implies the consistency of the *multiple* conclusion relation which is induced by **G**. That relation extends $\vdash^{seq}_{\mathbf{G}}$, and therefore **G** is consistent.

Since we only use the notions of *generalized* $\mathcal{L}$-frames and *generalized* $\mathcal{L}$-semiframes, we shall refer them as $\mathcal{L}$-frames and $\mathcal{L}$-semiframes.

Definition 27. Let $\mathcal{W} = \langle W, \leq, v\rangle$ be an $\mathcal{L}$-semiframe.

1. A sequent $\Gamma \Rightarrow E$ is *locally true* in $a \in W$ iff either $v(a, \psi) = f$ for some $\psi \in \Gamma$, or $E = \{\varphi\}$ and $v(a, \varphi) = t$.
2. A sequent is *true* in $a \in W$ iff it is locally true in every $b \geq a$.
3. $\mathcal{W}$ is a *model* of a sequent s if s is true in every $a \in W$ (iff s is locally true in every $a \in W$). It is a model of a set of sequents $\mathcal{S}$ if it is a model of every $s \in \mathcal{S}$.

Definition 28. Let $\mathcal{W} = \langle W, \leq, v\rangle$ be an $\mathcal{L}$-semiframe. An $\mathcal{L}$-substitution, σ, *(locally) satisfies* a Horn clause $\Pi \Rightarrow E$ in $a \in W$ iff $\sigma(\Pi) \Rightarrow \sigma(E)$ is (locally) true in a.

Remark 29. Because of the persistence condition, a definite Horn clause of the form $\Rightarrow q$ is satisfied in a by σ iff $v(a, \sigma(q)) = t$.

Definition 30. Let $\mathcal{W} = \langle W, \leq, v\rangle$ be an $\mathcal{L}$-semiframe.

1. An $\mathcal{L}$-substitution fulfils a strict canonical introduction rule in $a \in W$ iff it satisfies every premise of the rule in a.
2. An $\mathcal{L}$-substitution fulfils a strict canonical elimination rule in $a \in W$ iff it satisfies every definite premise of the rule in a, and locally satisfies every negative premise of the rule in a.

Definition 31. Let $\mathcal{W} = \langle W, \leq, v\rangle$ be an $\mathcal{L}$-semiframe, and let r be a strict canonical rule for $\diamond$. Assume that $W \times \mathcal{F}'$ is the domain of v. $\mathcal{W}$ *respects* r iff for every $a \in W$ and every substitution σ: if σ fulfils r in a and $\sigma(\diamond(p_1, \ldots, p_n)) \in \mathcal{F}'$ then σ locally satisfies r's conclusion in a.

Example 32 (Implication)**.** By definition, an $\mathcal{L}$-frame $\mathcal{W} = \langle W, \leq, v\rangle$ respects the rule ($\supset\Rightarrow$) iff for every $a \in W$, $v(a, \varphi \supset \psi) = f$ whenever $v(b, \varphi) = t$ for every $b \geq a$ and $v(a, \psi) = f$. Because of the persistence condition, this is equivalent to: $v(a, \varphi \supset \psi) = f$ whenever $v(a, \varphi) = t$ and $v(a, \psi) = f$. Again by the persistence condition, this is equivalent to: $v(a, \varphi \supset \psi) = f$ whenever there exists $b \geq a$ such that $v(b, \varphi) = t$ and $v(b, \psi) = f$. $\mathcal{W}$ respects ($\Rightarrow\supset$) iff for every $a \in W$, $v(a, \varphi \supset \psi) = t$ whenever for every $b \geq a$, either $v(b, \varphi) = f$ or $v(b, \psi) = t$. Hence the two rules together impose exactly the well-known Kripke semantics for intuitionistic implication ([12]).

Example 33 (Semi-Implication)**.** An $\mathcal{L}$-frame $\mathcal{W} = \langle W, \leq, v\rangle$ respects the rule ($\rightsquigarrow\Rightarrow$) under the same conditions it respects ($\supset\Rightarrow$). $\mathcal{W}$ respects ($\Rightarrow\rightsquigarrow$) iff for every $a \in W$, $v(a, \varphi \rightsquigarrow \psi) = t$ whenever $v(a, \psi) = t$ (recall that this is equivalent to: $v(b, \psi) = t$ for every $b \geq a$). Note that in this case the two rules for $\rightsquigarrow$ do not always determine the value assigned to $\varphi \rightsquigarrow \psi$: if $v(a, \psi) = f$, and there is no $b \geq a$ such that $v(b, \varphi) = t$ and $v(b, \psi) = f$, then $v(a, \varphi \rightsquigarrow \psi)$ is free to be either t or f. So the semantics of this connective is *non-deterministic*.

Example 34 (Affirmation)**.** An $\mathcal{L}$-frame $\mathcal{W} = \langle W, \leq, v\rangle$ respects the rule $(\rhd\Rightarrow)$ if $v(a, \rhd\ \psi) = f$ whenever $v(a,\psi) = f$. It respects $(\Rightarrow\rhd)$ if $v(a, \rhd\ \psi) = t$ whenever $v(a,\psi) = t$. This means that for every $a \in W$, $v(a, \rhd\ \psi)$ simply equals to $v(a,\psi)$.

Example 35 (Tonk)**.** An $\mathcal{L}$-frame $\mathcal{W} = \langle W, \leq, v\rangle$ respects the rule $(T \Rightarrow)$ if $v(a, \varphi T \psi) = f$ whenever $v(a,\varphi) = f$. It respects $(\Rightarrow T)$ if $v(a, \varphi T \psi) = t$ whenever $v(a,\psi) = t$. The two constraints contradict each other in case both $v(a,\varphi) = f$ and $v(a,\psi) = t$. This is a semantic explanation why Prior's "connective" T ("Tonk") is meaningless.

Definition 36. Let $\mathbf{G}$ be a strict canonical system for $\mathcal{L}$. An $\mathcal{L}$-semiframe is $\mathbf{G}$-*legal* iff it respects all the rules of $\mathbf{G}$.

We now can give the definition of the semantic relation induced by a strict canonical system:

Definition 37. Let $\mathbf{G}$ be a strict canonical constructive system, and let $\mathcal{S} \cup \{s\}$ be a set of definite sequents. $\mathcal{S} \vDash^{seq}_{\mathbf{G}} s$ iff every $\mathbf{G}$-legal $\mathcal{L}$-frame which is a model of $\mathcal{S}$ is also a model of s.

3.2.3 Soundness, Completeness, Cut-elimination

In this section we show that the two relations induced by a canonical constructive system $\mathbf{G}$ ($\vdash^{seq}_{\mathbf{G}}$ and $\vDash^{seq}_{\mathbf{G}}$) are identical. Half of this identity is given in the following theorem:

Theorem 38. *Every canonical constructive system* $\mathbf{G}$ *is strongly sound with respect to the semantics of* $\mathbf{G}$*-legal frames. In other words: If* $\mathcal{S} \vdash^{seq}_{\mathbf{G}} s$ *then* $\mathcal{S} \vDash^{seq}_{\mathbf{G}} s$.

Proof. Assume that $\mathcal{S} \vdash^{seq}_{\mathbf{G}} s$, and $\mathcal{W} = \langle W, \leq, v\rangle$ is a $\mathbf{G}$-legal model of $\mathcal{S}$. We show that s is locally true in every $a \in W$. Since the axioms of $\mathbf{G}$ and the assumptions of $\mathcal{S}$ trivially have this property, and the cut and weakening rules obviously preserve it, it suffices to show that the property of being locally true is preserved also by applications of the logical rules of $\mathbf{G}$.

- Suppose $\Gamma \Rightarrow \sigma(\diamond(p_1,\dots,p_n))$ is derived from $\{\Gamma, \sigma(\Pi_i) \Rightarrow \sigma(q_i)\}_{1\leq i\leq m}$ using the introduction rule $r = \{\Pi_i \Rightarrow q_i\}_{1\leq i\leq m} \Rightarrow \diamond(p_1,\dots,p_n)$. Assume that all the premises of this application have the required property. We show that so does its conclusion. Let $a \in W$. If $v(a,\psi) = f$ for some $\psi \in \Gamma$, then obviously $\Gamma \Rightarrow \sigma(\diamond(p_1,\dots,p_n))$ is locally true in a. Assume otherwise. Then the persistence condition implies that $v(b,\psi) = t$ for every $\psi \in \Gamma$ and $b \geq a$. Hence our assumption concerning $\{\Gamma, \sigma(\Pi_i) \Rightarrow \sigma(q_i)\}_{1\leq i\leq m}$ entails that for every $b \geq a$ and $1 \leq i \leq m$, either $v(b,\psi) = f$ for some $\psi \in \sigma(\Pi_i)$, or $v(b,\sigma(q_i)) = t$. It follows that for $1 \leq i \leq m$, $\Pi_i \Rightarrow q_i$ is satisfied in a by σ. Thus, σ fulfils r in a. Since $\mathcal{W}$ respects r, it follows that $v(a, \sigma(\diamond(p_1,\dots,p_n))) = t$, as required.

- Now we deal with the elimination rules. Suppose $\Gamma, \sigma(\diamond(p_1, \ldots, p_n)) \Rightarrow \theta$ is derived from $\{\Gamma, \sigma(\Pi_i) \Rightarrow \sigma(q_i)\}_{1 \leq i \leq m_1}$ and $\{\Gamma, \sigma(\Pi_i) \Rightarrow \theta\}_{m_1+1 \leq i \leq m}$, using the rule $r = \{\Pi_i \Rightarrow E_i\}_{1 \leq i \leq m} / \diamond(p_1, \ldots, p_n) \Rightarrow$ (where $E_i = \{q_i\}$ for $1 \leq i \leq m_1$ and E_i is empty for $m_1 + 1 \leq i \leq m$). Assume that all the premises of this application have the required property. We show that so does its conclusion. Let $a \in W$. If $v(a, \psi) = f$ for some $\psi \in \Gamma$ or $v(a, \theta) = t$, then we are done. Assume otherwise. Then $v(a, \theta) = f$, and (by the persistence condition) $v(b, \psi) = t$ for every $\psi \in \Gamma$ and $b \geq a$. Hence our assumption concerning $\{\Gamma, \sigma(\Pi_i) \Rightarrow \sigma(q_i)\}_{1 \leq i \leq m_1}$ entails that for every $b \geq a$ and $1 \leq i \leq m_1$, either $v(b, \psi) = f$ for some $\psi \in \sigma(\Pi_i)$, or $v(b, \sigma(q_i)) = t$. This immediately implies that the definite premises of r are satisfied in a by σ. Since $v(a, \theta) = f$, our assumption concerning $\{\Gamma, \sigma(\Pi_i) \Rightarrow \theta\}_{m_1+1 \leq i \leq m}$ entails that for every $m_1 + 1 \leq i \leq m$, $v(a, \psi) = f$ for some $\psi \in \sigma(\Pi_i)$. Hence the negative premises of r are locally satisfied in a by σ. Thus, σ fulfils r in a. Since $\mathcal{W}$ respects r, it follows that $v(a, \sigma(\diamond(p_1, \ldots, p_n))) = f$, as required.

□

For the converse, we first prove the following key result.

Theorem 39. *Let* **G** *be a strict canonical constructive system in* $\mathcal{L}$*, and let* $\mathcal{S} \cup \{s\}$ *be a set of definite sequents in* $\mathcal{L}$*. Then either there is an* $\mathcal{S}$*-proof of* s *from* $\mathcal{S}$*, or there is a* **G***-legal* $\mathcal{L}$*-frame which is a model of* $\mathcal{S}$ *but not a model of* s*.*

Proof. Assume that $s = \Gamma_0 \Rightarrow \varphi_0$ does not have an $\mathcal{S}$-proof in **G**. We construct a **G**-legal $\mathcal{L}$-frame $\mathcal{W}$ which is a model of $\mathcal{S}$ but not of s. Let $\mathcal{F}'$ be the set of subformulas of $\mathcal{S} \cup \{s\}$. Given a formula $\varphi \in \mathcal{F}'$, call a theory $\mathcal{T} \subseteq \mathcal{F}'$ φ-*maximal* if there is no finite $\Gamma \subseteq \mathcal{T}$ such that $\Gamma \Rightarrow \varphi$ has an $\mathcal{S}$-proof from $\mathcal{S}$, but every proper extension $\mathcal{T}' \subseteq \mathcal{F}'$ of $\mathcal{T}$ contains such a finite subset Γ. Obviously, if $\Gamma \cup \{\varphi\} \subseteq \mathcal{F}'$ and $\Gamma \Rightarrow \varphi$ has no $\mathcal{S}$-proof from $\mathcal{S}$, then Γ can be extended to a theory $\mathcal{T} \subseteq \mathcal{F}'$ which is φ-maximal. In particular, Γ_0 can be extended to a φ_0-maximal theory $\mathcal{T}_0$.

Now let $\mathcal{W} = \langle W, \subseteq, v \rangle$, where:

- W is the set of all extensions of $\mathcal{T}_0$ in $\mathcal{F}'$ which are φ-maximal for some $\varphi \in \mathcal{F}'$.
- v is defined inductively as follows. For atomic formulas:

  $$v(\mathcal{T}, p) = \begin{cases} t & p \in \mathcal{T} \\ f & p \notin \mathcal{T} \end{cases}$$

  Suppose $v(\mathcal{T}, \psi_i)$ has been defined for every $\mathcal{T} \in W$ and $1 \leq i \leq n$. We let $v(\mathcal{T}, \diamond(\psi_1, \ldots, \psi_n)) = t$ iff at least one of the following holds:

  1. There exists an introduction rule for $\diamond$ which is fulfilled in $\mathcal{T}$ by a substitution σ such that $\sigma(p_i) = \psi_i$ $(1 \leq i \leq n)$.

2. $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{T}$ and there does not exist $\mathcal{T}' \in W$, $\mathcal{T} \subseteq \mathcal{T}'$, and an elimination rule for $\diamond$ which is fulfilled in $\mathcal{T}'$ by a substitution σ such that $\sigma(p_i) = \psi_i$ $(1 \leq i \leq n)$. [4]

First we prove that $\mathcal{W}$ is an $\mathcal{L}$-frame:

- W is not empty because $\mathcal{T}_0 \in W$.
- We prove by structural induction that v is persistent:
  For atomic formulas v is trivially persistent since the order is $\subseteq$.
  Assume that v is persistent for $\psi_1, \ldots, \psi_n$. We prove its persistence for $\diamond(\psi_1, \ldots, \psi_n)$. So assume that $v(\mathcal{T}, \diamond(\psi_1, \ldots, \psi_n)) = t$ and $\mathcal{T} \subseteq \mathcal{T}^*$. By v's definition there are two possibilities:
  1. There exists an introduction rule for $\diamond$ which is fulfilled in $\mathcal{T}$ by a substitution σ such that $\sigma(p_i) = \psi_i$ $(1 \leq i \leq n)$. This is also true in $\mathcal{T}^*$, and so $v(\mathcal{T}^*, \diamond(\psi_1, \ldots, \psi_n)) = t$.
  2. $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{T}$ and there does not exist $\mathcal{T}' \in W$, $\mathcal{T} \subseteq \mathcal{T}'$, and an elimination rule for $\diamond$ which is fulfilled in $\mathcal{T}'$ by a substitution σ such that $\sigma(p_i) = \psi_i$ $(1 \leq i \leq n)$. Then $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{T}^*$ (since $\mathcal{T} \subseteq \mathcal{T}^*$), and there surely does not exist $\mathcal{T}' \in W$, $\mathcal{T}^* \subseteq \mathcal{T}'$, and an elimination rule for $\diamond$ whose which is fulfilled in $\mathcal{T}'$ by such σ (otherwise the same would hold for $\mathcal{T}$). Hence $v(\mathcal{T}^*, \diamond(\psi_1, \ldots, \psi_n)) = t$ in this case too.

Next we prove that $\mathcal{W}$ is **G**-legal:

1. The introduction rules are directly respected by the first condition in v's definition.
2. Let r be an elimination rule for $\diamond$, and suppose r is fulfilled by a substitution σ, such that $\sigma(p_i) = \psi_i$ $(1 \leq i \leq n)$. Then neither of the conditions under which $v(\mathcal{T}, \diamond(\psi_1, \ldots, \psi_n)) = t$ can hold:
   (a) The second condition explicitly excludes the option that r is fulfilled by σ (in any $\mathcal{T}' \in W$, $\mathcal{T} \subseteq \mathcal{T}'$, so also in $\mathcal{T}$ itself).
   (b) The first condition cannot be met because of **G**'s coherence, which does not allow the two sets of premises (of an introduction rule and an elimination rule for the same connective) to be locally satisfied together, and hence the two rules cannot be both fulfilled by the same substitution in the same element of W. To see this, assume by the way of contradiction that S_1 is the set of premises of an elimination rule for $\diamond$, S_2 is the set of premises of an introduction rule for $\diamond$, and there exists $\mathcal{T} \in W$ in which both sets of premises are locally satisfied by a substitution σ such that $\sigma(p_i) = \psi_i$ $(1 \leq i \leq n)$. Let u

[4] This inductive definition is not totally formal, since satisfaction by a substitution is defined for a $\mathcal{L}$-frame, which we are in the middle of constructing, but the intention should be clear.

be an assignment in $\{t, f\}$ in which $u(p_i) = v(\mathcal{T}, \psi_i)$. Since σ locally satisfies in $\mathcal{T}$ both sets of premises, u classically satisfies S_1 and S_2. This contradicts the coherence of **G**.

It follows that $v(\mathcal{T}, \diamond(\psi_1, \ldots, \psi_n)) = f$, as required.

It remains to prove that $\mathcal{W}$ is a model of $\mathcal{S}$ but not of s. For this we first prove that the following hold for every $\mathcal{T} \in W$ and every formula $\psi \in \mathcal{F}'$:

(a) If $\psi \in \mathcal{T}$ then $v(\mathcal{T}, \psi) = t$.

(b) If $\mathcal{T}$ is ψ-maximal then $v(\mathcal{T}, \psi) = f$.

We prove **(a)** and **(b)** together by a simultaneous induction on the complexity of ψ. For atomic formulas they easily follow from v's definition, and the fact that $p \Rightarrow p$ is an axiom. For the induction step, assume that **(a)** and **(b)** hold for $\psi_1, \ldots, \psi_n \in \mathcal{F}'$. We prove them for $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{F}'$.

- Assume that $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{T}$, but $v(\mathcal{T}, \diamond(\psi_1, \ldots, \psi_n)) = f$. By v's definition, since $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{T}$ there should exist $\mathcal{T}' \in W$, $\mathcal{T} \subseteq \mathcal{T}'$, and an elimination rule, $r = \{\Pi_i \Rightarrow E_i\}_{1 \leq i \leq m} / \diamond(p_1, \ldots, p_n) \Rightarrow$, which is fulfilled in $\mathcal{T}'$ by a substitution σ such that $\sigma(p_i) = \psi_i$ $(1 \leq i \leq n)$. Let $\{\Pi_i \Rightarrow\}_{1 \leq i \leq m_1}$ be the negative premises of r and, $\{\Pi_i \Rightarrow q_i\}_{m_1+1 \leq i \leq m}$ be the definite ones. Since σ locally satisfies in $\mathcal{T}'$ every sequent in $\{\Pi_i \Rightarrow\}_{1 \leq i \leq m_1}$, then for every $1 \leq i \leq m_1$ there exists $\psi_{j_i} \in \sigma(\Pi_i)$ such that $v(\mathcal{T}', \psi_{j_i}) = f$. By the induction hypothesis this implies that for every $1 \leq i \leq m_1$, there exists $\psi_{j_i} \in \sigma(\Pi_i)$ such that $\psi_{j_i} \notin \mathcal{T}'$. Let φ be the formula for which $\mathcal{T}'$ is maximal. Then for every $1 \leq i \leq m_1$ there is a finite $\Delta_i \subseteq \mathcal{T}'$ such that $\Delta_i, \psi_{j_i} \Rightarrow \varphi$ has an $\mathcal{S}$-proof from $\mathcal{S}$, and so $\Delta_i, \sigma(\Pi_i) \Rightarrow \varphi$ has such a proof. This in turn implies that there must exist $m_1 + 1 \leq i_0 \leq m$ such that $\Gamma, \sigma(\Pi_{i_0}) \Rightarrow \sigma(q_{i_0})$ has no $\mathcal{S}$-proof from $\mathcal{S}$ for any finite $\Gamma \subseteq \mathcal{T}'$. Indeed, if such a proof exists for every $m_1 + 1 \leq i \leq m$, we would use the m_1 proofs of $\Delta_i, \sigma(\Pi_i) \Rightarrow \varphi$ for $1 \leq i \leq m_1$, the $m - m_1$ proofs for $\Gamma_i, \sigma(\Pi_i) \Rightarrow \sigma(q_i)$ for $m_1 + 1 \leq i \leq m$, some trivial weakenings, and the elimination rule r to get an $\mathcal{S}$-proof from $\mathcal{S}$ of the sequent $\cup_{i=1}^{i=m_1} \Delta_i, \cup_{i=m_1+1}^{i=m} \Gamma_i, \diamond(\psi_1, \ldots, \psi_n) \Rightarrow \varphi$. Since $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{T}$, this would contradict $\mathcal{T}'$'s φ-maximality. Using this i_0, extend $\mathcal{T}' \cup \sigma(\Pi_{i_0})$ to a $\sigma(q_{i_0})$-maximal theory $\mathcal{T}''$. By the induction hypothesis, $v(\mathcal{T}'', \psi) = t$ for every $\psi \in \sigma(\Pi_{i_0})$ and $v(\mathcal{T}'', \sigma(q_{i_0})) = f$. Since $T' \subseteq \mathcal{T}''$, this contradicts the fact that σ satisfies $\Pi_{i_0} \Rightarrow q_{i_0}$ in $\mathcal{T}'$.

- Assume that $\mathcal{T}$ is $\diamond(\psi_1, \ldots, \psi_n)$-maximal, but $v(\mathcal{T}, \diamond(\psi_1, \ldots, \psi_n)) = t$. Obviously, $\diamond(\psi_1, \ldots, \psi_n) \notin \mathcal{T}$ (because $\diamond(\psi_1, \ldots, \psi_n) \Rightarrow \diamond(\psi_1, \ldots, \psi_n)$ is an axiom). Hence by v's definition there exists an introduction rule for $\diamond$, $r = \{\Pi_i \Rightarrow q_i\}_{1 \leq i \leq m} / \Rightarrow \diamond(p_1, \ldots, p_n)$, which is fulfilled in $\mathcal{T}$, by a substitution σ such that $\sigma(p_i) = \psi_i$ $(1 \leq i \leq n)$. As in the previous case, there must exist $1 \leq i_0 \leq m$ such that $\Gamma, \sigma(\Pi_{i_0}) \Rightarrow \sigma(q_{i_0})$ has no $\mathcal{S}$-proof from $\mathcal{S}$ for any finite $\Gamma \subseteq \mathcal{T}$ (if such a proof exists for every $1 \leq i \leq m$ with finite

$\Gamma_i \subseteq \mathcal{T}$ than we could have an $\mathcal{S}$-proof from $\mathcal{S}$ of $\cup_{i=1}^{i=m}\Gamma_i \Rightarrow \diamond(\psi_1,\dots,\psi_n)$ using the m proofs of $\Gamma_i, \sigma(\Pi_i) \Rightarrow \sigma(q_i)$, some weakenings and r). Using this i_0, extend $\mathcal{T} \cup \sigma(\Pi_{i_0})$ to a $\sigma(q_{i_0})$-maximal theory $\mathcal{T}'$. By the induction hypothesis, $v(\mathcal{T}',\psi) = t$ for every $\psi \in \sigma(\Pi_{i_0})$ and $v(\mathcal{T}',\sigma(q_{i_0})) = f$. Since $\mathcal{T} \subseteq \mathcal{T}'$, this contradicts the fact that σ satisfies $\Pi_{i_0} \Rightarrow q_{i_0}$ in $\mathcal{T}$.

Next we note that **(b)** can be strengthened as follows:

(c) If $\psi \in \mathcal{F}'$, $\mathcal{T} \in W$ and there is no finite $\Gamma \subseteq \mathcal{T}$ such that $\Gamma \Rightarrow \psi$ has an $\mathcal{S}$-proof from $\mathcal{S}$, then $v(\mathcal{T},\psi) = f$.

Indeed, under these conditions $\mathcal{T}$ can be extended to a ψ-maximal theory $\mathcal{T}'$. Now $\mathcal{T}' \in W$, $\mathcal{T} \subseteq \mathcal{T}'$, and by **(b)**, $v(\mathcal{T}',\psi) = f$. Hence also $v(\mathcal{T},\psi) = f$.

Now **(a)** and **(b)** together imply that $v(\mathcal{T}_0,\psi) = t$ for every $\psi \in \Gamma_0 \subseteq \mathcal{T}_0$, and $v(\mathcal{T}_0,\varphi_0) = f$. Hence $\mathcal{W}$ is not a model of s. We end the proof by showing that $\mathcal{W}$ is a model of $\mathcal{S}$. So let $\psi_1,\dots,\psi_n \Rightarrow \theta \in \mathcal{S}$ and let $\mathcal{T} \in W$, where $\mathcal{T}$ is φ-maximal. Assume by way of contradiction that $\psi_1,\dots,\psi_n \Rightarrow \theta \in \mathcal{S}$ is not locally true in $\mathcal{T}$. Therefore, $v(\mathcal{T},\psi_i) = t$ for $1 \le i \le n$, while $v(\mathcal{T},\theta) = f$. By **(c)**, for every $1 \le i \le n$ there is a finite $\Gamma_i \subseteq \mathcal{T}$ such that $\Gamma_i \Rightarrow \psi_i$ has an $\mathcal{S}$-proof from $\mathcal{S}$. On the other hand $v(\mathcal{T},\theta) = f$ implies (by **(a)**) that $\theta \notin \mathcal{T}$. Since $\mathcal{T}$ is φ-maximal, it follows that there is a finite $\Delta \subseteq \mathcal{T}$ such that $\Delta,\theta \Rightarrow \varphi$ has an $\mathcal{S}$-proof from $\mathcal{S}$. Now from $\Gamma_i \Rightarrow \psi_i$ $(1 \le i \le n)$, $\Delta,\theta \Rightarrow \varphi$, and $\psi_1,\dots,\psi_n \Rightarrow \theta$ one can infer $\Gamma_1,\dots,\Gamma_n,\Delta \Rightarrow \varphi$ by $n{+}1$ $\mathcal{S}$-cuts (on $\psi_1,\dots,\psi_n$ and θ). It follows that $\Gamma_1,\dots,\Gamma_n,\Delta \Rightarrow \varphi$ has an $\mathcal{S}$-proof from $\mathcal{S}$. Since $\Gamma_1,\dots,\Gamma_n,\Delta \subseteq \mathcal{T}$, this contradicts the φ-maximality of $\mathcal{T}$. □

Remark 40. In [2], Avron suggested a strengthening of the cut-elimination theorem for Gentzen's original systems. He defines the notion of a *hyper-resolution* rule (or *hyper-cut* rule), and shows that this special kind of cuts is the only one needed in derivations. We can show the same in our case. Let *hyper-cut* be the rule:

$$\frac{\psi_1,\dots,\psi_n \Rightarrow \theta \quad \Gamma_1 \Rightarrow \psi_1 \quad \dots \quad \Gamma_n \Rightarrow \psi_n \quad \Delta,\theta \Rightarrow \varphi}{\Gamma_1,\dots,\Gamma_n,\Delta \Rightarrow \varphi}$$

Call $\psi_1,\dots,\psi_n \Rightarrow \theta$ the *nucleus* of the rule. Obviously, this rule is a special case of the cut rule, since it can be produced by $n{+}1$ consecutive cuts. The last proof shows that this kind of derivations, whose nuclei are the initial assumptions, are the only ones that are needed in strict canonical constructive systems. Hence, the last theorem can be strengthened as follows: if there does not exist a **G**-legal $\mathcal{L}$-frame which is model of $\mathcal{S}$ and not a model of s, then there exists a proof of s from $\mathcal{S}$, which uses only canonical rules, weakening and hyper-cuts with elements of $\mathcal{S}$ as nuclei.

Theorem 41 (Soundness and Completeness)**.** *Every strict canonical constructive system* **G** *is strongly sound and complete with respect to the semantics of* **G***-legal frames. In other words:* $\mathcal{S} \vdash^{seq}_{\mathbf{G}} s$ *iff* $\mathcal{S} \models^{seq}_{\mathbf{G}} s$.

Proof. Immediate from Theorem 39 and Theorem 38. □

Corollary 42 (Compactness)**.** *Let* **G** *be a strict canonical constructive system. If* $\mathcal{S} \vDash^{seq}_{\mathbf{G}} s$ *then there exists a finite* $\mathcal{S}' \subseteq \mathcal{S}$ *such that* $\mathcal{S}' \vDash^{seq}_{\mathbf{G}} s$*.*

Theorem 43 (General Strong Cut-Elimination Theorem)**.** *Every strict canonical constructive system* **G** *admits strong cut-elimination (see definition 24).*

Proof. Follows from Theorem 41 and Theorem 39. □

Corollary 44. *The following conditions are equivalent for a strict canonical system* **G***:*

1. **G** *is consistent.*
2. **G** *is coherent.*
3. **G** *admits strong cut-elimination.*
4. **G** *admits cut-elimination.*

Proof. 1 implies 2 by Theorem 22. 2 implies 3 by Theorem 43. 3 trivially implies 4. Finally, without using cuts there is no way to derive $p_1 \Rightarrow p_2$ in a strict canonical system. Hence 4 implies 1. □

3.2.4 Analycity and Decidability

In general, in order for a denotational semantics of a propositional logic to be useful and effective, it should be *analytic*. This means that in order to determine whether a sequent s follows from a set of sequents $\mathcal{S}$, it suffices to consider *partial* valuations, defined on the set of all subformulas of the formulas in $\mathcal{S} \cup \{s\}$. Now we show that the semantics of **G**-legal frames is analytic in this sense.

Theorem 45 (Analycity)**.** *Let* **G** *be a strict canonical constructive system for* $\mathcal{L}$*. Then the semantics of* **G***-legal frames is* analytic *in the following sense: If* $\mathcal{W}' = \langle W, \leq, v' \rangle$ *is a* **G***-legal* semiframe*, then* v' *can be extended to a function* v *so that* $\mathcal{W} = \langle W, \leq, v \rangle$ *is a* **G***-legal frame.*

Proof. Let $\mathcal{W}' = \langle W, \leq, v' \rangle$ be a **G**-legal semiframe. We recursively extend v' to a total function v. For atomic p and for every $a \in W$ we let $v(a, p) = v'(a, p)$ if $v'(a, p)$ is defined, and $v(a, p) = t$ (say) otherwise. For $\varphi = \diamond(\psi_1, \ldots, \psi_n)$ and for every $a \in W$ we let $v(a, \varphi) = v'(a, \varphi)$ whenever the latter is defined, and otherwise we define $v(a, \varphi) = f$ iff there exists an elimination rule for $\diamond$, r, and an element $b \geq a$ of W, such that a substitution σ, such that $\sigma(p_j) = \psi_j$ $(1 \leq j \leq n)$, fulfils r in b (with respect to $\langle W, \leq, v \rangle$). Note that the satisfaction of the premises of r by σ in elements of W depends only on the values assigned by v to $\psi_1, \ldots, \psi_n$, so the recursion works, and v is well defined. From the definition of v and the assumption that $\mathcal{W}'$ is a **G**-legal semiframe, it immediately follows that v is an extension of v', that v is a persistent function defined on $W \times \mathcal{F}$ (so $\mathcal{W} = \langle W, \leq, v \rangle$ is a $\mathcal{L}$-frame), and that $\mathcal{W}$ respects all the elimination rules of

G. Hence it only remains to prove that it respects also the introduction rules of **G**. Let $r = \{\Pi_i \Rightarrow q_i\}_{1\leq i\leq m}/ \Rightarrow \diamond(p_1,\ldots,p_n)$ be such a rule, and assume that for every $1 \leq i \leq m$, $\sigma(\Pi_i) \Rightarrow \sigma(q_i)$ is true in a with respect to $\langle W,\leq,v\rangle$. We should show that $v(a,\diamond(\psi_1,\ldots,\psi_n)) = t$.

If $v'(a,\diamond(\psi_1,\ldots,\psi_n))$ is defined, then since its domain is closed under subformulas, for every $1 \leq i \leq n$ and every $b \in W$, $v'(b,\psi_i)$ is defined. In this case, our construction ensures that for every $1 \leq i \leq n$ and every $b \in W$ we have $v'(b,\psi_i) = v(b,\psi_i)$. Therefore, since for every $1 \leq i \leq m$, $\sigma(\Pi_i) \Rightarrow \sigma(q_i)$ is locally true in every $b \geq a$ with respect to $\langle W,\leq,v\rangle$, it is also locally true with respect to $\langle W,\leq,v'\rangle$. Since v' respects r, $v'(a,\diamond(\psi_1,\ldots,\psi_n)) = t$, and it implies $v(a,\diamond(\psi_1,\ldots,\psi_n)) = t$ as well, as required.

Now, assume $v'(a,\diamond(\psi_1,\ldots,\psi_n))$ is not defined, and assume by way of contradiction that $v(a,\diamond(\psi_1,\ldots,\psi_n)) = f$. So, there exists $b \geq a$ and an elimination rule $\{\Delta_i \Rightarrow E_i\}_{1\leq i\leq k}/ \diamond(p_1,\ldots,p_n) \Rightarrow$ such that $\sigma(\Delta_i) \Rightarrow \sigma(E_i)$ is locally true in b for $1 \leq i \leq k$. Since $b \geq a$, our assumption about a implies that $\sigma(\Pi_i) \Rightarrow \sigma(q_i)$ is locally true in b for $1 \leq i \leq m$. It follows that by defining $u(p) = v(b,\sigma(p))$ we get a valuation u in $\{t,f\}$ which satisfies all the clauses in the union of $\{\Pi_i \Rightarrow q_i\}_{1\leq i\leq m}$ and $\{\Delta_i \Rightarrow E_j\}_{1\leq i\leq k}$. This contradicts the coherence of **G**. □

The following two theorems are now easy consequences of Theorem 45 and the soundness and completeness theorems of the previous section:

Theorem 46. *Let* $\mathbf{G_1}$ *be a strict canonical constructive system in a language* $\mathcal{L}_1$*, and let* $\mathbf{G_2}$ *be a strict canonical constructive system in a language* $\mathcal{L}_2$*. Assume that* $\mathcal{L}_2$ *is an extension of* $\mathcal{L}_1$ *by some set of connectives, and that* $\mathbf{G_2}$ *is obtained from* $\mathbf{G_1}$ *by adding to the latter canonical rules for connectives in* $\mathcal{L}_2 - \mathcal{L}_1$*. Then* $\mathbf{G_2}$ *is a conservative extension of* $\mathbf{G_1}$ *(i.e.: if all sequents in* $\mathcal{S} \cup s$ *are in* $\mathcal{L}_1$ *then* $\mathcal{S} \vdash^{seq}_{\mathbf{G_1}} s$ *iff* $\mathcal{S} \vdash^{seq}_{\mathbf{G_2}} s$*).*

Proof. Suppose that $\mathcal{S} \not\vdash^{seq}_{\mathbf{G_1}} s$. Then there is $\mathbf{G_1}$-legal model $\mathcal{W}$ of $\mathcal{S}$ which is not a model of s. Since the set of formulas of $\mathcal{L}_1$ is a subset of the set of formulas of $\mathcal{L}_2$ which is closed under subformulas, Theorem 45 implies that $\mathcal{W}$ can be extended to a $\mathbf{G_2}$-legal model of $\mathcal{S}$ which is not a model of s. Hence $\mathcal{S} \not\vdash^{seq}_{\mathbf{G_2}} s$. □

Remark 47. Prior's "connective" Tonk ([14]) has made it clear that not every combination of "ideal" introduction and elimination rules can be used for defining a connective. Some constraints should be imposed on the set of rules. Such a constraint was indeed suggested by Belnap in his famous [6]: the rules for a connective $\diamond$ should be *conservative*, in the sense that if $\mathcal{S} \vdash^{seq}_{\mathbf{G}} s$ is derivable using them, and $\diamond$ does not occur in $\mathcal{S} \cup s$, then $\mathcal{S} \vdash^{seq}_{\mathbf{G}} s$ can also be derived without using the rules for $\diamond$. This solution to the problem has two problematic aspects:

1. Belnap did not provide any effective necessary and sufficient criterion for checking whether a given set of rules is conservative in the above sense.

Without such criterion every connective defined by inference rules (without an independent denotational semantics) is suspected of being a Tonk-like connective, and should not be used until a proof is given that it is "innocent".

2. Belnap formulated the condition of conservativity only with respect to the basic deduction framework, in which no connectives are assumed. But nothing in what he wrote excludes the possibility of a system **G** having two connectives, each of them "defined" by a set of rules which is conservative over the basic system **B**, while **G** itself is not conservative over **B**. If this happens then it will follow from Belnap's thesis that each of the two connectives is well-defined and meaningful, but they cannot exist together. Such a situation is almost as paradoxical as that described by Prior.

Now the first of these two objections is met, of course, by our coherence criterion for strict canonical systems, since coherence of a finite set of canonical rules can effectively be checked. The second is met by Theorem 46. That theorem shows that a very strong form of Belnap's conservativity criterion is valid for strict canonical constructive systems, and that a set of rules can be used as a definition for a connective in such systems since it is independent of the system in which it is included. This is a necessary demand, if we do want to see the rules themselves as definitions of a connective.

Theorem 48. *Let* **G** *be a strict canonical constructive system. Then* **G** *is strongly decidable: Given a finite set* $\mathcal{S}$ *of sequents, and a sequent* s*, it is decidable whether* $\mathcal{S} \vdash^{seq}_{\mathbf{G}} s$ *or not.*

Proof. Let $\mathcal{F}'$ be the set of subformulas in $\mathcal{S} \cup \{s\}$. From Theorem 45 and the proof of Theorem 39 it easily follows that in order to decide whether $\mathcal{S} \vdash^{seq}_{\mathbf{G}} s$ it suffices to check all triples of the form $\langle W, \subseteq, v' \rangle$ where $W \subseteq 2^{\mathcal{F}'}$ and $v' : \mathcal{F}' \times W \to \{t, f\}$, and see if any of them is a **G**-legal semiframe which is a model of $\mathcal{S}$ but not a model of s. □

Remark 49. The last two theorems can also be proved directly from the cut-elimination theorem for strict canonical constructive systems.

3.2.5 The Induced Consequence Relation

Originally, a consequence relation is a relation between formulas. The use of sequents is a tool for defining consequence relation between formulas. In this subsection we recall the definition of a consequence relation and its properties, and follow the usual way to obtain a consequence relation from $\vdash$ and $\vDash$. Then we state our major results in terms of formulas, without use of sequents. The propositions in this subsection are easily derived from our definitions, so we omit their proofs.

Definition 50. A *Tarskian consequence relation* (*tcr* for short) for $\mathcal{L}$ is a binary relation $\vdash$ between sets of formulas of $\mathcal{L}$ and formulas of $\mathcal{L}$ that satisfies the following conditions:

strong Reflexivity: if $\varphi \in \mathcal{T}$ then $\mathcal{T} \vdash \varphi$.
Monotonicity: if $\mathcal{T} \vdash \varphi$ and $\mathcal{T} \subseteq \mathcal{T}'$ then $\mathcal{T}' \vdash \varphi$.
Transitivity (*cut*): if $\mathcal{T} \vdash \psi$ and $\mathcal{T}, \psi \vdash \varphi$ then $\mathcal{T} \vdash \varphi$.

Definition 51. A tcr $\vdash$ for $\mathcal{L}$ is *structural* if for every $\mathcal{L}$-substitution σ and every $\mathcal{T}$ and φ, if $\mathcal{T} \vdash \varphi$ then $\sigma(\mathcal{T}) \vdash \sigma(\varphi)$. $\vdash$ is *finitary* if the following condition holds for every $\mathcal{T}$ and φ: if $\mathcal{T} \vdash \varphi$ then there exists a finite $\Gamma \subseteq \mathcal{T}$ such that $\Gamma \vdash \varphi$. $\vdash$ is *consistent* (or *non-trivial*) if $p_1 \not\vdash p_2$.

It is easy to see (see [4]) that there are exactly two inconsistent structural tcrs in any given language [5]. These tcrs are obviously trivial, so we exclude them from our definition of a *logic*:

Definition 52. A propositional *logic* is a pair $\langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ is a propositional language, and $\vdash$ is a tcr for $\mathcal{L}$ which is structural, finitary and consistent.

Definition 53. Let $\mathbf{G}$ be a strict canonical system. The tcr $\vdash_{\mathbf{G}}$ between *formulas* which is induced by $\mathbf{G}$ is defined by: $\mathcal{T} \vdash_{\mathbf{G}} \varphi$ iff there exists a finite $\Gamma \subseteq \mathcal{T}$ such that $\vdash^{seq}_{\mathbf{G}} \Gamma \Rightarrow \varphi$.

Proposition 54. *$\vdash_{\mathbf{G}}$ is a structural and finitary tcr for every strict canonical system $\mathbf{G}$.*

Proposition 55. *$\mathcal{T} \vdash_{\mathbf{G}} \varphi$ iff $\{\Rightarrow \psi \mid \psi \in \mathcal{T}\} \vdash^{seq}_{\mathbf{G}} \Rightarrow \varphi$.*

Proposition 56. *A strict canonical system $\mathbf{G}$ is consistent iff $\vdash_{\mathbf{G}}$ is consistent.*

A strict canonical system also induces a semantic consequence relation:

Definition 57. An $\mathcal{L}$-semiframe $\langle W, \leq, v \rangle$ is a *model* of a formula φ if $v(a, \varphi) = t$ for every $a \in W$. It is a model of a theory $\mathcal{T}$ if it is a model of every $\varphi \in \mathcal{T}$.

Remark 58. $\mathcal{W}$ is a model of a formula φ iff it is a model of the sequent $\Rightarrow \varphi$.

Definition 59. Let $\mathbf{G}$ be a strict canonical constructive system. The semantic tcr $\vDash_{\mathbf{G}}$ between *formulas* which is induced by $\mathbf{G}$ is defined by: $\mathcal{T} \vDash_{\mathbf{G}} \varphi$ if every $\mathbf{G}$-legal $\mathcal{L}$-frame which is a model of $\mathcal{T}$ is also a model of φ.

Again we have:

Proposition 60. *$\mathcal{T} \vDash_{\mathbf{G}} \varphi$ iff $\{\Rightarrow \psi \mid \psi \in \mathcal{T}\} \vDash^{seq}_{\mathbf{G}} \Rightarrow \varphi$.*

Corollary 61 (Soundness and Completeness)**.** *Every strict canonical constructive system $\mathbf{G}$ is strongly sound and complete with respect to the semantics of $\mathbf{G}$-legal frames. In other words: $\mathcal{T} \vdash_{\mathbf{G}} \varphi$ iff $\mathcal{T} \vDash_{\mathbf{G}} \varphi$.*

Proof. Immediate from Theorem 39, Theorem 38 and propositions 55, 60. □

Corollary 62. *If $\mathbf{G}$ is a strict canonical constructive system in $\mathcal{L}$ then $\langle \mathcal{L}, \vDash_{\mathbf{G}} \rangle$ (or equivalently $\langle \mathcal{L}, \vdash_{\mathbf{G}} \rangle$) is a logic.*

Strong conservativity and strong decidability of $\vdash_{\mathbf{G}}$ and $\vDash_{\mathbf{G}}$ (in the sense of Theorem 46 and Theorem 48) are also easy corollaries of the previous theorems and the two reductions.

[5] $\mathcal{T} \vdash \varphi$ for every $\mathcal{T}$ and φ, and $\mathcal{T} \vdash \varphi$ for every nonempty $\mathcal{T}$ and φ.

3.2.6 The Natural Deduction Version

We formulated this section in terms of Gentzen-type systems, in which each rule introduces a connective on the right side of the sequent or on the left side. However, we could have formulated them instead in terms of natural deduction systems, in which there are rules to eliminate a connective rather than to introduce it on the left side. An application of a strict canonical introduction rule in this context is defined exactly as before, while an application of an elimination rule of the form $\{\Pi_i \Rightarrow E_i\}_{1\leq i\leq m}/\diamond(p_1,\dots,p_n) \Rightarrow$ in the context of natural deduction is any inference step of the form:

$$\frac{\{\Gamma, \sigma(\Pi_i) \Rightarrow \sigma(E_i), F_i\}_{1\leq i\leq m} \quad \Gamma \Rightarrow \sigma(\diamond(p_1,\dots,p_n))}{\Gamma \Rightarrow \theta}$$

where Γ, σ, θ and F_i are as in the definition of a strict canonical elimination rule.

Translating our results to natural deduction systems is possible.

Example 63 (Conjunction)**.** The usual elimination rule for conjunction is:

$$\{p_1, p_2 \Rightarrow\ \} \ / \ p_1 \wedge p_2 \Rightarrow$$

In natural-deduction systems its applications have the form:

$$\frac{\Gamma, \psi, \varphi \Rightarrow \theta \quad \Gamma \Rightarrow \psi \wedge \varphi}{\Gamma \Rightarrow \theta}$$

Example 64 (Disjunction)**.** The usual elimination rule for disjunction is:

$$\{p_1 \Rightarrow\ ,\ \ p_2 \Rightarrow\} \ / \ p_1 \vee p_2 \Rightarrow$$

In natural deduction systems its applications have the form:

$$\frac{\Gamma, \psi \Rightarrow \theta \quad \Gamma, \varphi \Rightarrow \theta \quad \Gamma \Rightarrow \psi \vee \varphi}{\Gamma \Rightarrow \theta}$$

Example 65 (Implication)**.** The usual elimination rule for implication is:

$$\{\Rightarrow p_1\ ,\ \ p_2 \Rightarrow\} \ / \ p_1 \supset p_2 \Rightarrow$$

In natural-deduction systems its applications have the form:

$$\frac{\Gamma \Rightarrow \psi \quad \Gamma, \varphi \Rightarrow \theta \quad \Gamma \Rightarrow \psi \supset \varphi}{\Gamma \Rightarrow \theta}$$

This form of the rule is obviously equivalent to the more usual one (from $\Gamma \Rightarrow \psi$ and $\Gamma \Rightarrow \psi \supset \varphi$ infer $\Gamma \Rightarrow \varphi$).

Example 66 (Absurdity)**.** The usual rule for absurdity is:

$$\{\ \} \ / \ \ \bot \Rightarrow$$

In natural-deduction systems applications of the same rule have the form:

$$\frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow \theta}$$

3.3 Non-Strict Sequential Systems

In this section we adapt our results to sequential systems which do allow the use of negative sequents in their derivations (as done in most presentations of intuitionistic logic). Most of the definitions and the proofs are similar to the strict version. In this case we will just state this fact to avoid repetitions.

3.3.1 Non-Strict Canonical Constructive Systems

The use of negative sequents in derivations should be reflected in the structure of the logical rules of canonical systems. Here, negative sequents might serve as premises of the rules. This is simple in introduction rules, since they do not allow right context in a single-conclusion system (otherwise there would not be "enough space" for the conclusion). Therefore, the introduction rules formulation is simply changed to allow negative premises as well, and these premises are realized as negative sequents in derivations. The elimination rules are slightly more complicated, since right context might be added to the negative premises and penetrate into the conclusion. However, there also might be negative premises which do not allow adding a right context, as negative premises of the introduction rules. In order to support this kind of rule, we split the set of premises of the canonical elimination rule into two sets: *hard* premises which do not allow right context (including the definite premises), and *soft* premises, which do allow right context (of-course all of them are negative).

Definition 67.

1. A *non-strict canonical introduction rule* is an expression of the form:

$$\{\Pi_i \Rightarrow E_i\}_{1 \leq i \leq m} / \ \Rightarrow \diamond(p_1, \dots, p_n)$$

   where $m \geq 0$, $\diamond$ is a connective of arity n, and for every $1 \leq i \leq m$, $\Pi_i \Rightarrow E_i$ is a Horn clause such that $\Pi_i \cup E_i \subseteq \{p_1, \dots, p_n\}$.
   $\Pi_i \Rightarrow E_i$ $(1 \leq i \leq m)$ are called the *premises* of the rule.
   $\Rightarrow \diamond(p_1, \dots, p_n)$ is called the *conclusion* of the rule.

2. A *non-strict canonical elimination rule* is an expression of the form:

$$\langle\{\Pi_i \Rightarrow E_i\}_{1 \leq i \leq m}, \{\Sigma_i \Rightarrow\}_{1 \leq i \leq k}\rangle / \ \diamond(p_1, \dots, p_n) \Rightarrow$$

   where m, $\diamond$, Π_i, E_i are as above, and for every $1 \leq i \leq k$, $\Sigma_i \Rightarrow$ is a negative Horn clause such that $\Sigma_i \subseteq \{p_1, \dots, p_n\}$.
   $\Pi_i \Rightarrow E_i$ $(1 \leq i \leq m)$ and $\Sigma_i \Rightarrow$ $(1 \leq i \leq k)$ are called the *premises* of the rule.
   $\Pi_i \Rightarrow E_i$ $(1 \leq i \leq m)$ are called *hard premises*.
   $\Sigma_i \Rightarrow$ $(1 \leq i \leq k)$ are called *soft premises*.
   $\diamond(p_1, \dots, p_n) \Rightarrow$ is called the *conclusion* of the rule.

3. An *application* of the rule $\{\Pi_i \Rightarrow E_i\}_{1\leq i\leq m}/ \;\Rightarrow \diamond(p_1,\dots,p_n)$ is any inference step of the form:

$$\frac{\{\Gamma, \sigma(\Pi_i) \Rightarrow \sigma(E_i)\}_{1\leq i\leq m}}{\Gamma \Rightarrow \sigma(\diamond(p_1,\dots,p_n))}$$

where Γ is a finite set of formulas and σ is a substitution in $\mathcal{L}$.

4. An *application* of the rule $\langle\{\Pi_i \Rightarrow E_i\}_{1\leq i\leq m}, \{\Sigma_i \Rightarrow\}_{1\leq i\leq k}\rangle / \diamond(p_1,\dots,p_n) \Rightarrow$ is any inference step of the form:

$$\frac{\{\Gamma, \sigma(\Pi_i) \Rightarrow \sigma(E_i)\}_{1\leq i\leq m} \quad \{\Gamma, \sigma(\Sigma_i) \Rightarrow E\}_{1\leq i\leq k}}{\Gamma, \sigma(\diamond(p_1,\dots,p_n)) \Rightarrow E}$$

where Γ and σ are as above, E is either a singleton (of some formula) or empty.

Remark 68. The soft premises of a non-strict canonical elimination rule allow the addition of right context formula, but do not force it. No right context is optional (an empty E in the previous definition). This cannot be done in strict canonical systems, and for this reason, a strict canonical elimination rule has no exact equivalent non-strict canonical rule.

Example 69 (Negation)**.** Allowing negative premises in introduction rules makes it possible to deal with negation as a basic connective. The two usual rules for negation are:

$$\langle\{\Rightarrow p_1\}, \{\ \}\rangle \;/\; \neg p_1 \Rightarrow \quad \text{and} \quad \{p_1 \Rightarrow\} \;/\; \Rightarrow \neg p_1$$

Applications of these rules have the form:

$$\frac{\Gamma \Rightarrow \psi}{\Gamma, \neg\psi \Rightarrow E} \qquad \frac{\Gamma, \psi \Rightarrow}{\Gamma \Rightarrow \neg\psi}$$

Example 70 (The previous connectives)**.** Every connective in the examples in the previous section ($\wedge, \vee, \supset, \bot, \rightsquigarrow, \rhd$ and T) has also non-strict canonical introduction and elimination rules. The introduction rules and their applications are exactly the same. The elimination rules are also similar, except for splitting the definite premises and the negative premises to two sets (the definite premises are hard, and the negative premises are soft) and using E instead of θ in the applications. E.g., the elimination rule of conjunction is:

$$\langle\{\ \}, \{p_1, p_2 \Rightarrow\}\ \rangle \;/\; p_1 \wedge p_2 \Rightarrow \quad \text{instead of} \quad \{p_1, p_2 \Rightarrow\ \} \;/\; p_1 \wedge p_2 \Rightarrow$$

Applications of this rule have the form:

$$\frac{\Gamma, \psi, \varphi \Rightarrow E}{\Gamma, \psi \wedge \varphi \Rightarrow E} \quad \text{instead of} \quad \frac{\Gamma, \psi, \varphi \Rightarrow \theta}{\Gamma, \psi \wedge \varphi \Rightarrow \theta}$$

Example 71 (Bowen's new connectives)**.** Bowen's rules for his new connectives (see Section 2.2) are equivalent to the following non-strict canonical rules:

$$\langle\{p_2 \Rightarrow p_1\}, \{\ \}\rangle \ /\ p_1 \not\subset p_2 \Rightarrow \quad \text{and} \quad \{p_1 \Rightarrow\ ,\ \ \Rightarrow p_2\}\ /\ \Rightarrow p_1 \not\subset p_2$$

$$\langle\{\Rightarrow p_1\ ,\ \Rightarrow p_2\}, \{\ \}\rangle / p_1 \mid p_2 \Rightarrow \quad \text{and} \quad \{p_1 \Rightarrow\} / \Rightarrow p_1 \mid p_2 \quad \{p_2 \Rightarrow\} / \Rightarrow p_1 \mid p_2$$

Applications of these rules have the form:

$$\frac{\Gamma, \psi \Rightarrow \varphi}{\Gamma, \varphi \not\subset \psi \Rightarrow E} \qquad \frac{\Gamma, \varphi \Rightarrow \qquad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \not\subset \psi}$$

$$\frac{\Gamma \Rightarrow \varphi \qquad \Gamma \Rightarrow \psi}{\Gamma, \varphi \mid \psi \Rightarrow E} \qquad \frac{\Gamma, \varphi \Rightarrow}{\Gamma \Rightarrow \varphi \mid \psi} \quad \frac{\Gamma, \psi \Rightarrow}{\Gamma \Rightarrow \varphi \mid \psi}$$

Example 72 (Strong Affirmation)**.** Suppose we introduce a "strong affirmation" connective $\blacktriangleright$ with the following rules:

$$\langle\{p_1 \Rightarrow\ \}, \{\ \}\rangle \ /\ \ \blacktriangleright p_1 \Rightarrow \quad \text{and} \quad \{\Rightarrow p_1\}\ /\ \ \Rightarrow \blacktriangleright p_1$$

Applications of these rules have the form:

$$\frac{\Gamma, \varphi \Rightarrow}{\Gamma, \blacktriangleright \varphi \Rightarrow E} \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \blacktriangleright \varphi}$$

Non-strict canonical system is defined equivalently to strict canonical system, with non-strict canonical rules instead of strict canonical rules. Of course, the weakening and cut rules have to be modified, and we now take the following rules:

$$\frac{\Gamma \Rightarrow E}{\Gamma, \Delta \Rightarrow E} \quad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \psi} \qquad \frac{\Gamma \Rightarrow \varphi \qquad \Delta, \varphi \Rightarrow E}{\Gamma, \Delta \Rightarrow E}$$

The induced relation, $\vdash^{seq}_{\mathbf{G}}$, and the consistency of a non-strict canonical system are also defined equivalently.

As in the strict case, we would like to offer a simple criterion for non-triviality of a non-strict canonical system, i.e. for the consistency of **G**. However, in this case we cannot show an equivalent simple criterion, and we leave this as an open question. To overcome this obstacle, we define a stronger form of consistency, and show that our previous coherence criterion, applied to non-strict canonical systems, is equivalent to this stronger form.

Definition 73. A non-strict canonical system, **G**, is called *strongly consistent* if $\Rightarrow p_1\ ,\ p_2 \Rightarrow \nvdash^{seq}_{\mathbf{G}} \Rightarrow$.

Remark 74. This property is trivial in strict systems, since it assumes negative sequents in the derivations.

Proposition 75. *Every strongly consistent non-strict canonical system is also consistent.*

Proof. Let **G** be an inconsistent non-strict canonical system. Then $\vdash_{\mathbf{G}}^{seq} p_1 \Rightarrow p_2$. Using the premises $\Rightarrow p_1, p_2 \Rightarrow$ and two cuts we can have $\Rightarrow p_1, p_2 \Rightarrow \vdash_{\mathbf{G}}^{seq} \Rightarrow$ as needed. □

The next example shows that the other direction is not always true:

Example 76 (Circle)**.** Let **G** be a non-strict canonical system, containing only the following rules:

$$\langle \{\ \}, \{p_1 \Rightarrow\ \}\rangle\ /\ \circ p_1 \Rightarrow \quad \text{and} \quad \{p_1 \Rightarrow\ \}\ /\ \Rightarrow \circ p_1$$

Applications of these rules have the form:

$$\frac{\Gamma, \varphi \Rightarrow E}{\Gamma, \circ\varphi \Rightarrow E} \qquad \frac{\Gamma, \varphi \Rightarrow}{\Gamma \Rightarrow \circ\varphi}$$

In **G** there is no way to derive a negative sequent from no assumptions (this is proved by simple induction), and hence the introduction rule for $\circ$ cannot be used. For this trivial reason, **G** is consistent. However, it is easy to see that **G** is not strongly consistent.

Remark 77*.* This example also shows that strong cut-elimination and cut-elimination are not equivalent in the case of non-strict canonical systems. For the same reason, **G** admits cut-elimination, but it does not admit strong cut-elimination. Recall that these two properties are equivalent in *strict* canonical systems (Corollary 44).

Although it is stronger than the usual consistency, strong consistency is a very natural demand from a system. Intuitively a strongly inconsistent system is a system in which either everything is provable, or it only proves that formulas are true or only proves that formulas are false.

Coherence of sets of non-strict canonical rules is defined essentially the same as for strict rules (note that three sets of premises are now involved - the premises of the introduction rules, the soft premises of the elimination rule and the hard premises of the elimination rule). Coherence of non-strict canonical systems is defined exactly as in the strict case.

Example 78. All sets of rules for the $\wedge, \vee, \supset, \bot, \rightsquigarrow, \rhd, \neg, \not\subset, |$ and $\blacktriangleright$ from the previous examples are coherent. The sets of rules for the connectives T and $\circ$ from the previous examples are not coherent.

Theorem 79. *Every strongly consistent non-strict canonical system is coherent.*

Proof. Let **G** be an incoherent non-strict canonical system. This means that **G** includes two rules $\langle S_1, S_2\rangle / \diamond(p_1, \ldots, p_n) \Rightarrow$ and $S_3/ \Rightarrow \diamond(p_1, \ldots, p_n)$, such that the set of clauses $S_1 \cup S_2 \cup S_3$ is classically satisfiable. Let v be an assignment in $\{t, f\}$ that satisfies all the clauses in $S_1 \cup S_2 \cup S_3$. Define a substitution σ by:

$$\sigma(p) = \begin{cases} p_1 & v(p) = t \\ p_2 & v(p) = f \end{cases}$$

Let $\Pi \Rightarrow E \in S_1 \cup S_2 \cup S_3$. Since v satisfies all the clauses in $S_1 \cup S_2 \cup S_3$, for every $\Pi \Rightarrow E \in S_1 \cup S_2 \cup S_3$ we have $p_2 \in \sigma(\Pi)$ or $p_1 \in \sigma(E)$. Hence, every $\Pi \Rightarrow E \in S_1 \cup S_2$ can be derived from $\Rightarrow p_1, p_2 \Rightarrow$ by weakening. Now by applying $\langle S_1, S_2 \rangle / \diamond (p_1, \ldots, p_n) \Rightarrow$ and $S_3 / \; \Rightarrow \diamond(p_1, \ldots, p_n)$ to these provable sequents we get proofs from $\Rightarrow p_1, p_2 \Rightarrow$ of $\Rightarrow \sigma(\diamond(p_1, \ldots, p_n))$ and of $\sigma(\diamond(p_1, \ldots, p_n)) \Rightarrow$. That $\Rightarrow p_1, p_2 \Rightarrow \vdash^{seq}_{\mathbf{G}} \Rightarrow$ then follows using a cut. □

The last theorem shows that coherence is also a minimal demand from a non-strict canonical system. Again, in the sequel we show that coherence is also a sufficient demand for strong consistency (Corollary 93) [6]. Hence, a *non-strict canonical constructive system* is defined equivalently to a strict canonical constructive system.

Example 76 shows that while coherence is necessary for (usual) consistency of strict systems, this is not the case for non-strict systems. However we can impose a restriction on non-strict canonical systems, which gives them this property, as done in the next definition:

Definition 80. A non-strict canonical system is called *definite* if its introduction rules have only definite Horn clauses as premises, and its elimination rules have only definite Horn clauses as hard premises.

Theorem 81. *Every consistent definite non-strict canonical system is coherent.*

Proof. Similar to the proof of Theorem 22. □

An $\mathcal{S}$-*cut*, an $\mathcal{S}$-*proof*, *cut-elimination* and *strong cut-elimination* for non-strict canonical systems are defined the same as for strict canonical systems.

3.3.2 Semantics for Non-Strict Canonical Systems

We modify our *fulfil* definition (definition 30) in order to prove soundness and completeness in the non-strict case. The *respect* definition remains the same, using the new *fulfil* definition:

Definition 82. Let $\mathcal{W} = \langle W, \leq, v \rangle$ be an $\mathcal{L}$-semiframe.

1. An $\mathcal{L}$-substitution fulfils a non-strict canonical introduction rule in $a \in W$ iff it satisfies every premise of the rule in a.
2. An $\mathcal{L}$-substitution fulfils a non-strict canonical elimination rule in $a \in W$ iff it satisfies every hard premise of the rule in a, and locally satisfies every soft premise of the rule in a.

[6] Again, this can be derived from [3, 4]. See footnote 3.

Remark 83. The only difference is in the *elimination* rules. Instead of satisfaction of the *definite* premises and local satisfaction of the *negative* premises, we now require satisfaction of the *hard* premises and local satisfaction of the *soft* premises.

Example 84. The semantics that was presented in the previous section for $\supset$, $\leadsto$, $\triangleright$ and T remains exactly the same.

Example 85 (Negation)**.** An $\mathcal{L}$-frame $\mathcal{W} = \langle W, \leq, v\rangle$ respects the rule $(\neg \Rightarrow)$ if $v(a, \neg\psi) = f$ whenever $v(a, \psi) = t$. Because of the persistence condition, this is equivalent to: $v(a, \neg\psi) = f$ whenever $v(b, \psi) = t$ for some $b \geq a$. It respects $(\Rightarrow \neg)$ if $v(a, \neg) = t$ whenever $v(b, \psi) = f$ for every $b \geq a$. Hence the two rules together impose exactly the well-known Kripke semantics for intuitionistic negation.

Example 86 (Converse Non-Implication)**.** An $\mathcal{L}$-frame $\mathcal{W} = \langle W, \leq, v\rangle$ respects the rule $(\not\subset\Rightarrow)$ if $v(a, \varphi \not\subset \psi) = f$ whenever for every $b \geq a$ either $v(b, \varphi) = t$ or $v(b, \psi) = f$. Because of the persistence condition, this is equivalent to: $v(a, \varphi \not\subset \psi) = f$ whenever there exists some $b \geq a$ such that either $v(b, \varphi) = t$ or $v(c, \psi) = f$ for every $c \geq b$. It respects $(\Rightarrow\not\subset)$ if $v(a, \varphi \not\subset \psi) = t$ whenever $v(b, \varphi) = f$ and $v(b, \psi) = t$ for every $b \geq a$. Because of the persistence condition, this is equivalent to: $v(a, \varphi \not\subset \psi) = t$ whenever $v(a, \psi) = t$ and $v(b, \varphi) = f$ for every $b \geq a$. This implies that $v(a, \varphi \not\subset \psi)$ is free when $v(b, \varphi) = f$ for every $b \geq a$, $v(a, \psi) = f$, and there does not exist $b \geq a$ such that $v(c, \psi) = f$ for every $c \geq b$. Hence, the semantics for this connective is non-deterministic. For this reason, it does not have a semantics in the sense of McCullough (see Section 2.1).

Example 87 (Not Both)**.** An $\mathcal{L}$-frame $\mathcal{W} = \langle W, \leq, v\rangle$ respects the rule $(|\Rightarrow)$ if $v(a, \varphi \mid \psi) = f$ whenever $v(b, \varphi) = t$ and $v(b, \psi) = t$ for every $b \geq a$. Because of the persistence condition, this is equivalent to: $v(a, \varphi \mid \psi) = f$ whenever $v(b, \psi) = t$ and $v(b, \varphi) = t$ for some $b \geq a$. It respects $(\Rightarrow|)_1$ if $v(a, \varphi \mid \psi) = t$ whenever $v(b, \varphi) = f$ for every $b \geq a$. It respects $(\Rightarrow|)_2$ if $v(a, \varphi \mid \psi) = t$ whenever $v(b, \psi) = f$ for every $b \geq a$. This implies that $v(a, \varphi \mid \psi)$ is free when $v(a, \varphi) = f$ and $v(a, \psi) = f$ but $v(b_1, \varphi) = t$ and $v(b_1, \psi) = f$ for some $b_1 \geq a$, and $v(b_2, \varphi) = f$ and $v(b_2, \psi) = t$ for some $b_2 \geq a$ (this is possible because the order relation does not have to be linear). Again, we obtain non-deterministic semantics.

Example 88 (Strong Affirmation)**.** An $\mathcal{L}$-frame $\mathcal{W} = \langle W, \leq, v\rangle$ respects the rule $(\blacktriangleright\Rightarrow)$ if $v(a, \blacktriangleright \psi) = f$ whenever $v(b, \psi) = f$ for every $b \geq a$. It respects $(\Rightarrow\blacktriangleright)$ if $v(a, \blacktriangleright \psi) = t$ whenever $v(b, \psi) = t$ for every $b \geq a$. Because of the persistence condition, this is equivalent to: $v(a, \blacktriangleright \psi) = t$ whenever $v(a, \psi) = t$. This implies that $v(a, \blacktriangleright \psi)$ is free when $v(a, \psi) = f$ and $v(b, \psi) = t$ for some $b \geq a$. Again, we obtain non-deterministic semantics.

A **G**-*legal frame* and the relation $\vDash^{seq}_{\mathbf{G}}$ are defined exactly as in the strict case.

3.3.3 Soundness, Completeness, Cut-elimination

The proofs of the next theorems are essentially similar to their strict counterparts, using the non-strict semantics definitions. For the sake of brevity, we exclude them.

Theorem 89 (Soundness and Completeness)**.** *Every non-strict canonical constructive system* **G** *is strongly sound and complete with respect to the semantics of* **G***-legal frames. In other words:* $\mathcal{S} \vdash^{seq}_{\mathbf{G}} s$ *iff* $\mathcal{S} \vDash^{seq}_{\mathbf{G}} s$*.*

Theorem 90 (General Strong Cut-Elimination Theorem)**.** *Every non-strict canonical constructive system* **G** *admits strong cut-elimination.*

Remark 91*.* Again (see remark 40) we can prove a strengthening of the last theorem. Let *hyper-cut*$_1$ and *hyper-cut*$_2$ be the rules which allow the following two derivations:

$$\frac{\psi_1, \ldots, \psi_n \Rightarrow \theta \quad \Gamma_1 \Rightarrow \psi_1 \quad \ldots \quad \Gamma_n \Rightarrow \psi_n \quad \Delta, \theta \Rightarrow F}{\Gamma_1, \ldots, \Gamma_n, \Delta \Rightarrow F}$$

$$\frac{\psi_1, \ldots, \psi_n \Rightarrow \quad \Gamma_1 \Rightarrow \psi_1 \quad \ldots \quad \Gamma_n \Rightarrow \psi_n}{\Gamma_1, \ldots, \Gamma_n \Rightarrow}$$

Call $\psi_1, \ldots, \psi_n \Rightarrow E$, where $E = \theta$ in the first derivation and empty in the second, the *nucleus* of the rule. The last theorem can be strengthened as follows: If there does not exist a **G**-legal $\mathcal{L}$-frame which is model of $\mathcal{S}$ and not a model of s, then there exists a proof of s from $\mathcal{S}$, which uses only canonical rules, weakening and hyper-cuts with elements of $\mathcal{S}$ as nuclei. The full proof also shows that we can restrict the use of weakening on the right side of sequents to apply only on negative assumptions, immediately after hyper-cut$_2$ was applied.

Corollary 92 (Compactness)**.** *Let* **G** *be a non-strict canonical constructive system. If* $\mathcal{S} \vDash^{seq}_{\mathbf{G}} s$ *then there exists a finite* $\mathcal{S}' \subseteq \mathcal{S}$ *such that* $\mathcal{S}' \vDash^{seq}_{\mathbf{G}} s$*.*

Corollary 93. *The following conditions are equivalent for a non-strict canonical system* **G***:*

1. **G** *is strongly consistent.*
2. **G** *is coherent.*
3. **G** *admits strong cut-elimination.*

Proof. 1 implies 2 by Theorem 79. 2 implies 3 by Theorem 90. Finally, in a non-strict canonical system there is no way to derive $\Rightarrow$ from $\Rightarrow p_1, p_2 \Rightarrow$, without using other cuts than cuts on p_1 or p_2. Hence 3 implies 1. □

Corollary 94. *The following conditions are equivalent for a* definite *non-strict canonical system* **G***:*

1. **G** *is consistent.*

2. **G** *is coherent.*

3. **G** *admits strong cut-elimination.*

4. **G** *admits cut-elimination.*

5. **G** *is strongly consistent.*

Proof. 1 implies 2 by Theorem 81. 2 implies 3 by Theorem 90. 3 trivially implies 4. 4 implies 1 since without using cuts there is no way to derive $p_1 \Rightarrow p_2$ in a non-strict canonical system. 5 implies 1 by proposition 75. 3 implies 5 since there is no way to derive $\Rightarrow$ from $\Rightarrow p_1$ and $p_2 \Rightarrow$ with cuts only on p_1 and p_2 in a non-strict canonical system. □

3.3.4 Analycity and Decidability

The semantics of **G**-legal frames is analytic (see Theorem 45) also in the non-strict case. Strong conservativity and strong decidability (in the sense of Theorem 46 and Theorem 48) are again easy consequences of the analycity and the soundness and completeness theorems of the previous subsection. The proofs of these theorems are identical to the proofs in the strict case.

3.3.5 The Induced Consequence Relation

In this section we give our results for non-strict canonical systems in terms of formulas. The main ideas are similar to those of Subsection 3.2.5.

It is natural to extend definition 50 in the non-strict case, as follows:

Definition 95. An *Extended Tarskian consequence relation* (*etcr* for short) for $\mathcal{L}$ is a binary relation $\vdash$ between sets of formulas of $\mathcal{L}$ and singletons or empty sets of formulas of $\mathcal{L}$ that satisfies the following conditions:

strong Reflexivity:	if $\varphi \in \mathcal{T}$ then $\mathcal{T} \vdash \varphi$.
Monotonicity:	if $\mathcal{T} \vdash E$ and $\mathcal{T} \subseteq \mathcal{T}'$ then $\mathcal{T}' \vdash E$.
Transitivity (*cut*):	if $\mathcal{T} \vdash \psi$ and $\mathcal{T}, \psi \vdash E$ then $\mathcal{T} \vdash E$.

Intuitively, $\mathcal{T} \vdash$ means that $\mathcal{T}$ contains a self-contradiction. Structurality, finitarity and consistency of an etcr are defined equivalently as for a tcr (see definition 51). Notice that in this case (see [4]) there are exactly *four* inconsistent structural etcrs in any given language [7]. These etcrs are obviously trivial, so we exclude them from our definition of an *extended logic*:

Definition 96. A propositional *extended logic* (*elogic* for short) is a pair $\langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ is a propositional language, and $\vdash$ is an etcr for $\mathcal{L}$ which is structural, finitary and consistent.

[7] $\mathcal{T} \vdash E$ for every $\mathcal{T}$ and E, $\mathcal{T} \vdash E$ for every E and nonempty $\mathcal{T}$, $\mathcal{T} \vdash E$ for every $\mathcal{T}$ and nonempty E, and $\mathcal{T} \vdash E$ for every nonempty $\mathcal{T}$ and nonempty E.

The etcr $\vdash_{\mathbf{G}}$ between *formulas* which is induced by a non-strict canonical system,$\mathbf{G}$, is defined equivalently to the tcr which is induced by a strict canonical system.

Proposition 97. $\vdash_{\mathbf{G}}$ *is a structural and finitary tcr for every non-strict canonical system* $\mathbf{G}$.

Proposition 98. $\mathcal{T} \vdash_{\mathbf{G}} E$ *iff* $\{\Rightarrow \psi \mid \psi \in \mathcal{T}\} \vdash^{seq}_{\mathbf{G}} \Rightarrow E$.

Proposition 99. *A non-strict canonical system* $\mathbf{G}$ *is consistent iff* $\vdash_{\mathbf{G}}$ *is consistent.*

Again, a non-strict canonical system also induces a semantic consequence relation.

Definition 100. Let $\mathbf{G}$ be a non-strict canonical constructive system. The semantic tcr $\vDash_{\mathbf{G}}$ between *formulas* which is induced by $\mathbf{G}$ is defined by: $\mathcal{T} \vDash_{\mathbf{G}} E$ if every $\mathbf{G}$-legal $\mathcal{L}$-frame which is a model of $\mathcal{T}$ is also a model of E.

Again we have:

Proposition 101. $\mathcal{T} \vDash_{\mathbf{G}} E$ *iff* $\{\Rightarrow \psi \mid \psi \in \mathcal{T}\} \vDash^{seq}_{\mathbf{G}} \Rightarrow E$.

Corollary 102 (Soundness and Completeness)**.** *Every non-strict canonical constructive system* $\mathbf{G}$ *is strongly sound and complete with respect to the semantics of* $\mathbf{G}$*-legal frames. In other words:* $\mathcal{T} \vdash_{\mathbf{G}} E$ *iff* $\mathcal{T} \vDash_{\mathbf{G}} E$.

Corollary 103. *If* $\mathbf{G}$ *is a non-strict canonical constructive system in* $\mathcal{L}$ *then* $\langle \mathcal{L}, \vDash_{\mathbf{G}} \rangle$ *(or equivalently* $\langle \mathcal{L}, \vdash_{\mathbf{G}} \rangle$*) is an elogic.*

Strong conservativity and strong decidability of $\vdash_{\mathbf{G}}$ and $\vDash_{\mathbf{G}}$ (in the sense of Theorem 46 and Theorem 48) are also easy corollaries of the previous theorems and the two reductions.

Chapter 4

Conclusions and Further Work

Definition. A *constructive connective* is a connective defined by a set of rules in some canonical constructive system.

This is our answer to the initial question, "What is a constructive connective?" Theorem 46 (as well as its non-strict counterpart) ensures that a coherent set of canonical rules can be seen as a definition of a connective, since it shows that in canonical constructive systems the same set of rules defines the same connective regardless of the rules for the other connectives of the system. Note that we have suggested two kinds of canonical constructive systems: strict and non-strict. We did not deal with the connections between them. We leave this issue to a further work.

In the following we return to the four papers that were described in Chapter 2, and explain the connections between their results and ours. In the last subsection we list some more directions for further work.

Logical Connectives for Intuitionistic Propositional Logic, Dean P. McCullough

McCullough's semantic definition of a constructive connective, as formulated in Section 2.1, does not cover the whole range of connectives that can be defined in canonical constructive systems. McCullough's semantics is deterministic, adhering to the principle of truth-functionality. Hence, non-deterministic connectives are not captured by McCullough's approach. In the current work, truth-functionality is rejected, and thus many *new*, perhaps useful, connectives can be defined.

Unlike McCullough, our basic definition of a connective is proof-theoretical. However, we do give Kripke frame semantics for each connective in the form of conditions which are imposed on Kripke frames concerning formulas that contain that connective. In our definitions these conditions are given using natural language. However, it is not hard to see that they can be formalized as MLO formulas, of the same kind that was considered in McCullough's semantics. The crucial difference is that while McCullough uses one MLO formula to define a connective, we use two formulas: one expressing the truth-conditions, and the other expressing the falsity-conditions. The MLO formulas that we obtain by formalizing our semantics satisfy McCullough's conditions. This can be seen as a justification for McCullough's unjustified assertion on the structure of defining formulas, since this structure is what one obtains following our *independent* approach for defining connectives.

An Extension of the Intuitionistic Propositional Calculus, K. A. Bowen

As Bowen suspected (see Section 2.2), we showed that his two new connectives are two specific examples of much broader family of connectives that can be defined in single-conclusion sequential systems, which admit cut-elimination. Unlike Bowen, we provided semantics for all these connectives (and particularly for his two new connectives), and used this semantics to prove our main results.

Although we did not deal with expressibility questions, it is easy to see from our results that Bowen's connectives cannot be expressed by the four basic intuitionistic connectives. The reason for this is that Bowen's connectives are essentially different, as they have non-deterministic nature. This argument avoids Bowen's complicated syntactic arguments and the use of McCullough's result.

Nonstandard Connectives for Intuitionistic Propositional Logic, M. Kaminski

As described in Section 2.3, Kaminski has proved a cut-elimination theorem for any "consistent" extension of LJ of some certain form. The templates of the logical rules for the new connective in Kaminski's work match our definition of a non-strict canonical rule [1]. However, our work has an advantage over his result, since our coherence criterion is much simpler and easy to verify than his consistency demands.

Another crucial difference between Kaminski's work and ours is the nature of the cut-elimination proof. Kaminski's proof is syntactic. It extends Gentzen's original proof by adding another level of induction. We avoid this, and integrate the cut-elimination proof in the completeness proof. Thus, this makes our proof simpler (and perhaps safer...).

[1] Note that in order to deal with the full variety of Kaminski's rules, we had to allow negative hard premises in elimination rules.

Towards a Semantic Characterization of Cut-Elimination, A. Ciabattoni and K. Terui

Our notion of non-strict canonical systems falls in the category of *simple calculi* that was presented in [8] (see Section 2.4). However, our presentation is simpler, since while a connective has essentially infinitely many introduction (and elimination) rules in a simple calculus, our definition of a canonical rule treats this infinite set of rules as one rule. Hence, our coherence criterion is simpler than Ciabattoni's and Terui's reductivity criterion, but they can be shown to be equivalent.

The notions of *strong cut-elimination* and Ciabattoni's and Terui's *reductive cut-elimination* are also connected. While they give structural conditions on the required applications of the cut rule in a proof, we simply limit the set of possible cut formulas. In order to obtain our limitation from their conditions, one should inductively apply their conditions in a certain proof, cut by cut.

Another crucial similarity is that both works use non-deterministic semantic frameworks (in [8] this is only implicit). However, while we use the concrete framework of intuitionistic-like Kripke frames (which leads to decision procedures for all the systems we consider), variants of the significantly more abstract phase semantics are used in [8].

Further Work

The next list is a short collection of related issues which require further work:

- Investigate the connections between the strict case and the non-strict case, and perhaps deal with both of them in a unified framework.
- Give an independent semantic definition of a constructive connective using generalized frames, and prove its equivalence to our definition.
- Expressibility issues: When can one connective be expressed by a composition of other connectives? Is there a functional complete set of constructive connectives?
- Give simple criterions for consistency and (usual) cut-elimination in non-strict canonical systems.
- Investigate the connections between invertibility, axiom-expansion and determinism of connectives in canonical constructive systems.
- Extend our results about propositional constructive logic to first-order constructive logic.
- Extend our results to signed calculi with more than two signs.

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