# From Frame Properties to Hypersequent Rules in Modal Logics

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# Motivation

- Ever since the introduction of relational semantics for modal logics, the assembling of a new modal logic for particular applications often begins by locating relevant frame properties.
- Examples:

  $\mathbf{K4} = \mathbf{K} \ + \ \textit{transitivity}$
  $\mathbf{S4} = \mathbf{K4} \ + \ \textit{reflexivity}$
  $\mathbf{S4.3} = \mathbf{S4} \ + \ \textit{linearity}$
  $\mathbf{KDBC_8} = \mathbf{K} + \textit{seriality} \ + \ \textit{cardinality} \leq 8$

  and many more...
- Our goal is to uniformly obtain proof-theoretic characterizations for modal logics defined by frame properties.

# Sequent Calculi

- There is a well-studied strong correspondence between frame properties and Hilbert-style systems (*correspondence theory*).
- Hilbert-style systems are hardly useful for proof-search and proof-theoretic investigations.
- On the other hand, Gentzen-style calculi are particularly suitable for these purposes.

# Sequent Calculi

- There is a well-studied strong correspondence between frame properties and Hilbert-style systems (*correspondence theory*).
- Hilbert-style systems are hardly useful for proof-search and proof-theoretic investigations.
- On the other hand, Gentzen-style calculi are particularly suitable for these purposes.

**Definition**

A *sequent* is an ordered pair of finite set of formulas, denoted by

$$\Gamma \Rightarrow \Delta.$$

Intuitively,

$$\Gamma \Rightarrow \Delta \quad \leftrightsquigarrow \quad \bigwedge \Gamma \supset \bigvee \Delta.$$

# Gentzen Calculi for Modal Logics

$$(IW\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad (\Rightarrow IW)\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$$

$$(id)\ \frac{}{A \Rightarrow A} \qquad (cut)\ \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$(\supset\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, A_1 \quad \Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \supset A_2 \Rightarrow \Delta} \qquad (\Rightarrow\supset)\ \frac{\Gamma, A_1 \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \supset A_2, \Delta}$$

$$(\bot\Rightarrow)\ \frac{}{\bot \Rightarrow} \qquad (\Rightarrow\Box)\ \frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}$$

The calculus $\mathbf{G_K}$

**Facts**

1. $A$ is a theorem of $\mathbf{K}$ iff $\Rightarrow A$ is provable in $\mathbf{G_K}$.
2. $(cut)$ is admissible.

# Hypersequent Calculi

1. For many important modal logics (e.g. **S5** = universal accessibility relation) there is no known (cut-free) sequent calculus.
2. It is possible to characterize **S5** by going "one step further" — from sequents to hypersequents [Pottinger '83, Avron '87].

**Definition**

A *hypersequent* is a finite set of sequents, denoted by

$$\Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n.$$

Intuitively, a hypersequent is a disjunction of sequents:
$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ is true in some Kripke model if *some* $\Gamma_i \Rightarrow \Delta_i$ is true in all worlds.

# Hypersequent Calculus for **S5**

$$(IW\Rightarrow)\ \frac{H \mid \Gamma \Rightarrow \Delta}{H \mid \Gamma, A \Rightarrow \Delta} \qquad (\Rightarrow IW)\ \frac{H \mid \Gamma \Rightarrow \Delta}{H \mid \Gamma \Rightarrow \Delta, A} \qquad (EW)\ \frac{H}{H \mid \Gamma \Rightarrow \Delta}$$

$$(id)\ \frac{}{H \mid A \Rightarrow A} \qquad (cut)\ \frac{H \mid \Gamma \Rightarrow A, \Delta \quad H \mid \Gamma, A \Rightarrow \Delta}{H \mid \Gamma \Rightarrow \Delta}$$

$$(\supset\Rightarrow)\ \frac{H \mid \Gamma \Rightarrow \Delta, A_1 \quad H \mid \Gamma, A_2 \Rightarrow \Delta}{H \mid \Gamma, A_1 \supset A_2 \Rightarrow \Delta} \qquad (\Rightarrow\supset)\ \frac{H \mid \Gamma, A_1 \Rightarrow A_2, \Delta}{H \mid \Gamma \Rightarrow A_1 \supset A_2, \Delta}$$

$$(\bot\Rightarrow)\ \frac{}{H \mid \bot \Rightarrow} \qquad (\Rightarrow \Box)\ \frac{H \mid \Gamma \Rightarrow A}{H \mid \Box\Gamma \Rightarrow \Box A} \qquad (S5)\ \frac{H \mid \Gamma, \Gamma' \Rightarrow \Delta}{H \mid \Box\Gamma' \Rightarrow \mid \Gamma \Rightarrow \Delta}$$

The calculus $\mathbf{HG_{S5}}$

## Facts

1. $A$ is a theorem of **S5** iff $\Rightarrow A$ is provable in $\mathbf{HG_{S5}}$.
2. $(cut)$ is admissible.

# Derivation of $\Rightarrow \Diamond A \supset \Box\Diamond A$ in $\mathbf{HG_{S5}}$

$$\dfrac{}{A \supset \bot \Rightarrow A \supset \bot}\ (id)$$

$$\dfrac{A \supset \bot \Rightarrow A \supset \bot}{\Rightarrow A \supset \bot \mid \Box(A \supset \bot) \Rightarrow}\ (S5)$$

$$\dfrac{\Rightarrow A \supset \bot \mid \Box(A \supset \bot) \Rightarrow}{\Rightarrow \Box(A \supset \bot) \mid \Box(A \supset \bot) \Rightarrow}\ (\Rightarrow \Box)$$

$$\vdots\ (\bot \Rightarrow), (\supset\Rightarrow), (\Rightarrow IW), (\Rightarrow\supset)$$

$$\dfrac{\Box(A \supset \bot) \supset \bot \Rightarrow \mid \Rightarrow \Box(A \supset \bot) \supset \bot}{\Box(A \supset \bot) \supset \bot \Rightarrow \mid \Rightarrow \Box((\Box(A \supset \bot)) \supset \bot)}\ (\Rightarrow \Box)$$

$$\vdots\ (IW \Rightarrow), (\Rightarrow IW)$$

$$\dfrac{\Box(A \supset \bot) \supset \bot \Rightarrow \Box((\Box(A \supset \bot)) \supset \bot)}{\Rightarrow (\Box(A \supset \bot) \supset \bot) \supset \Box((\Box(A \supset \bot)) \supset \bot)}\ (\Rightarrow\supset)$$

# Main Contribution

## Questions

- What is the full power of hypersequent calculi for modal logics?
- What frame properties can be characterized?

## (Partial) Answers:

- We recognize a class of frame properties, called *simple frame properties*, for which it is possible to construct a hypersequent calculus.
- We provide the method to construct these calculi, and uniformly prove soundness and completeness, and cut-admissibility.

# Remark

There are other proof-theoretical frameworks for modal logics. E.g.:

- Semantic tableaus
- Display calculi
- Tree-hypersequent calculi and nested sequent calculi
- Labelled calculi

We are interested in (fully-structural) hypersequent calculi:

- Very close to Gentzen's approach.
- Kripke semantics is not explicitly involved in derivations.
- Useful for many other logics of different natures.
- Decidability follows from cut-admissibility (in the propositional level).

# Simple Frame Properties

- We use classical first-order language to formulate the frame properties.
  - For example, $\forall w . wRw$ captures reflexivity.

*Simple frame properties* are formulated by formulas of the form

$$\forall w_1 \ldots \forall w_n \exists u \varphi$$

where $\varphi$ consists of:

- Atomic formulas of the form $w_i R u$ or $w_i = u$.
- Conjunctions and disjunctions.

- Reflexivity is simple:

$$\forall w_1 \exists u \, (w_1 R u \wedge w_1 = u)$$

# Examples

| | |
|---|---|
| Seriality | $\forall w_1 \exists u\,(w_1 R u)$ |
| Directedness | $\forall w_1 \forall w_2 \exists u (w_1 R u) \wedge (w_2 R u)$ |
| Degenerateness | $\forall w_1 \forall w_2 \exists u\,(w_1 = u \wedge w_2 = u)$ |
| Universality | $\forall w_1 \forall w_2 \exists u\,(w_1 R u \wedge w_2 = u)$ |
| Linearity | $\forall w_1 \forall w_2 \exists u (w_1 R u \wedge w_2 = u) \vee (w_2 R u \wedge w_1 = u)$ |
| Bounded Cardinality | $\forall w_1 \ldots \forall w_n \exists u \bigvee_{1 \le i < j \le n} (w_i = u \wedge w_j = u)$ |
| Bounded Top Width | $\forall w_1 \ldots \forall w_n \exists u \bigvee_{1 \le i < j \le n} (w_i R u \wedge w_j R u)$ |
| Bounded Width | $\forall w_1 \ldots \forall w_n \exists u \bigvee_{1 \le i, j \le n; i \neq j} (w_i R u \wedge w_j = u)$ |

# From Simple Frame Properties to Hypersequent Rules

(1) Extract the *normal form* of $\forall w_1 \dots \forall w_n \exists u \varphi$

A set $\{\langle R_1, E_1\rangle, \dots, \langle R_m, E_m\rangle\}$ such that

$$\varphi \equiv \bigvee_{1 \leq i \leq m} \Big( \bigwedge_{j \in R_i} w_j R u \ \wedge \ \bigwedge_{j \in E_i} w_j = u \Big)$$

| | |
|---|---|
| $\forall w_1 \forall w_2 \exists u (w_1 R u) \wedge (w_2 R u)$ | $\{\langle \{1,2\}, \emptyset \rangle\}$ |
| $\forall w_1 \forall w_2 \exists u \, (w_1 R u \wedge w_2 = u)$ | $\{\langle \{1\}, \{2\} \rangle\}$ |
| $\forall w_1 \forall w_2 \exists u (w_1 R u \wedge w_2 = u) \vee (w_2 R u \wedge w_1 = u)$ | $\{\langle \{1\}, \{2\} \rangle, \langle \{2\}, \{1\} \rangle\}$ |
| $\forall w_1 \dots \forall w_n \exists u \bigvee_{1 \leq i < j \leq n} (w_i = u \wedge w_j = u)$ | $\{\langle \emptyset, \{i,j\} \rangle \mid 1 \leq i < j \leq n\}$ |

# From Simple Frame Properties to Hypersequent Rules

(2) For a normal form $\{\langle R_1, E_1\rangle, \dots, \langle R_m, E_m\rangle\}$ construct the following rule and add it to $\mathbf{HG_K}$:

$$\frac{H \mid \Gamma_{E_1}, \Gamma'_{R_1} \Rightarrow \Delta_{E_1} \quad \dots \quad H \mid \Gamma_{E_m}, \Gamma'_{R_m} \Rightarrow \Delta_{E_m}}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n, \Box\Gamma'_n \Rightarrow \Delta_n}$$

Notation: $\Pi_{\{i_1,,\dots,i_k\}} := \Pi_{i_1}, \dots, \Pi_{i_k}$

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Notation: $\Pi_{\{i_1,,\ldots,i_k\}} := \Pi_{i_1}, \ldots, \Pi_{i_k}$

*In the presence of the weakening rules, $\Gamma_i, \Gamma'_i, \Delta_i$'s that appear only in the conclusion can be discarded.*

$$\frac{H \mid \Gamma'_1, \Gamma'_2 \Rightarrow}{H \mid \Box\Gamma'_1 \Rightarrow \mid \Box\Gamma'_2 \Rightarrow}$$

Directedness

$$\frac{H \mid \Gamma_2, \Gamma'_1 \Rightarrow \Delta_2}{H \mid \Box\Gamma'_1 \Rightarrow \mid \Gamma_2 \Rightarrow \Delta_2}$$

Universality

$$\frac{H \mid \Gamma_2, \Gamma'_1 \Rightarrow \Delta_2 \quad H \mid \Gamma_1, \Gamma'_2 \Rightarrow \Delta_1}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1 \mid \Gamma_2, \Box\Gamma'_2 \Rightarrow \Delta_2}$$

Linearity

$$\frac{\{H \mid \Gamma_i, \Gamma_j \Rightarrow \Delta_i, \Delta_j \mid 1 \leq i < j \leq n\}}{H \mid \Gamma_1, \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n}$$

Bounded Cardinality

# Main Result

**Theorem**

*The constructed hypersequent calculus is sound and complete for the modal logic, and it enjoys cut-admissibility.*

- Uniform semantic proof for all calculi of this form.

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- Uniform semantic proof for all calculi of this form.

**Strong Soundness and Completeness**

| | | |
|---|---|---|
| $\Gamma \vdash_{Local} A$ | $A$ holds in every world in which $\Gamma$ holds | $\vdash \Gamma \Rightarrow A$ |
| $\Gamma \vdash_{Global} A$ | $A$ holds in every world if $\Gamma$ holds in every world | $\{ \Rightarrow B \mid B \in \Gamma\} \vdash \Rightarrow A$ |

**Strong Cut-Admissibility**

($cut$) can be confined to apply only on formulas that appear in the assumptions.

# Decidability

**Corollary**

*All modal logics characterized by finite sets of simple frame properties are decidable.*

**Proof.**

Cut-admissibility $\longrightarrow$ Subformula property $\longrightarrow$
We can check one by one all possible proofs candidates. $\square$

# Transitivity and Symmetry

*Simple frame properties* are formulated by formulas of the form

$$\forall w_1 \ldots \forall w_n \exists u \varphi$$

where $\varphi$ consists of:

- Atomic formulas of the form $w_i R u$ or $w_i = u$.
- Conjunctions and disjunctions.

- Simple properties are *monotone increasing* (preserved under enrichment of $R$).
- Transitivity and symmetry are not simple.

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- Atomic formulas of the form $w_i R u$ or $w_i = u$.
- Conjunctions and disjunctions.

- Simple properties are *monotone increasing* (preserved under enrichment of $R$).
- Transitivity and symmetry are not simple.
- We have to change the basic calculus:

$$\frac{H \mid \Gamma \Rightarrow A}{H \mid \Box\Gamma \Rightarrow \Box A} \quad \frac{H \mid \Gamma, \Box\Gamma \Rightarrow A}{H \mid \Box\Gamma \Rightarrow \Box A} \quad \frac{H \mid \Gamma \Rightarrow A, \Box\Delta}{H \mid \Box\Gamma \Rightarrow \Box A, \Delta}$$

**K** **K4** **KB**

# Transitivity

For a normal form $\{\langle R_1, E_1\rangle, \ldots, \langle R_m, E_m\rangle\}$ construct the rule:

$$\frac{H \mid \Gamma_{E_1}, \Gamma'_{R_1}, \Box\Gamma'_{R_1} \Rightarrow \Delta_{E_1} \quad \ldots \quad H \mid \Gamma_{E_m}, \Gamma'_{R_m}, \Box\Gamma'_{R_m} \Rightarrow \Delta_{E_m}}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n, \Box\Gamma'_n \Rightarrow \Delta_n}$$

For example:

$$\frac{H \mid \Gamma_2, \Gamma'_1, \Box\Gamma'_1 \Rightarrow \Delta_2 \quad H \mid \Gamma_1, \Gamma'_2, \Box\Gamma'_2 \Rightarrow \Delta_1}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1 \mid \Gamma_2, \Box\Gamma'_2 \Rightarrow \Delta_2}$$

Linearity

Again, we show:

- Strong soundness and completeness.
- Strong cut-admissibility.
- Decidability.

# Symmetry

For a normal form $\{\langle R_1, E_1\rangle, \ldots, \langle R_m, E_m\rangle\}$ construct the rule:

$$\frac{H \mid \Gamma_{E_1}, \Gamma'_{R_1} \Rightarrow \Delta_{E_1}, \Box\Delta'_{R_1} \quad \ldots \quad H \mid \Gamma_{E_m}, \Gamma'_{R_m} \Rightarrow \Delta_{E_m}, \Box\Delta'_{R_m}}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1, \Delta'_1 \mid \ldots \mid \Gamma_n, \Box\Gamma'_n \Rightarrow \Delta_n, \Delta'_n}$$

For example:

$$\frac{H \mid \Gamma'_1 \Rightarrow \Box\Delta'_1}{H \mid \Box\Gamma'_1 \Rightarrow \Delta'_1} \qquad \frac{H \mid \Gamma_1, \Gamma'_1 \Rightarrow \Delta_1, \Box\Delta'_1}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1, \Delta'_1}$$

Seriality — Reflexivity

- Cut-admissibility does not hold (even for the basic calculus).
- All constructed calculi still enjoy the subformula property.
- Decidability still follows.

# Conclusions and Further Research

- Correspondence between Kripke semantics and Gentzen-type calculi:

  simple frame property $\Leftrightarrow$ simple hypersequent rule

- Well-behaved Gentzen-type calculi can be constructed for all (transitive) (symmetric) modal logics characterized by simple frame properties.

  *E.g.* **KT**, **KD**, **S4**, **S5**, **K4D**, **K4.2**, **K4.3**, **S4.3**, **KBD**, **KBT**, $\mathbf{KBC_n}$, $\mathbf{KBW_n}$, $\mathbf{KBTW_n}$.
- These calculi may be helpful for investigating and using the logics (e.g. decidability).

Further work and open questions:

- Proof-search.
- Multi-modal logics.
- Non-simple properties. E.g. **K5** (euclidean), $\mathbf{BD_n}$ (bounded depth).
- Negative results?

Thank you!