Effective Finite-valued Semantics for Labelled Calculi

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The Big Picture

Goals:

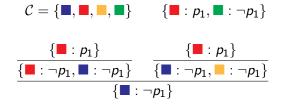
- Characterization of important proof-theoretic properties of calculi: *cut-admissibility, the subformula property, invertibility of rules,...*
- Understanding the dependencies between them
- Tool: Non-deterministic semantics
 - Goes back to [Schütte 1960], [Tait 1966]
 - Formalized and studied in [Avron,Lev 2001]
- Framework: Canonical labelled sequent calculi
 - Labelled = many-sided

- A propositional language \mathcal{L}
- A finite set of labels C $C \subseteq \{\blacksquare, \blacksquare, \blacksquare, \blacksquare, ...\}$
- Labelled formula:= $\Box : A$ where $A \in Frm_{\mathcal{L}}$ and $\Box \in C$
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 $p_1, p_1 \supset p_2 \Rightarrow p_2 \iff \{\blacksquare : p_1, \blacksquare : p_1 \supset p_2, \blacksquare : p_2\}$

Canonical Labelled Calculi

- All standard structural rules (exchange, contraction, weakening)
- A finite set of primitive rules
- In the set of canonical logical rules

Primitive Rules

Manipulate labels. Have the form (\Box 's are replaced by labels)

$$\frac{\{\Box:A,\ldots,\Box:A\}\cup s \quad \ldots \quad \{\Box:A,\ldots,\Box:A\}\cup s}{\{\Box:A,\ldots,\Box:A\}\cup s}$$

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Examples:

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 $\{\blacksquare:A,\blacksquare:A\}\cup s$

Canonical Rules

- "Ideal" logical introduction rules [Avron, Lev 2001]:
 - Introduce *exactly one connective*.
 - The active formulas are *immediate subformulas* of the principal formula.
 - The application is *context-independent*.

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• May introduce a connective with more than one label.

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Semantics

Intuition

- The value of A determines which of the labelled formulas
 ■: A, ■: A, ■: A, ... is true.
- In general, there are $2^{|\mathcal{C}|}$ possible options.
- Primitive rules forbid some of them.
- Logical rules are used to determine the values of compound formulas.

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- Logical rules are used to determine the values of compound formulas.

Formalization

- The set of truth-values $\mathcal{T}_G \subseteq P(\mathcal{C})$ is determined according to the primitive rules of G.
- A valuation $v : Frm_{\mathcal{L}} \to \mathcal{T}_{\mathbf{G}}$ is a model of $\Box : A$ if $\Box \in v(A)$.
- A valuation is a model of a sequent *s* if it is a model of some labelled formula in *s*.





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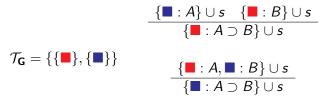
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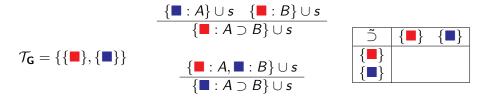
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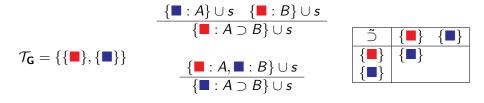
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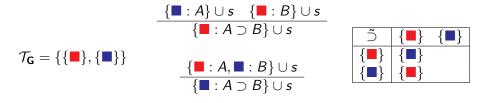
Theorem

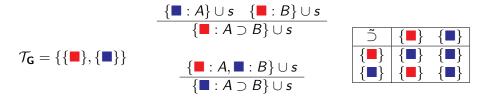
Given a canonical calculus **G** without logical rules, $\Omega \vdash_{\mathbf{G}} s$ iff every valuation $v : \operatorname{Frm}_{\mathcal{L}} \to \mathcal{T}_{\mathbf{G}}$ which is a model of every sequent in Ω is also a model of s.











For example:



A legal valuation should respect the table: $v(\diamond(A_1,\ldots,A_n)) = \tilde{\diamond}(v(A_1),\ldots,v(A_n))$

Non truth-functional connectives, a.g. primal implication [Gurevich, Neema]

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How to determine $\tilde{\supset}(\{\blacksquare\}, \{\blacksquare\})$?

• More than one option satisfies the conclusion, e.g.

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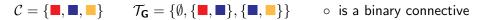
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Solution: Non-deterministic Truth-Tables [Avron, Lev 2001]

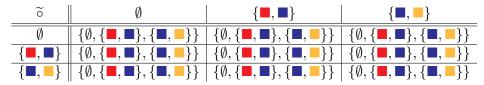
A table of an *n*-ary connective \diamond is a function $\tilde{\diamond} : \mathcal{T}^n \to P^+(\mathcal{T})$. A legal valuation satisfies: $v(\diamond(A_1, \ldots, A_n)) \in \tilde{\diamond}(v(A_1), \ldots, v(A_n))$

Example: Construction of a Non-deterministic Truth-Table



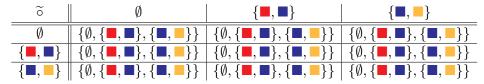
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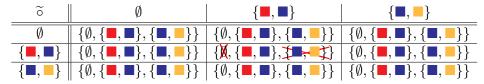
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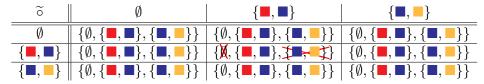
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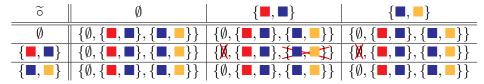
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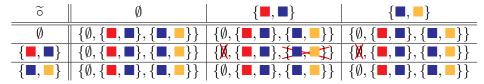
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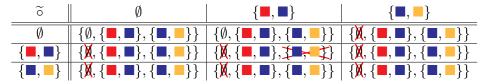
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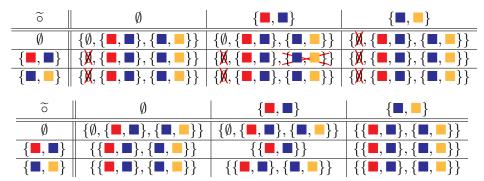
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• Contradictions between rules, e.g.

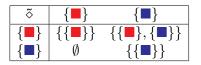
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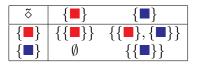
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Solution: Partial Truth-Tables

Allow empty entries: $\tilde{\diamond} : \mathcal{T}^n \to P(\mathcal{T})$.

The Semantic Framework

Partial Non-deterministic Matrices

- A PNmatrix **M** for \mathcal{L} and \mathcal{C} consists of:
 - A set \mathcal{T} of truth-values.
 - A function D : C → P(T) assigning a set of designated truth-values for every label.
 - A truth-table $\tilde{\diamond} : \mathcal{T}^n \to P(\mathcal{T})$ for every *n*-ary connective of \mathcal{L} .

A valuation $v : Frm_{\mathcal{L}} \to \mathcal{T}$ is:

- a model (in **M**) of a sequent s if $v(A) \in \mathcal{D}(\Box)$ for some $\Box : A$ in s.
- M-legal if $v(\diamond(A_1,\ldots,A_n)) \in \tilde{\diamond}(v(A_1),\ldots,v(A_n))$ for every $\diamond(A_1,\ldots,A_n) \in Frm_{\mathcal{L}}$.

Main Result

Theorem

For every canonical labelled calculus **G**, there exists a strongly characteristic *PNmatrix* $\mathbf{M}_{\mathbf{G}}$ (i.e. $\Omega \vdash_{\mathbf{G}} s$ iff every $\mathbf{M}_{\mathbf{G}}$ -legal valuation which is a model of every sequent in Ω is also a model of s).

Moreover, we provide a uniform algorithm to obtain M_G from G.

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Moreover, we provide a uniform algorithm to obtain M_G from G.

In many cases, the obtained semantics coincides with a known one:

- Propositional fragment of LK
- LK without cut [Girard 1987]
- LK without identity axiom [Hösli, Jäger 1994]
- Two-sided canonical systems [Avron,Lev 2001]
- Labelled calculi studied in [Baaz et al. 1998] and [Avron,Zamansky 2009]



Theorem

Semantic consequence relations induced by PNmatrices are decidable.

Corollary

All canonical labelled calculi are decidable.

Effectiveness

Theorem

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Proof Outline.

- Usual method: To decide whether Ω ⊢_M s, check one-by-one all M-legal partial valuations defined sub[Ω, s].
- Hidden assumption: All M-legal partial valuations can be extended to full ones (semantic analyticity).
 But, it does not hold for PNmatrices (recall õ({■}, {■}) = Ø !).
- Lemma: It is decidable whether an **M**-legal partial valuations can be extended to a full one.
- Solution: Check one-by-one all extendable M-legal partial valuations defined sub[Ω, s].

A cut is a primitive rule of the form:

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Strong Cut-Admissibility

 $\Omega \vdash_{\mathbf{G}} s \Longrightarrow$ there is a derivation of s from Ω in **G** in which: the cut-formula of each cut occurs either in the cut-context or in Ω .

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Theorem

For every canonical labelled calculus G:

G enjoys strong cut-admissibility iff M_G does not include empty entries.



- We provided effective and modular semantic characterization for labelled canonical sequent calculi using partial non-deterministic matrices.
- Application: semantic characterization of proof-theoretic properties.
- Similar ideas can be applied for: single-conclusion canonical calculi, sequent calculi for modal logics, canonical Gödel hypersequent calculi...
- Future research directions:
 - First-order
 - Less restrictive primitive and introduction rules

Thank you!