Effective Semantics for the Modal Logics K and KT via Non-deterministic Matrices*

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Abstract. A four-valued semantics for the modal logic K is introduced. Possible worlds are replaced by a hierarchy of four-valued valuations, where the valuations of the first level correspond to valuations that are legal w.r.t. a basic non-deterministic matrix, and each level further restricts its set of valuations. The semantics is proven to be effective, and to precisely capture derivations in a sequent calculus for K of a certain form. Similar results are then obtained for the modal logic KT, by simply deleting one of the truth values.

1 Introduction

Propositional modal logics extend classical logic with *modalities*, intuitively interpreted as necessity, knowledge, or temporal operators. Such extensions have several applications in computer science and artificial intelligence (see, e.g., [7, 9, 13]).

The most common and successful semantic framework for modal logics is the so called *possible worlds semantics*, in which each world is equipped with a two-valued valuation, and the semantic constraints regarding the modal operators consider the valuations in *accessible* worlds. While this has been the gold standard for modal logic semantics for many years, alternative semantic frameworks have been proposed. One of these approaches, initiated by Kearns [10], is based on an infinite sequence of sets of valuations in a non-deterministic many-valued semantics. Since then, several non-deterministic many-valued semantics, without possible worlds, were developed for modal logics (see, e.g., [14, 4, 8, 12]). The current paper is a part of that body of work. Having an alternative semantic framework for modal logics, different than the common possible worlds semantics, has the potential of exposing new intuitions and understandings of modal logics, and also to form the basis to new decision procedures.

Our main contribution is a four-valued semantics for the modal logic `K`. The key characteristic of the semantics that we present is *effectiveness*: when checking for the entailment of a formula φ from a set Γ of formulas in `K`, it suffices to only consider *partial* models, defined over the subformulas of Γ and φ. To the

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best of our knowledge, this is the first effective Nmatrices-based semantics for `K`. Such a semantics has the potential of being subject to reductions to classical satisfiability [3], as it is based on finite-valued truth tables, and thus improving the performance of solvers for modal logic by utilizing off-the-shelf SAT solvers. Another advantage of this semantics is that it precisely captures derivations in a sequent calculus for `K` that admit a certain property. Following Kearns, models of this semantics are based on the concept of *levels*—valuations of level 0 are the ordinary valuations of Nmatrices, while each level $m > 0$ introduces more constraints. We show that valuations of level m correspond to derivations in the calculus whose largest number of applications of the rule that correspond to the axiom (K) in any branch of the derivation is at most m. Our restrictions between the levels are more complex than the original restrictions in Kearns' work, in order to obtain effectiveness. Another precise correspondence between the semantics and the proof system that we prove, is between the domains of valuations and the formulas allowed to be used in derivations.

Finally, we observe that by deleting one of the truth values, a three-valued semantics for the modal logic `KT` is obtained, which is similar to the one presented in [8]. Like the case of `K`, the resulting semantics is effective, and tightly correspond to derivations in a sequent calculus for `KT`.

Outline. The paper is organized as follows: Section 2 reviews standard notions in non-deterministic matrices. In §3, we present our semantics for the modal logic `K`, as well as the sequent calculus our investigation will be based on, which is coupled with the notion of (K)-depth of derivations. In §4, we prove soundness and completeness theorems between the sequent calculus and the semantics. In §5, we prove that the semantics that we provide is effective, not only for deciding entailment, but also for producing countermodels when an entailment does not hold. In §6 we establish similar results for the modal logic `KT`. We conclude with §7, where directions for future research are outlined.

Related Work. In [10], Kearns initiated the study of modal semantics without possible worlds. This work was recently revisited by Skurt and Omori [14], who generalized Kearns' work and reframed his framework within the framework of logical Non-deterministic matrices. As indicated in [14], it was not clear how to make this semantics effective, as it requires checking truth values of infinitely many formulas when considering the validity of a given formula (see, e.g., Remark 42 of [14]). In [4], Coniglio et al. develop a similar framework for modal logics, and some bound over the formulas that need to be considered was achieved. However, in [5], the authors clarified that it is unclear how to effectively use the resulting semantics. A semantics based on Nmatrices for the modal logics `KT` and `S4` was presented in [8] by Grätz, that includes a method to extend a partial model in that semantics into a total one, which results in an effective semantics. We chose here to focus on `K`, which is a weaker logic, forming a common basis to all other normal modal logics. By deleting one out of four truth values, we obtain corresponding results for `KT` as well. The semantics that we present here is similar in nature to the one presented in [8], however: (i) the truth tables are different, as we intentionally enforced the many-valued tables of the classical connectives to be obtained by a straightforward duplication of truth values from the original two-valued truth tables; and (*ii*) the semantic condition for levels of valuations that we define here is inductive, where each level relies on lower levels (thus refraining from a definition of a more cyclic nature as the one in [8], that is better understood operationally). A variant of the semantics from [14] was also introduced and studied in [12], but without considering the ability to perform effective automated reasoning but instead focusing on infinite valuations rather than on partial ones. A complete proof theoretic characterization in terms of sequent calculi to the various levels of valuations was not given in any of the above works. Also, an effective semantics for K, which is the most basic modal logic, was not given in any of the above works.

Non-deterministic matrices were introduced in [2], and have since became a useful tool for investigating non-classical logics and proof systems (see [1] for a survey). They generalize (deterministic) matrices [15] by allowing a non-deterministic choice of truth values in the truth tables. Like matrices, Nmatrices enjoy the semantic *analyticity* property, which allows one to extend a partial valuation into a full one. Our semantic framework can be viewed as a further refinement of non-deterministic matrices, namely *restricted* non-deterministic matrices, introduced in [6].

2 Preliminaries

In this section we provide the necessary definitions about Nmatrices following [1]. We assume a propositional language $\mathcal{L}$ with countably infinitely many atomic variables $p_1, p_2, \ldots$. When there is no room for confusion, we identify $\mathcal{L}$ with its set of well-formed formulas (e.g., when writing $\varphi \in \mathcal{L}$). We write $sub(\varphi)$ for the set of subformulas of a formula φ. This notation is extended to sets of formulas in the natural way.

Valuations. In the context of a set $\mathcal{V}$ of "truth values", a *valuation* is a function v from some domain $\mathsf{Dom}(v) \subseteq \mathcal{L}$ to $\mathcal{V}$. For a set $\mathcal{F} \subseteq \mathcal{L}$, an *$\mathcal{F}$-valuation* is a valuation with domain $\mathcal{F}$. (In particular, an $\mathcal{L}$-valuation is defined on all formulas.) For $X \subseteq \mathcal{V}$, we write $v^{-1}[X]$ for the set $\{\varphi \mid v(\varphi) \in X\}$. For $x \in \mathcal{V}$, we also write $v^{-1}[x]$ for the set $\{\varphi \mid v(\varphi) = x\}$.

Definition 1. Let $\mathcal{D} \subseteq \mathcal{V}$ be a set of "designated truth values". A valuation v *$\mathcal{D}$-satisfies* a formula φ, denoted by $v \models_{\mathcal{D}} \varphi$, if $v(\varphi) \in \mathcal{D}$. For a set Σ of formulas, we write $v \models_{\mathcal{D}} \Sigma$ if $v \models_{\mathcal{D}} \varphi$ for every $\varphi \in \Sigma$.

Notation 2 Let $\mathcal{D} \subseteq \mathcal{V}$ be a set of designated truth values and $\mathbb{V}$ be a set of valuations. For sets L, R of formulas, we write $L \vdash^{\mathbb{V}}_{\mathcal{D}} R$ if for every $v \in \mathbb{V}$, $v \models_{\mathcal{D}} L$ implies that $v \models_{\mathcal{D}} \varphi$ for some $\varphi \in R$. We omit L or R in this notation when they are empty (e.g., when writing $\vdash^{\mathbb{V}}_{\mathcal{D}} R$), and set parentheses for singletons (e.g., when writing $L \vdash^{\mathbb{V}}_{\mathcal{D}} \varphi$).

Nmatrices. An Nmatrix M for $\mathcal{L}$ is a triple of the form $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V}$ is a set of *truth values*, $\mathcal{D} \subseteq \mathcal{V}$ is a set of *designated truth values*, and $\mathcal{O}$ is a function assigning a *truth table* $\mathcal{V}^n \to P(\mathcal{V}) \setminus \{\emptyset\}$ to every n-ary connective $\diamond$ of $\mathcal{L}$ (which assigns a set of possible values to each tuple of values). In the context of an Nmatrix $M = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, we often denote $\mathcal{O}(\diamond)$ by $\tilde{\diamond}$.

An $\mathcal{F}$-valuation v is M-*legal* if $v(\varphi) \in \mathsf{pos\text{-}val}(\varphi, M, v)$ for every formula $\varphi \in \mathcal{F}$ whose immediate subformulas are contained in $\mathcal{F}$, where $\mathsf{pos\text{-}val}(\varphi, M, v)$ is defined by:

1. $\mathsf{pos\text{-}val}(p, M, v) = \mathcal{V}$ for every atomic formula p.
2. $\mathsf{pos\text{-}val}(\diamond(\psi_1, \ldots, \psi_n), M, v) = \tilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$ for every non-atomic formula $\diamond(\psi_1, \ldots, \psi_n)$.

In other words, there is no restriction regarding the values assigned to atomic formulas, whereas the values of compound formulas should respect the truth tables.

Lemma 1 ([1]). *Let $\mathcal{F} \subseteq \mathcal{L}$ be a set closed under subformulas and M an Nmatrix for $\mathcal{L}$. Then every M-legal $\mathcal{F}$-valuation v can be extended to an M-legal $\mathcal{L}$-valuation.*

3 The Modal Logic K

In this section we introduce a novel effective semantics for the model logic K. We first present a known proof system for this logic (§3.1), and then our semantics (§3.2). From here on, we assume that the language $\mathcal{L}$ consists of the connectives $\supset, \wedge, \vee, \neg$ and $\Box$ with their usual arities. The standard $\Diamond$ operator can be defined as a macro $\Diamond\varphi \overset{\text{def}}{=} \neg\Box\neg\varphi$. Obviously, using De-Morgan rules, fewer connectives can be used. However, we chose this set of connectives in order to have a primitive language rich enough for the examples that we include along the paper.

3.1 Proof System

Figure 1 presents a Gentzen-style calculus, denoted by $\mathsf{G_K}$, for the modal logic K that was proven to be equivalent to the original formulation of the logic as a Hilbert system (see, e.g., [16]). We take *sequents* to be pairs $\langle \Gamma, \Delta \rangle$ of finite *sets* of formulas. For readability, we write $\Gamma \Rightarrow \Delta$ instead of $\langle \Gamma, \Delta \rangle$ and use standard notations such as $\Gamma, \varphi \Rightarrow \psi$ instead of $(\Gamma \cup \{\varphi\}) \Rightarrow \{\psi\}$.

The (CUT) rule is included in $\mathsf{G_K}$ for convenience, but applications of (CUT) can be eliminated from derivations (see, e.g., [11]). Since the focus of this paper is semantics rather than cut-elimination, we allow ourselves to use cut freely and do not distinguish derivations that use it from derivations that do not. We write $\vdash_{\mathsf{G_K}} \Gamma \Rightarrow \Delta$ if there is a derivation of a sequent $\Gamma \Rightarrow \Delta$ in the calculus $\mathsf{G_K}$.

In the sequel, we provide a semantic characterization of $\vdash_{\mathsf{G_K}}$. It is based on a more refined notion of derivability that takes into account: (i) the set $\mathcal{F}$ of formulas used in the derivation; and (ii) the (K)-*depth* of the derivation, as defined next.

$$(\text{WEAK})\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \qquad (\text{ID})\ \frac{}{\Gamma, \varphi \Rightarrow \varphi, \Delta} \qquad (\text{CUT})\ \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \Delta} \qquad (K)\ \frac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}$$

$$(\neg\Rightarrow)\ \frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma, \neg\varphi \Rightarrow \Delta} \qquad (\Rightarrow\neg)\ \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\varphi, \Delta} \qquad (\supset\Rightarrow)\ \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \supset \psi, \Delta} \qquad (\Rightarrow\supset)\ \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \supset \psi, \Delta}$$

$$(\wedge\Rightarrow)\ \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta} \quad (\Rightarrow\wedge)\ \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta} \quad (\vee\Rightarrow)\ \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta} \quad (\Rightarrow\vee)\ \frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}$$

Fig. 1. The sequent calculus $\mathsf{G_K}$

Definition 3. A *derivation* of a sequent $\Gamma \Rightarrow \Delta$ in $\mathsf{G_K}$ is a tree in which the nodes are labeled with sequents, the root is labeled with $\Gamma \Rightarrow \Delta$, and every node is the result of an application of some rule of $\mathsf{G_K}$ where the premises are the labels of its children in the tree. A derivation is called an $\mathcal{F}$*-derivation* if it employs only sequents composed of formulas from $\mathcal{F}$. The (K)*-depth* of a derivation is the maximal number of applications of rule (K) in any of the branches of the derivation.

Notation 4 We write $\vdash^{\mathcal{F},m}_{\mathsf{G_K}} \Gamma \Rightarrow \Delta$ if there is a derivation of $\Gamma \Rightarrow \Delta$ in $\mathsf{G_K}$ in which only $\mathcal{F}$-sequents occur and that has (K)-depth at most m. We drop $\mathcal{F}$ from this notation when $\mathcal{F} = \mathcal{L}$; and drop m to dismiss the restriction regarding the (K)-depth.

Example 1. Let $\varphi \stackrel{\mathsf{def}}{=} \Box(p_1 \wedge p_2) \supset (\Box p_1 \wedge \Box p_2)$ and $\mathcal{F} = sub(\varphi)$. The following is a derivation of $\Rightarrow \varphi$ in $\mathsf{G_K}$ that only uses $\mathcal{F}$-formulas and has (K)-depth of 1 (though the number of applications of (K) in the derivation is 2):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p_1, p_2 \Rightarrow p_1}(\text{ID})}{p_1 \wedge p_2 \Rightarrow p_1}(\wedge\Rightarrow)}{\Box(p_1 \wedge p_2) \Rightarrow \Box p_1}(\text{K}) \quad \dfrac{\dfrac{\dfrac{}{p_1, p_2 \Rightarrow p_2}(\text{ID})}{p_1 \wedge p_2 \Rightarrow p_2}(\wedge\Rightarrow)}{\Box(p_1 \wedge p_2) \Rightarrow \Box p_2}(\text{K})}{\Box(p_1 \wedge p_2) \Rightarrow \Box p_1 \wedge \Box p_2}(\Rightarrow\wedge)}{\Rightarrow \Box(p_1 \wedge p_2) \supset \Box p_1 \wedge \Box p_2}(\Rightarrow\supset)$$

3.2 Semantics

The semantics is based on a four-valued Nmatrix stratified with "levels", where for every m, legal valuations of level $m+1$ are a subset of legal valuations of level m. The underlying Nmatrix, denoted by $\mathsf{M_K}$, is obtained by duplicating the classical truth values. Thus, the sets of truth values and of designated truth values are given by:

$$\mathcal{V}_4 \stackrel{\mathsf{def}}{=} \{\mathsf{T}, \mathsf{t}, \mathsf{f}, \mathsf{F}\} \qquad \mathcal{D} \stackrel{\mathsf{def}}{=} \{\mathsf{T}, \mathsf{t}\}$$

The truth tables are as follows (we have $\overline{\mathcal{D}} = \{\mathsf{f}, \mathsf{F}\}$):

$x \tilde{\supset} y$	T	t	F	f
T	$\mathcal{D}$	$\mathcal{D}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$
t	$\mathcal{D}$	$\mathcal{D}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$
F	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
f	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$

$x \tilde{\wedge} y$	T	t	F	f
T	$\mathcal{D}$	$\mathcal{D}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$
t	$\mathcal{D}$	$\mathcal{D}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$
F	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$
f	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$

$x \tilde{\vee} y$	T	t	F	f
T	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
t	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
F	$\mathcal{D}$	$\mathcal{D}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$
f	$\mathcal{D}$	$\mathcal{D}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$

x	$\tilde{\neg} x$
T	$\overline{\mathcal{D}}$
t	$\overline{\mathcal{D}}$
F	$\mathcal{D}$
f	$\mathcal{D}$

x	$\tilde{\Box} x$
T	$\mathcal{D}$
t	$\overline{\mathcal{D}}$
F	$\mathcal{D}$
f	$\overline{\mathcal{D}}$

We employ the following notations for subsets of truth values:

$$\mathsf{TF} \stackrel{\mathsf{def}}{=} \{\mathsf{T}, \mathsf{F}\} \qquad \mathsf{tf} \stackrel{\mathsf{def}}{=} \{\mathsf{t}, \mathsf{f}\}$$

For the classical connectives, the truth tables of M_K treat t just like T, and f just like F, and are essentially two-valued—the result is either $\mathcal{D}$ or $\overline{\mathcal{D}}$, and it depends solely on whether the inputs are elements of $\mathcal{D}$ or $\overline{\mathcal{D}}$. Thus, for the language without $\Box$, this Nmatrix provides a (non-economic) four-valued semantics for classical logic.

While the output for $\Box$ is also always $\mathcal{D}$ or $\overline{\mathcal{D}}$, it differentiates between T (that results in $\mathcal{D}$) and t (that results in $\overline{\mathcal{D}}$), and similarly between F and f. In fact, this table is captured by the condition: $\tilde{\Box}(x) \in \mathcal{D}$ iff $x \in \mathsf{TF}$.

Example 2. Let $\mathcal{F} = sub(\varphi)$ where φ is the formula from Example 1. The following valuation v is an $\mathcal{F}$-valuation that is M_K-legal:

$$v(p_1) = v(p_2) = \mathsf{f} \quad v(p_1 \wedge p_2) = \mathsf{F} \quad v(\Box p_1) = v(\Box p_2) = v(\Box p_1 \wedge \Box p_2) = \mathsf{F}$$

$$v(\Box(p_1 \wedge p_2)) = \mathsf{T} \quad v(\Box(p_1 \wedge p_2) \supset (\Box p_1 \wedge \Box p_2)) = \mathsf{F}$$

To show that it is M_K-legal, one needs to verify that $v(\psi) \in \mathsf{pos\text{-}val}(\psi, \mathsf{M}_\mathsf{K}, v)$ for each $\psi \in \mathcal{F}$. For example, $v(p_1) = \mathsf{f} \in \mathcal{V}_4 = \mathsf{pos\text{-}val}(p_1, \mathsf{M}_\mathsf{K}, v)$. As another example, since $v(p_1) = \mathsf{f}$, we have that $\mathsf{pos\text{-}val}(\Box p_1, \mathsf{M}_\mathsf{K}, v) = \tilde{\Box}(\mathsf{f}) = \{\mathsf{F}, \mathsf{f}\}$, and hence $v(\Box p_1) = \mathsf{F} \in \mathsf{pos\text{-}val}(\Box p_1, \mathsf{M}_\mathsf{K}, v)$. Notice that v does not satisfy φ.

The truth table for $\Box$ can be understood via "possible worlds" intuition. Our four truth values are intuitively captured as follows, assuming a given formula ψ and a world w:

- T: ψ holds in w and in every world accessible from w;
- t: ψ holds in w but it does not hold in some world accessible from w;
- F: ψ does not hold in w but does hold in every world accessible from w; and
- f: ψ does not hold in w and it does not hold in some world accessible from w.

In the possible worlds semantics, $\Box\psi$ holds in some world w iff ψ holds in every world that is accessible from w, which intuitively explains the table for $\Box$. Note that non-determinism is inherent here. For example, if ψ holds in w and in every world accessible from w (i.e., ψ has value T), we know that $\Box\psi$ holds in w, but we do not know whether $\Box\psi$ holds in every world accessible from w (thus $\Box\psi$ has value T or t).

Now, the Nmatrix $\mathsf{M}_{\mathtt{K}}$ by itself is not adequate for the modal logic $\mathtt{K}$ (as Examples 1 and 2 demonstrate). What is missing is the relation between the choices we make to resolve non-determinism for different formulas. Continuing with the possible worlds intuition, we observe that if a formula φ follows from a set of formulas Σ that hold in all accessible worlds (i.e., φ follows from formulas whose truth value is T or F), then φ itself should hold in all accessible worlds (i.e., φ's truth value should be T or F). Directly encoding this condition requires us to consider a set $\mathbb{V}$ of $\mathsf{M}_{\mathtt{K}}$-legal $\mathcal{F}$-valuations for which the following holds (recall Notation 2 from §2):

$$\forall v \in \mathbb{V}.\, \forall \varphi \in \mathcal{F}.\, (v^{-1}[\mathsf{TF}] \vdash^{\mathbb{V}}_{\mathcal{D}} \varphi \implies v(\varphi) \in \mathsf{TF}) \qquad (\textit{necessitation})$$

In turn, to obtain completeness we take a maximal set $\mathbb{V}$ that satisfies the *necessitation* condition. While it is possible to define this set of valuations as the greatest fixpoint of *necessitation*, following previous work, we find it convenient to reach this set using "levels":

Definition 5. The set $\mathbb{V}^{\mathcal{F},m}_{\mathtt{K}}$ is inductively defined as follows:

- $\mathbb{V}^{\mathcal{F},0}_{\mathtt{K}}$ is the set of $\mathsf{M}_{\mathtt{K}}$-legal $\mathcal{F}$-valuations.
- $\mathbb{V}^{\mathcal{F},m+1}_{\mathtt{K}} \stackrel{\text{def}}{=} \left\{ v \in \mathbb{V}^{\mathcal{F},m}_{\mathtt{K}} \mid \forall \varphi \in \mathcal{F}.\, v^{-1}[\mathsf{TF}] \vdash^{\mathbb{V}^{\mathcal{F},m}_{\mathtt{K}}}_{\mathcal{D}} \varphi \implies v(\varphi) \in \mathsf{TF} \right\}$

We also define:

$$\mathbb{V}^{\mathcal{F}}_{\mathtt{K}} \stackrel{\text{def}}{=} \bigcap_{m \geq 0} \mathbb{V}^{\mathcal{F},m}_{\mathtt{K}} \qquad \mathbb{V}^{m}_{\mathtt{K}} \stackrel{\text{def}}{=} \mathbb{V}^{\mathcal{L},m}_{\mathtt{K}} \qquad \mathbb{V}_{\mathtt{K}} \stackrel{\text{def}}{=} \bigcap_{m \geq 0} \mathbb{V}^{\mathcal{L},m}_{\mathtt{K}}$$

Similarly to the idea originated by Kearns in [10], valuations are partitioned into *levels*, which are inductively defined. The first level, $\mathbb{V}^{\mathcal{F},0}_{\mathtt{K}}$, consists solely of the $\mathsf{M}_{\mathtt{K}}$-legal valuations with domain $\mathcal{F}$. For each $m > 0$, the m'th level is defined as a subset of the $(m-1)$'th level, with an additional constraint: a valuation v from level $m-1$ remains in level m, only if every formula $\varphi \in \mathcal{F}$ entailed (at the $m-1$ level) from the set of formulas that were assigned a value from TF by v, is itself assigned a value from TF by v. As we show below, in the "end" of this process, by taking $\bigcap_{m \geq 0} \mathbb{V}^{\mathcal{F},m}_{\mathtt{K}}$, one obtains the greatest set $\mathbb{V}$ satisfying the *necessitation* condition

Remark 1. The *necessitation* condition is similar to the one provided in [8] to the modal logics $\mathtt{KT}$ and $\mathtt{S4}$. In contrast, the condition from [10, 14, 4] is simpler and does not involve $v^{-1}[\mathsf{TF}]$ at all, but also does not give rise to decision procedures.

Example 3. Following Example 2, while the formula φ is not satisfied by all valuations in $\mathbb{V}^{\mathcal{F},0}_{\mathtt{K}}$, it is satisfied by all valuations in $\mathbb{V}^{\mathcal{F},m}_{\mathtt{K}}$ for every $m > 0$. In particular, the valuation v from Example 2 is not in $\mathbb{V}^{\mathcal{F},1}_{\mathtt{K}}$: we have $p_1 \wedge p_2 \vdash^{\mathbb{V}^{\mathcal{F},0}_{\mathtt{K}}}_{\mathcal{D}} p_1$ and $v(p_1 \wedge p_2) = \mathsf{F}$ (so $p_1 \wedge p_2 \in v^{-1}[\mathsf{TF}]$), but $v(p_1) = \mathsf{f} \notin \mathsf{TF}$.

For each set $\mathcal{F} \subseteq \mathcal{L}$ and $m \geq 0$, we obtain a consequence relation $\vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F},m}}$ between sets of $\mathcal{F}$-formulas. Disregarding m, we also obtain the relation $\vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F}}}$ (for every $\mathcal{F}$), which we will show to be sound and complete for K. We note that all these relations are compact. The proof of the following theorem relies on the completeness theorems that we prove in §4.

Theorem 1 (Compactness).

1. *For every $m \geq 0$, if $L \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F},m}} R$, then $\Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F},m}} \Delta$ for some finite $\Gamma \subseteq L$ and $\Delta \subseteq R$.*
2. *If $L \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F}}} R$, then $\Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F}}} \Delta$ for some finite $\Gamma \subseteq L$ and $\Delta \subseteq R$.*

Now, to show that $\mathbb{V}_{\mathtt{K}}^{\mathcal{F}}$ is indeed the largest set $\mathbb{V}$ of $\mathsf{M}_{\mathtt{K}}$-legal $\mathcal{F}$-valuations that satisfies *necessitation*, we use the following two lemmas. The first is a general construction that relies only on the use of *finite-valued* valuation functions.

Lemma 2. *Let $v_0, v_1, v_2, ...$ be an infinite sequence of valuations over a common domain $\mathcal{F}$. Then, there exists some v such that for every finite set $\mathcal{F}' \subseteq \mathcal{F}$ of formulas and $m \geq 0$, we have $v|_{\mathcal{F}'} = v_k|_{\mathcal{F}'}$ for some $k \geq m$.*

Proof (Outline). First, if $\mathcal{F}$ is finite, then there is only a finite number of $\mathcal{F}$-valuations, and there must exists some $\mathcal{F}$-valuation v_m that occurs infinitely often in the sequence $v_0, v_1, ...$ We take $v = v_m$, and the required property trivially holds. Now, assume that $\mathcal{F}$ is infinite, and let $\varphi_0, \varphi_1, ...$ be an enumeration of the formulas in $\mathcal{F}$. For every $i \geq 0$, let $\mathcal{F}_i = \{\varphi_0, ..., \varphi_i\}$. We construct a sequence of infinite sets $A_0, A_1, ... \subseteq \mathbb{N}$ such that:

- For every $i \geq 0$, $A_{i+1} \subseteq A_i$.
- For every $0 \leq j \leq i$, $a \in A_j$, and $b \in A_i$, $v_a(\varphi_j) = v_b(\varphi_j)$.

To do so, take some infinite set $A_0 \subseteq \mathbb{N}$ such that $v_a(\varphi_0) = v_b(\varphi_0)$ for every $a, b \in A_0$ (such set must exist since we have a finite number of truth values). Then, given A_i, we let A_{i+1} be some infinite subset of A_i such that $v_a(\varphi_{i+1}) = v_b(\varphi_{i+1})$ for every $a, b \in A_{i+1}$. The valuation v is defined by $v(\varphi_i) = v_a(\varphi_i)$ for some $a \in A_i$. The properties of the A_i's ensure that v is well defined, and it can be shown that it also satisfies the required property. □

Using Lemma 2 and the compactness property, we can show the following:

Lemma 3. *Let $v_0, v_1, ...$ be a sequence of valuations over a common domain $\mathcal{F}$ such that $v_m \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F},m}$ for every $m \geq 0$. Then, there exists some $v \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F}}$ such that for every $\varphi \in \mathcal{F}$, $v(\varphi) = v_m(\varphi)$ for some $m \geq 0$.*

Proof (Outline). By Lemma 2, there exists some v such that for every finite set $\mathcal{F}'$ of formulas, $v|_{\mathcal{F}'} = v_m|_{\mathcal{F}'}$ for some $m \geq 0$. It is easy to verify that v satisfies the required properties. In particular, one shows that $v \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F},m}$ for every $m \geq 0$ by induction on m. In that proof we use Theorem 1 to obtain a finite $\Gamma \subseteq v^{-1}[\mathsf{TF}]$ such that $\Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F},m-1}} \varphi$ from the assumption that $v^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F},m-1}} \varphi$. Then, the above property of v is applied with $\mathcal{F}' = \Gamma \cup \{\varphi\}$. □

Now, our characterization theorem easily follows:

Theorem 2. *The set $\mathbb{V}_{\mathsf{K}}^{\mathcal{F}}$ is the largest set $\mathbb{V}$ of M_{K}-legal $\mathcal{F}$-valuations that satisfies necessitation.*

Proof (Outline). To prove that $\mathbb{V}_{\mathsf{K}}^{\mathcal{F}}$ satisfies *necessitation*, one needs to prove that if $v^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F}}} \varphi$, then also $v^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}} \varphi$ for some $m \geq 0$. This is done using Lemma 3. For maximality, given a set $\mathbb{V}$, we assume by contradiction that there is some m such that $\mathbb{V} \not\subseteq \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}$, take a minimal such m, and show that it cannot be 0. Then, from $\mathbb{V} \subseteq \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m-1}$, it follows that actually $\mathbb{V} \subseteq \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}$, and thus we obtain a contradiction. □

Finite Domain. By definition we have $\mathbb{V}_{\mathsf{K}}^{\mathcal{F},0} \supseteq \mathbb{V}_{\mathsf{K}}^{\mathcal{F},1} \supseteq \mathbb{V}_{\mathsf{K}}^{\mathcal{F},2} \supseteq \ldots$ (and so, $\vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},0}} \subseteq \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},1}} \subseteq \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},2}} \subseteq \ldots$). Next, we show that when $\mathcal{F}$ is finite, then this sequence must converge.

Lemma 4. *Suppose that $\mathbb{V}_{\mathsf{K}}^{\mathcal{F},m} = \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m+1}$ for some $m \geq 0$. Then, $\mathbb{V}_{\mathsf{K}}^{\mathcal{F}} = \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}$.*

Lemma 5. *For a* finite *set $\mathcal{F}$ of formulas, $\mathbb{V}_{\mathsf{K}}^{\mathcal{F}} = \mathbb{V}_{\mathsf{K}}^{\mathcal{F},4^{|\mathcal{F}|}}$.*

Proof. The left-to-right inclusion follows from our definitions. For the right-to-left inclusion, note that by Lemma 4, $\mathbb{V}_{\mathsf{K}}^{\mathcal{F},m} = \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m+1}$ implies that $\mathbb{V}_{\mathsf{K}}^{\mathcal{F},m} = \mathbb{V}_{\mathsf{K}}^{\mathcal{F},k}$ for every $k \geq m$. Thus, it suffices to show that $\mathbb{V}_{\mathsf{K}}^{\mathcal{F},m} = \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m+1}$ for some $0 \leq m \leq 4^{|\mathcal{F}|} + 1$. Indeed, otherwise we have $\mathbb{V}_{\mathsf{K}}^{\mathcal{F},0} \supset \mathbb{V}_{\mathsf{K}}^{\mathcal{F},1} \supset \mathbb{V}_{\mathsf{K}}^{\mathcal{F},2} \supset \ldots \supset \mathbb{V}_{\mathsf{K}}^{\mathcal{F},4^{|\mathcal{F}|}+1}$, but this is impossible since there are only $4^{|\mathcal{F}|}$ functions from $\mathcal{F}$ to $\mathcal{V}_4$. □

Optimized Tables. Starting from level 1, the condition on valuations allows us to refine the truth tables of M_{K}, and reduce the search space for countermodels. For instance, since $\psi \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},0}} \varphi \supset \psi$ (for every $\mathcal{F}$ with $\{\psi, \varphi, \varphi \supset \psi\} \subseteq \mathcal{F}$), at level 1 we have that if $\psi \in v^{-1}[\mathsf{TF}]$, then $v(\varphi \supset \psi) \in \mathsf{TF}$. This allows us to remove t and f from the first and third columns (when $y \in \mathsf{TF}$) in the table presenting $\tilde{\supset}$. The following entailments (at level 0), all with a single occurrence of some connective, lead to similar refinements, resulting in the optimized tables below for $\supset$, $\wedge$ and $\vee$:

$$\varphi, \varphi \supset \psi \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},0}} \psi \qquad \varphi, \psi \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},0}} \varphi \wedge \psi \qquad \varphi \wedge \psi \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},0}} \varphi \qquad \varphi \wedge \psi \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},0}} \psi$$

$$\varphi \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},0}} \varphi \vee \psi \qquad \psi \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},0}} \varphi \vee \psi$$

$x \tilde{\supset} y$	T	t	F	f
T	{T}	{t}	{F}	{f}
t	{T}	$\mathcal{D}$	{F}	$\overline{\mathcal{D}}$
F	{T}	{t}	{T}	{t}
f	{T}	$\mathcal{D}$	{T}	$\mathcal{D}$

$x \tilde{\wedge} y$	T	t	F	f
T	{T}	{t}	{F}	{f}
t	{t}	{t}	{f}	{f}
F	{F}	{f}	{F}	{f}
f	{f}	{f}	{f}	{f}

$x \tilde{\vee} y$	T	t	F	f
T	{T}	{T}	{T}	{T}
t	{T}	$\mathcal{D}$	{T}	$\mathcal{D}$
F	{T}	{T}	{F}	{F}
f	{T}	$\mathcal{D}$	{F}	$\overline{\mathcal{D}}$

We note that level 1 valuations are not fully captured by these tables. For example, they must assign T to every formula of the form $\varphi \supset \varphi$, while the table above allows also t when $v(\varphi) \in \mathsf{tf}$. A decision procedure for $\mathtt{K}$ can benefit from relying on these optimized tables instead of the original ones, starting from level 1.

4 Soundness and Completeness

In this section we establish the soundness and completeness of the proposed semantics. For that matter, we first extend the notion of satisfaction to sequents:

Definition 6. An $\mathcal{F}$-valuation v *$\mathcal{D}$-satisfies* an $\mathcal{F}$-sequent $\Gamma \Rightarrow \Delta$, denoted by $v \models_{\mathcal{D}} \Gamma \Rightarrow \Delta$, if $v \not\models_{\mathcal{D}} \varphi$ for some $\varphi \in \Gamma$ or $v \models_{\mathcal{D}} \varphi$ for some $\varphi \in \Delta$.

To prove soundness, we first note that except for (K), the soundness of each derivation rule easily follows from the Nmatrix semantics:

Lemma 6 (Local Soundness). *Consider an application of a rule of G_{K} other than* (K) *deriving a sequent $\Gamma \Rightarrow \Delta$ from sequents $\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_n \Rightarrow \Delta_n$, such that $\Gamma \cup \Gamma_1 \cup \dots \cup \Gamma_n \cup \Delta \cup \Delta_1 \cup \dots \cup \Delta_n \subseteq \mathcal{F}$. Let $v \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}$ for some $m \geq 0$. If $v \models_{\mathcal{D}} \Gamma_i \Rightarrow \Delta_i$ for every $1 \leq i \leq n$, then $v \models_{\mathcal{D}} \Gamma \Rightarrow \Delta$.*

For (K), we make use of the level requirement, and prove the following lemma.

Lemma 7 (Soundness of (K)). *Suppose that $\Gamma \cup \Box\Gamma \cup \{\varphi, \Box\varphi\} \subseteq \mathcal{F}$, and $\Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},m-1}} \varphi$. Then, $\Box\Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}} \Box\varphi$.*

Proof. Let $v \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}$ such that $v \models_{\mathcal{D}} \Box\Gamma$. We prove that $v \models_{\mathcal{D}} \Box\varphi$. By the truth table of $\Box$, we have that $v(\psi) \in \mathsf{TF}$ for every $\psi \in \Gamma$, and we need to show that $v(\varphi) \in \mathsf{TF}$. Since $v(\psi) \in \mathsf{TF}$ for every $\psi \in \Gamma$, we have $\Gamma \subseteq v^{-1}[\mathsf{TF}]$. Since $\Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},m-1}} \varphi$, we have $v^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},m-1}} \varphi$. Since $v \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}$, it follows that $v(\varphi) \in \mathsf{TF}$. □

The above two lemmas together establish soundness, and from soundness for each level, we easily derive soundness for arbitrary (K)-depth.

Theorem 3 (Soundness for m). *If $\vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m} \Gamma \Rightarrow \Delta$, then $\Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}} \Delta$.*

Theorem 4 (Soundness without m). *If $\vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F}} \Gamma \Rightarrow \Delta$, then $\Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F}}} \Delta$.*

By taking $\mathcal{F} = \mathcal{L}$ in Theorem 4 we get that if $\vdash_{\mathsf{G}_{\mathsf{K}}} \Gamma \Rightarrow \Delta$, then $\Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}} \Delta$.

Next, we prove the following two completeness theorems:

Theorem 5 (Completeness for m). *Let $\mathcal{F} \subseteq \mathcal{L}$ closed under subformulas and $\Gamma \Rightarrow \Delta$ an $\mathcal{F}$-sequent. If $\Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}} \Delta$, then $\vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m} \Gamma \Rightarrow \Delta$.*

Theorem 6 (Completeness without m). *Let $\mathcal{F} \subseteq \mathcal{L}$ closed under subformulas and $\Gamma \Rightarrow \Delta$ an $\mathcal{F}$-sequent. If $\Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F}}} \Delta$, then $\vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F}} \Gamma \Rightarrow \Delta$.*

In fact, since $\mathcal{F}$ may be infinite, we need to prove stronger theorems than Theorems 5 and 6, that incorporate infinite sequents.

Definition 7. An ω-sequent is a pair $\langle L, R\rangle$, denoted by $L \Rightarrow R$, such that L and R are (possibly infinite) sets of formulas. We write $\Vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m} L \Rightarrow R$ if $\vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m} \Gamma \Rightarrow \Delta$ for some finite $\Gamma \subseteq L$ and $\Delta \subseteq R$.

Other notions for sequents (e.g., being an $\mathcal{F}$-sequent) are extended to ω-sequents in the obvious way. In particular, $v \models_{\mathcal{D}} L \Rightarrow R$ if $v(\psi) \notin \mathcal{D}$ for some $\psi \in L$ or $v(\psi) \in \mathcal{D}$ for some $\psi \in R$.

Theorem 7 (ω-Completeness for m). *Let $\mathcal{F} \subseteq \mathcal{L}$ closed under subformulas and $L \Rightarrow R$ an ω-$\mathcal{F}$-sequent. If $L \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}} R$, then $\Vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m} L \Rightarrow R$.*

Theorem 8 (ω-Completeness without m). *Let $\mathcal{F} \subseteq \mathcal{L}$ closed under subformulas and $L \Rightarrow R$ an ω-$\mathcal{F}$-sequent. If $L \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F}}} R$, then $\Vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F}} L \Rightarrow R$.*

Theorem 5 is a consequence of Theorem 7. Indeed, by Theorem 7, $\Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}} \Delta$ implies that $\vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m} \Gamma' \Rightarrow \Delta'$ for some (finite) $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$. Using (WEAK), we obtain that $\vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m} \Gamma \Rightarrow \Delta$. Similarly, Theorem 6 is a consequence of Theorem 8. Also, using Lemma 3, we obtain Theorem 8 from Theorem 7. Hence in the remainder of this section we focus on the proof of Theorem 7.

Proof of Theorem 7 We start by defining maximal and consistent ω-sequents, and proving their existence.

Definition 8 (Maximal and consistent ω-sequent). Let $\mathcal{F} \subseteq \mathcal{L}$ and $m \geq 0$. An $\mathcal{F}$-ω-sequent $L \Rightarrow R$ is called:

1. *$\mathcal{F}$-maximal* if $\mathcal{F} \subseteq L \cup R$.
2. *$\langle \mathsf{G}_{\mathsf{K}}, \mathcal{F}, m\rangle$-consistent* if $\nVdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m} L \Rightarrow R$.
3. *$\langle \mathsf{G}_{\mathsf{K}}, \mathcal{F}, m\rangle$-maximal-consistent* (in short, *$\langle \mathsf{G}_{\mathsf{K}}, \mathcal{F}, m\rangle$-max-con*) if it is $\mathcal{F}$-maximal and $\langle \mathsf{G}_{\mathsf{K}}, \mathcal{F}, m\rangle$-consistent.

Lemma 8. *Let $\mathcal{F} \subseteq \mathcal{L}$ and $L \Rightarrow R$ an $\mathcal{F}$-ω-sequent. Suppose that $\nVdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m} L \Rightarrow R$. Then, there exist sets $L_{MC(\mathsf{G}_{\mathsf{K}},\mathcal{F},m,L\Rightarrow R)}$ and $R_{MC(\mathsf{G}_{\mathsf{K}},\mathcal{F},m,L\Rightarrow R)}$ such that the following hold:*

- *$L \subseteq L_{MC(\mathsf{G}_{\mathsf{K}},\mathcal{F},m,L\Rightarrow R)}$ and $R \subseteq R_{MC(\mathsf{G}_{\mathsf{K}},\mathcal{F},m,L\Rightarrow R)}$.*
- *$L_{MC(\mathsf{G}_{\mathsf{K}},\mathcal{F},m,L\Rightarrow R)} \cup R_{MC(\mathsf{G}_{\mathsf{K}},\mathcal{F},m,L\Rightarrow R)} \subseteq \mathcal{F}$.*
- *$L_{MC(\mathsf{G}_{\mathsf{K}},\mathcal{F},m,L\Rightarrow R)} \Rightarrow R_{MC(\mathsf{G}_{\mathsf{K}},\mathcal{F},m,L\Rightarrow R)}$ is $\langle \mathsf{G}_{\mathsf{K}}, \mathcal{F}, m\rangle$-max-con.*

Thus, given an underivable ω-sequent, we can extend it to a $\langle \mathsf{G}_{\mathsf{K}}, \mathcal{F}, m\rangle$-max-con ω-sequent. This ω-sequent induces the canonical countermodel, as defined next.

Algorithm 1 Deciding $\Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}} \varphi$.

$\mathcal{F} \leftarrow sub(\Gamma \cup \{\varphi\})$
$m \leftarrow 4^{|\mathcal{F}|}$
for $v \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}$ **do**
  if $v \models_{\mathcal{D}} \Gamma$ and $v \not\models_{\mathcal{D}} \varphi$ **then**
    return ("NO", v)
return "YES"

Notation 9 We denote the set $\{\psi \in \mathcal{F} \mid \Box\psi \in X\}$ by $\mathbb{B}_{\mathcal{F}}^{X}$.

Definition 10. Suppose that $L \uplus R = \mathcal{F}$. The *canonical model w.r.t.* $L \Rightarrow R$, $\mathcal{F}$*, and* m, denoted by $v(\mathcal{F}, L \Rightarrow R, m)$, is the $\mathcal{F}$-valuation defined as follows (in λ notation):

For $m = 0$:

$$\lambda\varphi \in \mathcal{F}. \begin{cases} \mathsf{T} & \varphi \in L \text{ and } \Box\varphi \in L \\ \mathsf{t} & \varphi \in L \text{ and } \Box\varphi \notin L \\ \mathsf{F} & \varphi \in R \text{ and } \Box\varphi \in L \\ \mathsf{f} & \varphi \in R \text{ and } \Box\varphi \notin L \end{cases}$$

For $m > 0$:

$$\lambda\varphi \in \mathcal{F}. \begin{cases} \mathsf{T} & \varphi \in L \text{ and } \Vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m-1} \mathbb{B}_{\mathcal{F}}^{L} \Rightarrow \varphi \\ \mathsf{t} & \varphi \in L \text{ and } \nVdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m-1} \mathbb{B}_{\mathcal{F}}^{L} \Rightarrow \varphi \\ \mathsf{F} & \varphi \in R \text{ and } \Vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m-1} \mathbb{B}_{\mathcal{F}}^{L} \Rightarrow \varphi \\ \mathsf{f} & \varphi \in R \text{ and } \nVdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m-1} \mathbb{B}_{\mathcal{F}}^{L} \Rightarrow \varphi \end{cases}$$

Clearly, $v(\mathcal{F}, L \Rightarrow R, m) \not\models_{\mathcal{D}} L \Rightarrow R$. The proof of Theorem 7 is done by induction on m, and then carries on by showing that if $L \Rightarrow R$ is $\langle \mathsf{G}_{\mathsf{K}}, \mathcal{F}, m\rangle$-max-con, then $v(\mathcal{F}, L \Rightarrow R, m)$ belongs to $\mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}$ for every m.

Concretely, let $v \overset{\text{def}}{=} v(\mathcal{F}, L \Rightarrow R, m)$. We show that $v \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F},k}$ for every $k \leq m$ by induction on k. The base case $k = 0$ is straightforward. For $k > 0$, we have $v \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F},k-1}$ by the induction hypothesis. Let $\varphi \in \mathcal{F}$, and suppose that $v^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},k-1}} \varphi$. To show that $v(\varphi) \in \mathsf{TF}$, we prove that $\Vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m-1} \mathbb{B}_{\mathcal{F}}^{L} \Rightarrow \varphi$. By the outer induction hypothesis (regarding the completeness theorem itself), $v^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},k-1}} \varphi$ implies that $\Vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},k-1} v^{-1}[\mathsf{TF}] \Rightarrow \varphi$, which implies that $\Vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m-1} v^{-1}[\mathsf{TF}] \Rightarrow \varphi$. Hence, there is a finite set $\{\varphi_1, \dots, \varphi_n\} \subseteq v^{-1}[\mathsf{TF}]$ such that $\vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m-1} \{\varphi_1, \dots, \varphi_n\} \Rightarrow \varphi$. For every $1 \leq i \leq n$, since $\varphi_i \in v^{-1}[\mathsf{TF}]$, we have that $\Vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m-1} \mathbb{B}_{\mathcal{F}}^{L} \Rightarrow \varphi_i$ and hence $\vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m-1} \Gamma_i \Rightarrow \varphi_i$ for some $\Gamma_i \subseteq \mathbb{B}_{\mathcal{F}}^{L}$. Using n applications of (CUT) on these sequents and $\vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m-1} \{\varphi_1, \dots, \varphi_n\} \Rightarrow \varphi$, we obtain that $\vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m-1} \Gamma_1, \dots, \Gamma_n \Rightarrow \varphi$, and so $\Vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m-1} \mathbb{B}_{\mathcal{F}}^{L} \Rightarrow \varphi$.

5 Effectiveness of the Semantics

In this section we study the effectiveness of the semantics introduced in Definition 5 for deciding $\vdash_{\mathsf{M}_{\mathsf{K}}}$. Roughly speaking, a semantic framework is said to be *effective* if it induces a decision procedure that decides its underlying logic.

Consider Algorithm 1. Given a finite set Γ of formulas and a formula φ, it checks whether any valuations in $\mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}$ is a countermodel. The correctness of this algorithm relies on the analyticity of G_{K}, namely:

Lemma 9 ([11]). *If* $\vdash_{\mathsf{G_K}} \Gamma \Rightarrow \Delta$*, then* $\vdash_{\mathsf{G_K}}^{sub(\Gamma\cup\{\varphi\})} \Gamma \Rightarrow \Delta$*.*

Using Lemma 9, we show that the algorithm is correct.

Lemma 10. *Algorithm 1 always terminates, and returns "YES" iff* $\Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}} \varphi$*.*

Proof. Termination follows from the fact that $\mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}$ is finite. Suppose that the result is "YES" and assume for contradiction that $\Gamma \nvdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}} \varphi$. Hence, there exists some $u \in \mathbb{V}_{\mathsf{K}}$ such that $u \models_{\mathcal{D}} \Gamma$ and $u \not\models_{\mathcal{D}} \varphi$. Consider $v \stackrel{\text{def}}{=} u|_{\mathcal{F}}$. Then, $v \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F}} \subseteq \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}$, which contradicts the fact that the algorithm returns "YES". Now, suppose that the result is "NO". Then, there exists some $v \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}$ such that $v \models_{\mathcal{D}} \Gamma$ and $v \not\models_{\mathcal{D}} \varphi$. By Lemma 5, $v \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F}}$. Hence, $\Gamma \nvdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F}}} \varphi$. By Theorem 3, we have $\nvdash_{\mathsf{G_K}}^{\mathcal{F}} \Gamma \Rightarrow \varphi$. By Lemma 9, we have $\nvdash_{\mathsf{G_K}} \Gamma \Rightarrow \varphi$. By Theorem 6, we have $\Gamma \nvdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}} \varphi$. ☐

Lemma 10 shows that Algorithm 1 is a decision procedure for $\vdash_{\mathsf{M_K}}$, when ignoring the additional output provided in Line 5. However, it is typical in applications that a "YES" or "NO" answer is not enough, and often it is expected that a "NO" result is accompanied with a countermodel. Algorithm 1 returns a valuation v in case the answer is "NO", but Lemma 10 does not ensure that v is indeed a countermodel for $\Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}} \varphi$. The issue is that the valuation v from the proof of Lemma 10 witnesses the fact that $\nvdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}}$ only in a non-constructive way. Indeed, using the soundness and completeness theorems, we are able to deduce that $v' \models_{\mathcal{D}} \Gamma$ and $v' \not\models_{\mathcal{D}} \varphi$ for some $v' \in \mathbb{V}_{\mathsf{K}}$, but the relation between v and v' is unclear. Most importantly, it is not clear whether v and v' agree on $\mathcal{F}$-formulas. In the remainder of this section we prove that v' extends v, and so the returned countermodel of Line 5 can be trusted.

We say that a valuation v' *extends* a valuation v if $\mathsf{Dom}(v) \subseteq \mathsf{Dom}(v')$ and $v'(\varphi) = v(\varphi)$ for every $\varphi \in \mathsf{Dom}(v)$ (identifying functions with sets of pairs, this means $v \subseteq v'$). Clearly, for a $\mathsf{Dom}(v)$-formula ψ we have that $v' \models_{\mathcal{D}} \psi$ iff $v \models_{\mathcal{D}} \psi$. We first show how to extend a given valuation $v \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}$ by a single formula ψ such that $sub(\psi) \setminus \{\psi\} \subseteq \mathcal{F}$, obtaining a valuation $v' \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F}\cup\{\psi\},m}$ that agrees with v on all formulas in $\mathcal{F}$.

Lemma 11. *Let* $m \geq 0$*,* $\mathcal{F} \subseteq \mathcal{L}$*, and* $v \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}$*. Let* $\psi \in \mathcal{L} \setminus \mathcal{F}$ *such that* $sub(\psi) \setminus \{\psi\} \subseteq \mathcal{F}$*. Then,* v *can be extended to some* $v' \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F}\cup\{\psi\},m}$*.*

We sketch the proof of Lemma 11.

When $m = 0$, v' exists from Lemma 1. For $m > 0$, we define v' as follows:[3]

$$v' \stackrel{\text{def}}{=} \lambda\varphi \in \mathcal{F}\cup\{\psi\} . \begin{cases} v(\varphi) & \varphi \in \mathcal{F} \\ \min(\mathsf{pos\text{-}val}(\psi, \mathsf{M_K}, v) \cap \mathsf{TF}) & \varphi = \psi \wedge v^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F}\cup\{\psi\},m-1}} \psi \\ \min(\mathsf{pos\text{-}val}(\psi, \mathsf{M_K}, v) \cap \mathsf{tf}) & \text{otherwise} \end{cases}$$

[3] The use of min here assumes an arbitrary order on truth values. It is used here only to choose *some* element from a non-empty set of truth values.

The proof of Lemma 11 then carries on by showing that $v' \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F}\cup\{\psi\},m}$.

Next, Lemma 11 is used in order to extend partial valuations into total ones.

Lemma 12. *Let $v \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}$ for some $\mathcal{F}$ closed under subformulas. Then, v can be extended to some $v' \in \mathbb{V}_{\mathsf{K}}^{m}$.*

Finally, Lemmas 3 and 12 can be used in order to extend any partial valuation in $\mathbb{V}_{\mathsf{K}}^{\mathcal{F}}$ into a total one.

Lemma 13. *Let $v \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F}}$ for some set $\mathcal{F}$ closed under subformulas. Then, v can be extended to some $v' \in \mathbb{V}_{\mathsf{K}}$.*

We conclude by showing that when Algorithm 1 returns ("NO", v), then v is a finite representation of a true countermodel for $\Gamma \vdash_{\mathsf{M}_{\mathsf{K}}} \varphi$.

Corollary 1. *If $\Gamma \nvdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}} \varphi$. Then Algorithm 1 returns ("NO", v) for some v for which there exists $v' \in \mathbb{V}_{\mathsf{K}}$ such that $v = v'|_{sub(\Gamma\cup\{\varphi\})}$, $v' \models_{\mathcal{D}} \Gamma$, and $v' \not\models_{\mathcal{D}} \varphi$.*

Proof. Suppose that $\Gamma \nvdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}} \varphi$. Then by Lemma 10, Algorithm 1 does not return "YES". Therefore, it returns ("NO", v) for some $v \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}$ such that $v \models_{\mathcal{D}} \Gamma$ and $v \not\models_{\mathcal{D}} \varphi$, where $\mathcal{F} = sub(\Gamma \cup \{\varphi\})$ and $m = 4^{|\mathcal{F}|}$. By Lemma 5, $v \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F}}$. By Lemma 13, v can be extended to some $v' \in \mathbb{V}_{\mathsf{K}}$. Therefore, $v = v'|_{sub(\Gamma\cup\{\varphi\})}$, $v' \models_{\mathcal{D}} \Gamma$, and $v' \not\models_{\mathcal{D}} \varphi$. □

Remark 2. Notice that in scenarios where model generation is not important, m can be set to a much smaller number in Line 2 of Algorithm 1, namely, the "modal depth" of the input.[4] The reason for that is that for such m, it can be shown that $\vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m} \Gamma \Rightarrow \varphi$ iff $\vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F}} \Gamma \Rightarrow \varphi$, by reasoning about the applications of rule (K). Using the soundness and completeness theorems, we can get $\Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}} \varphi$ iff $\Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F}}} \varphi$, and so limiting to such m is enough. Notice however, that we do not necessarily get $\mathbb{V}_{\mathsf{K}}^{\mathcal{F},m} = \mathbb{V}_{\mathsf{K}}^{\mathcal{F}}$ for such m, and so the valuation returned in Line 5 might not be an element of $\mathbb{V}_{\mathsf{K}}^{\mathcal{F}}$.

6 The Modal Logic KT

In this section we obtain similar results for the modal logic KT. First, the calculus G_{KT} is obtained from G_{K} by adding the following rule (see, e.g., [16]):

$$(T)\ \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \Box\varphi \Rightarrow \Delta}$$

Derivations are defined as before. (In particular, the (K)-depth of a derivation still depends on applications of rule (K), not of rule (T).) We write $\vdash_{\mathsf{G}_{\mathsf{KT}}}^{\mathcal{F},m} \Gamma \Rightarrow \Delta$

[4] The *modal depth* of an atomic formula p is 0. The modal depth of $\Box\varphi$ is the modal depth of φ plus 1. The modal depth of $\diamond(\varphi_1, \dots, \varphi_n)$ for $\diamond \neq \Box$ is the maximum among the modal depths of $\varphi_1, \dots, \varphi_n$.

if there is a derivation of $\Gamma \Rightarrow \Delta$ in $\mathsf{G}_{\mathtt{KT}}$ in which only $\mathcal{F}$-sequents occur and that has (K)-depth at most m.

Next, we consider the semantics. For a valuation $v \in \mathbb{V}_{\mathtt{K}}$ to respect rule (T), we must have that if $v \models_{\mathcal{D}} \Gamma, \varphi \Rightarrow \Delta$, then $v \models_{\mathcal{D}} \Gamma, \Box\varphi \Rightarrow \Delta$. In particular, when $v \not\models_{\mathcal{D}} \Gamma \Rightarrow \Delta$, we get that if $v(\varphi) \notin \mathcal{D}$, then $v(\Box\varphi) \notin \mathcal{D}$. Now, if $v(\varphi) = \mathsf{F}$, then $v(\Box\varphi) \in \mathcal{D}$ according to the truth table of $\Box$ in $\mathsf{M}_{\mathtt{K}}$. But, we must have $v(\Box\varphi) \notin \mathcal{D}$. This leads us to remove F from $\mathsf{M}_{\mathtt{K}}$.

We thus obtain the following Nmatrix $\mathsf{M}_{\mathtt{KT}}$: The sets of truth values and of designated truth values are given by[5]

$$\mathcal{V}_3 \stackrel{\text{def}}{=} \{\mathsf{T}, \mathsf{t}, \mathsf{f}\} \qquad \mathcal{D} \stackrel{\text{def}}{=} \{\mathsf{T}, \mathsf{t}\}$$

and the truth tables are as follows:

$x \tilde{\supset} y$	T	t	f
T	$\mathcal{D}$	$\mathcal{D}$	$\{\mathsf{f}\}$
t	$\mathcal{D}$	$\mathcal{D}$	$\{\mathsf{f}\}$
f	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$

$x \tilde{\wedge} y$	T	t	f
T	$\mathcal{D}$	$\mathcal{D}$	$\{\mathsf{f}\}$
t	$\mathcal{D}$	$\mathcal{D}$	$\{\mathsf{f}\}$
f	$\{\mathsf{f}\}$	$\{\mathsf{f}\}$	$\{\mathsf{f}\}$

$x \tilde{\vee} y$	T	t	f
T	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
t	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
f	$\mathcal{D}$	$\mathcal{D}$	$\{\mathsf{f}\}$

x	$\tilde{\neg} x$
T	$\{\mathsf{f}\}$
t	$\{\mathsf{f}\}$
f	$\mathcal{D}$

x	$\tilde{\Box} x$
T	$\mathcal{D}$
t	$\{\mathsf{f}\}$
f	$\{\mathsf{f}\}$

Again, one may gain intuition from the possible worlds semantics. There, the logic $\mathtt{KT}$ is characterized by frames with *reflexive* accessibility relation. Thus, for instance, if ψ holds in w but not in some world accessible from w (i.e., ψ has value t), we know that $\Box\psi$ does not hold in w, and the reflexivity of the accessibility relation implies that $\Box\psi$ does not hold in some world accessible from w (thus $\Box\psi$ has value f).

Example 4. Let $\varphi \stackrel{\text{def}}{=} \Box\Box(p_1 \wedge p_2) \supset \Box p_1$ and $\mathcal{F} \stackrel{\text{def}}{=} sub(\varphi)$. The sequent $\Rightarrow \varphi$ has a derivation in $\mathsf{G}_{\mathtt{KT}}$ using only $\mathcal{F}$ formulas of (K)-depth of 1. However, it is not satisfied by all $\mathsf{M}_{\mathtt{KT}}$-legal $\mathcal{F}$-valuations. For example, the following valuation is an $\mathsf{M}_{\mathtt{KT}}$-legal valuation that does not satisfy φ:

$$v(p_1) = v(p_2) = \mathsf{t} \quad v(\Box p_1) = \mathsf{f}$$

$$v(p_1 \wedge p_2) = v(\Box(p_1 \wedge p_2)) = v(\Box\Box(p_1 \wedge p_2)) = \mathsf{T} \quad v(\varphi) = \mathsf{f}$$

Next, we define the levels of valuations for $\mathsf{M}_{\mathtt{KT}}$. These are obtained from Definition 5 by removing the value F:

Definition 11. The set $\mathbb{V}_{\mathtt{KT}}^{\mathcal{F},m}$ is recursively defined as follows:

- $\mathbb{V}_{\mathtt{KT}}^{\mathcal{F},0}$ is the set of $\mathsf{M}_{\mathtt{KT}}$-legal $\mathcal{F}$-valuations.
- $\mathbb{V}_{\mathtt{KT}}^{\mathcal{F},m+1} \stackrel{\text{def}}{=} \left\{ v \in \mathbb{V}_{\mathtt{KT}}^{\mathcal{F},m} \mid \forall \varphi \in \mathcal{F}.\, v^{-1}[\mathsf{T}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{KT}}^{\mathcal{F},m}} \varphi \implies v(\varphi) = \mathsf{T} \right\}$

We also define:

$$\mathbb{V}_{\mathtt{KT}}^{\mathcal{F}} \stackrel{\text{def}}{=} \bigcap_{m \geq 0} \mathbb{V}_{\mathtt{KT}}^{\mathcal{F},m} \qquad \mathbb{V}_{\mathtt{KT}}^{m} \stackrel{\text{def}}{=} \mathbb{V}_{\mathtt{KT}}^{\mathcal{L},m} \qquad \mathbb{V}_{\mathtt{KT}} \stackrel{\text{def}}{=} \bigcap_{m \geq 0} \mathbb{V}_{\mathtt{KT}}^{\mathcal{L},m}$$

[5] In this section we denote the set $\{\mathsf{T}\}$ by TF.

Example 5. Following Example 4, we note that for every $v \in \mathbb{V}_{\mathtt{KT}}^{\mathcal{F},m}$ with $m > 0$, we have $v \models_{\mathcal{D}} \varphi$. In particular, the valuation v from Example 4 does not belong to $\mathbb{V}_{\mathtt{KT}}^{\mathcal{F},m}$: $\Box(p_1 \wedge p_2) \in v^{-1}[\mathsf{T}]$, $\Box(p_1 \wedge p_2) \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{KT}}^{\mathcal{F},0}} p_1$, but $v(p_1) = \mathsf{t}$.

Similarly to Theorem 2, the levels of valuations converge to a maximal set that satisfies the following condition:

$$\forall v \in \mathbb{V}. \forall \varphi \in \mathcal{F}. v^{-1}[\mathsf{T}] \vdash_{\mathcal{D}}^{\mathbb{V}} \varphi \implies v(\varphi) = \mathsf{T} \qquad (\textit{necessitation}_{\mathtt{KT}})$$

Theorem 9. *The set $\mathbb{V}_{\mathtt{KT}}^{\mathcal{F}}$ is the largest set $\mathbb{V}$ of $\mathsf{M}_{\mathtt{KT}}$-legal $\mathcal{F}$-valuations that satisfies necessitation$_{\mathtt{KT}}$.*

The proof of Theorem 9 is analogous to that of Theorem 2.

Remark 3. The *necessitation*$_{\mathtt{KT}}$ condition is equivalent to the one given in [8], except that the underlying truth table is different. Theorem 9 proves that our gradual way of defining $\mathbb{V}_{\mathtt{KT}}^{\mathcal{F}}$ via levels coincides with the semantic condition from [8].

As we demonstrated for K, starting from level 1, the condition on valuations allows us to refine the truth tables of $\mathsf{M}_{\mathtt{KT}}$, and reduce the search space. Simple entailments (at level 0) lead to the optimized tables below for $\supset$, $\wedge$ and $\vee$:

$x \tilde{\supset} y$	T	t	f
T	{T}	{t}	{f}
t	{T}	$\mathcal{D}$	{f}
f	{T}	$\mathcal{D}$	$\mathcal{D}$

$x \tilde{\wedge} y$	T	t	f
T	{T}	{t}	{f}
t	{t}	{t}	{f}
f	{f}	{f}	{f}

$x \tilde{\vee} y$	T	t	f
T	{T}	{T}	{T}
t	{T}	$\mathcal{D}$	$\mathcal{D}$
f	{T}	$\mathcal{D}$	{f}

Soundness and completeness for $\mathsf{G}_{\mathtt{KT}}$ are obtained analogously to $\mathsf{G}_{\mathtt{K}}$, keeping in mind that $\mathsf{M}_{\mathtt{KT}}$ is obtained from $\mathsf{M}_{\mathtt{K}}$ by deleting the value F. For soundness, this is captured by the rule (T). For completeness, the same construction of a countermodel is performed , while rule (T) ensures that it is three-valued.

Theorem 10 (Soundness and Completeness). *Let $\mathcal{F} \subseteq \mathcal{L}$ closed under subformulas and $\Gamma \Rightarrow \Delta$ an $\mathcal{F}$-sequent.*

1. *For every $m \geq 0$, $\Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{KT}}^{\mathcal{F},m}} \Delta$ iff $\vdash_{\mathsf{G}_{\mathtt{KT}}}^{\mathcal{F},m} \Gamma \Rightarrow \Delta$.*
2. *$\Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{KT}}^{\mathcal{F}}} \Delta$ iff $\vdash_{\mathsf{G}_{\mathtt{KT}}}^{\mathcal{F}} \Gamma \Rightarrow \Delta$.*

Effectiveness is also shown similarly to K. For that matter, we use the following main lemma, whose proof is similar to Lemma 13. The only component that is added to that proof is making sure that the constructed model is three-valued.

Lemma 14. *Let $v \in \mathbb{V}_{\mathtt{KT}}^{\mathcal{F}}$ for some set $\mathcal{F}$ closed under subformulas. Then, v can be extended to some $v' \in \mathbb{V}_{\mathtt{KT}}$.*

Let Algorithm 2 be obtained from Algorithm 1 by setting m to $3^{|\mathcal{F}|}$ in Line 2, and taking $v \in \mathbb{V}_{\mathtt{KT}}^{\mathcal{F},m}$ in Line 3. Similarly to Lemma 10 and Corollary 1, we get that Algorithm 2 is a model-producing decision procedure for $\vdash_{\mathsf{M}_{\mathtt{KT}}}$.

Lemma 15. *Algorithm 2 always terminates, and returns "YES" iff $\Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{KT}}} \varphi$. Further, if $\Gamma \nvdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{KT}}} \varphi$, then it returns ("NO", v) for some v for which there exists $v' \in \mathbb{V}_{\mathtt{KT}}$ such that $v = v'|_{sub(\Gamma \cup \{\varphi\})}$, $v' \models_{\mathcal{D}} \Gamma$, and $v' \not\models_{\mathcal{D}} \varphi$.*

7 Future Work

We have introduced a new semantics for the modal logic K, based on levels of valuations in many-valued non-deterministic matrices. Our semantics is effective, and was shown to tightly correspond to derivations in a sequent calculus for K. We also adapted these results for the modal logic KT.

There are two main directions for future work. The first is to establish similar semantics for other normal modal logics, such as KD, K4, S4 and S5, and to investigate $\Diamond$ as an independent modality. The second is to analyze the complexity, implement and experiment with decision procedures for K and KT based on the proposed semantics. In particular, we plan to consider SAT-based decision procedures that would encode this semantics in SAT, directly or iteratively.

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A Full Proofs

A.1 Proof of Lemma 2

First, if $\mathcal{F}$ is finite, then there is only a finite number of $\mathcal{F}$-valuations, and there must exists some $\mathcal{F}$-valuation v_m that occurs infinitely often in the sequence $v_0, v_1, \ldots$. We take $v = v_m$, and the required property trivially holds.

Now, assume that $\mathcal{F}$ is infinite, and let $\varphi_0, \varphi_1, \ldots$ be an enumeration of the formulas in $\mathcal{F}$. For every $i \geq 0$, let $\mathcal{F}_i = \{\varphi_0, \ldots, \varphi_i\}$. We construct a sequence of infinite sets $A_0, A_1, \ldots \subseteq \mathbb{N}$ such that:

- For every $i \geq 0$, $A_{i+1} \subseteq A_i$.
- For every $0 \leq j \leq i$, $a \in A_j$, and $b \in A_i$, $v_a(\varphi_j) = v_b(\varphi_j)$.

To do so, take some infinite set $A_0 \subseteq \mathbb{N}$ such that $v_a(\varphi_0) = v_b(\varphi_0)$ for every $a, b \in A_0$ (such set must exist since we have a finite number of truth values). Then, given A_i, we let A_{i+1} be some infinite subset of A_i such that $v_a(\varphi_{i+1}) = v_b(\varphi_{i+1})$ for every $a, b \in A_{i+1}$. By construction, $A_{i+1} \subseteq A_i$ for $i \geq 0$. In addition, given $0 \leq j \leq i$, $a \in A_j$, and $b \in A_i$, we have $b \in A_j$ as well and so $v_a(\varphi_j) = v_b(\varphi_j)$.

The valuation v is defined by $v(\varphi_i) = v_a(\varphi_i)$ for some $a \in A_i$. The properties of the A_i's ensure that v is well defined. We show that v satisfies the required property. Let $\mathcal{F}'$ be a finite set of formulas and let $m \geq 0$. Let $i \geq 0$ such that $\mathcal{F}' \subseteq \mathcal{F}_i$. Since A_i is infinite, there exists some $k \in A_i$ such that $k \geq m$. Now, for every $\varphi \in \mathcal{F}'$, there exists some $j \leq i$ such that $\varphi = \varphi_j$, and $a \in A_j$ such that $v(\varphi) = v(\varphi_j) = v_a(\varphi_j) = v_k(\varphi_j)$ (by the second property of the sets $A_0, A_1, \ldots$). Hence, for every $\varphi \in \mathcal{F}'$, we have $v(\varphi) = v_k(\varphi)$.

A.2 Proof of Lemma 3

By Lemma 2, there exists some v such that for every finite set $\mathcal{F}'$ of formulas, $v|_{\mathcal{F}'} = v_m|_{\mathcal{F}'}$ for some $m \geq 0$. We show that v satisfies the required properties.

First, for every $\varphi \in \mathcal{F}$, we have $v|_{\{\varphi\}} = v_m|_{\{\varphi\}}$ (i.e., $v(\varphi) = v_m(\varphi)$) for some $m \geq 0$. Next, we prove that $v \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F}}$. For that matter, we show that $v \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F},m}$ for every $m \geq 0$ by induction on m.

For $m = 0$, let $\psi_1, \ldots, \psi_n, \diamond(\psi_1, \ldots, \psi_n) \in \mathcal{F}$. Let $\mathcal{F}' = \{\psi_1, \ldots, \psi_n, \diamond(\psi_1, \ldots, \psi_n)\}$, and let $k \geq 0$ such that $v|_{\mathcal{F}'} = v_k|_{\mathcal{F}'}$. Since $v_k \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F},0}$, it follows that $v \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F},0}$ as well.

Now, let $m > 0$. By the induction hypothesis, we have $v \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F},m-1}$. Let $\varphi \in \mathcal{F}$ such that $v^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F},m-1}} \varphi$. By Theorem 1, there exists some finite $\Gamma \subseteq v^{-1}[\mathsf{TF}]$ such that $\Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F},m-1}} \varphi$. Let $\mathcal{F}' = \Gamma \cup \{\varphi\}$, and let $k \geq m$ such that $v|_{\mathcal{F}'} = v_k|_{\mathcal{F}'}$. Hence, $\Gamma \subseteq v_k^{-1}[\mathsf{TF}]$, which means that $v_k^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F},m-1}} \varphi$. Since $k \geq m$, we have $v_k \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F},m}$, and therefore $v(\varphi) = v_k(\varphi) \in \mathsf{TF}$.

A.3 Proof of Theorem 2

First, we prove that $\mathbb{V}_{\mathsf{K}}^{\mathcal{F}}$ is such a set, and then we prove maximality. Let $v \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F}}$ and $\varphi \in \mathcal{F}$. Suppose that $v^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F}}} \varphi$. Since $v \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F}}$, we have $v \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m+1}$ for every $m \geq 0$. Thus, to prove that $v(\varphi) \in \mathsf{TF}$, it suffices therefore to prove that $v^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}} \varphi$ for some $m \geq 0$. Assume otherwise. Then there exists a sequence $v_0, v_1, ...$ of $\mathcal{F}$-valuations such that for every $m \geq 0$, $v_m \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}$, $v_m \models_{\mathcal{D}} v^{-1}[\mathsf{TF}]$, and $v_m \not\models_{\mathcal{D}} \varphi$. By Lemma 3, there exists some $v \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F}}$ such that for every $\varphi \in \mathcal{F}$, $v(\varphi) = v_m(\varphi)$ for some $m \geq 0$. We prove that $v \models_{\mathcal{D}} v^{-1}[\mathsf{TF}]$, and $v \not\models_{\mathcal{D}} \varphi$, which leads to a contradiction to the fact that $v^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F}}} \varphi$. Indeed, for every $\psi \in v^{-1}[\mathsf{TF}]$, $v(\psi) = v_m(\psi)$ for some m. Since $v_m \models_{\mathcal{D}} v^{-1}[\mathsf{TF}]$, we have $v \models \psi$. Also, $v(\varphi) = v_m(\varphi)$ for some m, and since $v_m \not\models_{\mathcal{D}} \varphi$, we also have $v \not\models_{\mathcal{D}} \varphi$. Thus, $v \models_{\mathcal{D}} v^{-1}[\mathsf{TF}]$, and $v \not\models_{\mathcal{D}} \varphi$.

Next, we prove maximality. Let $\mathbb{V}$ be a set of M_{K}-legal $\mathcal{F}$-valuations such that for every $v \in \mathbb{V}$ and $\varphi \in \mathcal{F}$, we have $v^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}} \varphi$ implies $v(\varphi) \in \mathsf{TF}$. We prove that $\mathbb{V} \subseteq \mathbb{V}_{\mathsf{K}}^{\mathcal{F}}$. Let $m \geq 0$. We prove that $\mathbb{V} \subseteq \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}$. Assume otherwise, and let m be minimal such that $\mathbb{V} \not\subseteq \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}$. $m > 0$, since $\mathbb{V}$ is a set of M_{K}-legal $\mathcal{F}$-valuations and so $\mathbb{V} \subseteq \mathbb{V}_{\mathsf{K}}^{\mathcal{F},0}$. Hence $m - 1 \geq 0$, and by the minimality of m, $\mathbb{V} \subseteq \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m-1}$. We prove that in fact $\mathbb{V} \subseteq \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}$, obtaining a contradiction. Let $v \in \mathbb{V}$ and $\varphi \in \mathcal{F}$, and suppose $v^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},m-1}} \varphi$. We prove that $v(\varphi) \in \mathsf{TF}$. Since $\mathbb{V} \subseteq \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m-1}$, we also have $v^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}} \varphi$. Since $v \in \mathbb{V}$, $v(\varphi) \in \mathsf{TF}$.

A.4 Proof of Lemma 4

The left-to-right direction is trivial. Let $v \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}$ and $n \geq 0$. We prove $v \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F},n}$. If $n \leq m$, then this holds by definition. Otherwise, we prove by induction on $n-m$. If $n = m+1$, then this holds since $\mathbb{V}_{\mathsf{K}}^{\mathcal{F},m} = \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m+1}$. Now let $v \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}$. We prove $v \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F},n+1}$. Suppose that $v^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},n}} \varphi$. By the induction hypothesis, $v^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}} \varphi$. Hence $v(\varphi) \in \mathsf{TF}$. Therefore, $v \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F},n+1}$.

A.5 Proof of Theorem 3

We prove the claim by induction on m. Suppose that the claim holds for every $k < m$. We prove it for m by an inner induction on the length of the derivation of $\Gamma \Rightarrow \Delta$. In the base case, $\Gamma \Rightarrow \Delta$ was proven using (ID) and then $\Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}} \Delta$ trivially holds. For all rules except (K), the (inner) induction step follows from Lemma 6. For (K), suppose that $\Gamma \Rightarrow \Delta$ was derived using (K) (in particular, $m > 0$). Then, $\Gamma \Rightarrow \Delta$ has the form $\Box\Gamma' \Rightarrow \Box\varphi$, and we have $\vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m-1} \Gamma' \Rightarrow \varphi$, and $\Gamma' \cup \Box\Gamma' \cup \{\varphi, \Box\varphi\} \subseteq \mathcal{F}$. By the (outer) induction hypothesis, $\Gamma' \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},m-1}} \varphi$. Then, $\Box\Gamma' \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},m-1}} \Box\varphi$ follows by Lemma 7.

A.6 Proof of Theorem 4

Clearly, $\vdash^{\mathcal{F}}_{\mathsf{G_K}} \Gamma \Rightarrow \Delta$ implies that $\vdash^{\mathcal{F},m}_{\mathsf{G_K}} \Gamma \Rightarrow \Delta$ for some $m \geq 0$. By Theorem 3, we have $\Gamma \vdash^{\mathbb{V}^{\mathcal{F},m}_{\mathsf{K}}}_{\mathcal{D}} \Delta$. The claim follows from the fact that $\mathbb{V}^{\mathcal{F}}_{\mathsf{K}} \subseteq \mathbb{V}^{\mathcal{F},m}_{\mathsf{K}}$.

A.7 Proof of Lemma 8

If $\mathcal{F}$ is finite, let $\varphi_1, \varphi_2, \ldots, \varphi_n$ be its formulas and $\varphi_1, \varphi_2, \ldots$ be an infinite sequence of formulas such that $\varphi_m = \varphi_n$ for every $m \geq n$. If $\mathcal{F}$ is infinite, then let $\varphi_1, \varphi_2, \ldots$ be an enumeration of its formulas. Define two sequences of sets: $L_0, L_1, \ldots$ and $R_0, R_1, \ldots$ such that $L_0 \overset{\mathsf{def}}{=} L$, $R_0 \overset{\mathsf{def}}{=} R$, and for every $i > 0$: if $\nVdash^{\mathcal{F},m}_{\mathsf{G_K}} L_{i-1}, \varphi_i \Rightarrow R_{i-1}$, then $L_i \overset{\mathsf{def}}{=} L_{i-1} \cup \{\varphi_i\}$ and $R_i \overset{\mathsf{def}}{=} R_{i-1}$; otherwise, $L_i \overset{\mathsf{def}}{=} L_{i-1}$; $R_i \overset{\mathsf{def}}{=} R_{i-1} \cup \{\varphi_i\}$. We prove by induction on i that $\nVdash^{\mathcal{F},m}_{\mathsf{G_K}} L_i \Rightarrow R_i$. For $i = 0$ this follows from our assumption. Suppose the claim holds for $i-1$. if $\nVdash^{\mathcal{F},m}_{\mathsf{G_K}} L_{i-1}, \varphi_i \Rightarrow R_{i-1}$, then $\nVdash^{\mathcal{F},m}_{\mathsf{G_K}} L_i \Rightarrow R_i$ by construction. Otherwise, $\Vdash^{\mathcal{F},m}_{\mathsf{G_K}} L_{i-1}, \varphi_i \Rightarrow R_{i-1}$. Suppose for contradiction that $\Vdash^{\mathcal{F},m}_{\mathsf{G_K}} L_i \Rightarrow R_i$. In this case, $L_i \overset{\mathsf{def}}{=} L_{i-1}$; $R_i \overset{\mathsf{def}}{=} R_{i-1} \cup \{\varphi_i\}$. Hence, $\Vdash^{\mathcal{F},m}_{\mathsf{G_K}} L_{i-1} \Rightarrow R_{i-1}, \varphi_i$. Using (CUT), the (K)-depth of the derivations does not change, and so we get $\Vdash^{\mathcal{F},m}_{\mathsf{G_K}} L_{i-1} \Rightarrow R_{i-1}$, contradicting the induction hypothesis.

Next, we set $L^* \overset{\mathsf{def}}{=} \bigcup_{i \in \mathbb{N}} L_i$ and $R^* \overset{\mathsf{def}}{=} \bigcup_{i \in \mathbb{N}} R_i$.

- By construction, $L \subseteq L^*$ and $R \subseteq R^*$;
- All formulas were added from $\mathcal{F}$, and so $L^* \cup R^* \subseteq \mathcal{F}$;
- $\mathcal{F}$-maximal: by construction, each formula in the enumeration of $\mathcal{F}$ was added to either L^* or R^*. $\langle \mathsf{G_K}, \mathcal{F}, m \rangle$-consistent: suppose for contradiction that $\Vdash^{\mathcal{F},m}_{\mathsf{G_K}} L^* \Rightarrow R^*$. Then there exist some finite $\Gamma \subseteq L^*$ and $\Delta \subseteq R^*$ such that $\vdash^{\mathcal{F},m}_{\mathsf{G_K}} \Gamma \Rightarrow \Delta$. Let i such that $\Gamma \cup \Delta \subseteq \{\varphi_1, \ldots, \varphi_i\}$. Then $\Vdash^{\mathcal{F},m}_{\mathsf{G_K}} L_i \Rightarrow R_i$, which contradicts the above claim.

We set $L_{MC(\mathsf{G_K}, \mathcal{F}, m, L \Rightarrow R)}$ to be L^* and $R_{MC(\mathsf{G_K}, \mathcal{F}, m, L \Rightarrow R)}$ to be R^*.

A.8 Proof of Theorem 7

Lemma 16. *Let $\mathcal{F} \subseteq \mathcal{L}$, $m \geq 0$, and $L \Rightarrow R$ be a $\langle \mathsf{G_K}, \mathcal{F}, m \rangle$-max-con ω-sequent. $v(\mathcal{F}, L \Rightarrow R, m) \not\models_{\mathcal{D}} L \Rightarrow R$.*

Proof. Let $v = v(\mathcal{F}, L \Rightarrow R, m)$. For every $\varphi \in L$, $v(\varphi) \in \mathcal{D}$ and for every $\varphi \in R$, $v(\varphi) \notin \mathcal{D}$. □

Lemma 17. *Let $\mathcal{F} \subseteq \mathcal{L}$ and $L \Rightarrow R$ a $\langle \mathsf{G_K}, \mathcal{F}, 0 \rangle$-max-con ω-sequent. Then, $v(\mathcal{F}, L \Rightarrow R, 0) \in \mathbb{V}^{\mathcal{F},0}_{\mathsf{K}}$.*

Proof. This can be shown by a routine check that the truth tables of $\mathsf{M_K}$ are satisfied. We show the cases of $\neg$ and $\Box$. Let $v \overset{\mathsf{def}}{=} v(\mathcal{F}, L \Rightarrow R, m)$. Suppose $\varphi, \neg\varphi \in \mathcal{F}$. If $v(\varphi) \in \mathcal{D}$, then $\varphi \in L$, and hence $\neg\varphi \notin L$, as otherwise we would get a contradiction to $\langle \mathsf{G}_{\mathsf{K}}, \mathcal{F}, m\rangle$-consistency of $L \Rightarrow R$. By $\mathcal{F}$-maximality of $L \Rightarrow R$, it follows that $\neg\varphi \in R$ and hence $v(\neg\varphi) \notin \mathcal{D}$. The case where $v(\varphi) \notin \mathcal{D}$ is symmetric.

For $\Box$, suppose that $v(\varphi) \in \mathsf{TF}$. Then, $\Box\varphi \in L$, and hence $v(\Box\varphi) \in \mathcal{D}$. Now, suppose that $\varphi \notin \mathsf{TF}$. Then, $\Box\varphi \notin L$. By $\mathcal{F}$-maximality, it follows that $\Box\varphi \in R$, and hence $v(\Box\varphi) \notin \mathcal{D}$.

$\Box$

Lemma 18. *Let $\mathcal{F} \subseteq \mathcal{L}$, $m \geq 0$, and $\Gamma \Rightarrow \Delta$ be a $\langle \mathsf{G}_{\mathsf{K}}, \mathcal{F}, m\rangle$-max-con sequent. Then, $v(\mathcal{F}, \Gamma \Rightarrow \Delta, m) \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F},0}$.*

Proof. For $m = 0$, this is Lemma 17. Suppose $m > 0$. Let $v \overset{\mathsf{def}}{=} v(\mathcal{F}, L \Rightarrow R, m)$. Clearly, v is an $\mathcal{F}$-valuation. We show that it respects the truth tables of M_{K}.

The classical connectives are handled in a standard way. We do here only the case of $\neg$. Let $\varphi \in \mathcal{L}$ such that $\varphi, \neg\varphi \in \mathcal{F}$. If $v(\varphi) \in \mathcal{D}$, then $\varphi \in L$. Then, $\neg\varphi \notin L$ as otherwise we would have $\varphi, \neg\varphi \in L$, contradicting the $\langle \mathsf{G}_{\mathsf{K}}, \mathcal{F}, m\rangle$-consistency of $L \Rightarrow R$. Since $L \Rightarrow R$ is $\mathcal{F}$-maximal, we have $\neg\varphi \in R$, which means that $v(\neg\varphi) \notin \mathcal{D}$. The case where $v(\varphi) \in \mathcal{D}$ is symmetrical.

For the truth table of $\Box$, let $\varphi \in \mathcal{L}$ such that $\varphi, \Box\varphi \in \mathcal{F}$. We show that $v(\varphi) \in \mathsf{TF}$ iff $v(\Box\varphi) \in \mathcal{D}$:

- If $v(\varphi) \in \mathsf{TF}$, then $\Vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m-1} \mathbb{B}_{\mathcal{F}}^{L} \Rightarrow \varphi$. Notice that $\Box\varphi \in \mathcal{F}$ and for every $\psi \in \mathbb{B}_{\mathcal{F}}^{L}$, $\Box\psi \in \mathcal{F}$. Thus, we can use rule (K) and obtain $\Vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m} \Box\mathbb{B}_{\mathcal{F}}^{L} \Rightarrow \Box\varphi$. Clearly, $\Box\mathbb{B}_{\mathcal{F}}^{L} \subseteq L$. Thus, $\Box\varphi \notin R$, as otherwise we would get a contradiction to the fact that $L \Rightarrow R$ is $\langle \mathsf{G}_{\mathsf{K}}, \mathcal{F}, m\rangle$-consistent. Since $L \Rightarrow R$ is $\mathcal{F}$-maximal, we get that $\Box\varphi \in L$, which means that $v(\Box\varphi) \in \mathcal{D}$.
- If $v(\varphi) \notin \mathsf{TF}$, then $\nVdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m-1} \mathbb{B}_{\mathcal{F}}^{L} \Rightarrow \varphi$. In particular, using (ID), $\varphi \notin \mathbb{B}_{\mathcal{F}}^{L}$. Since $\varphi \in \mathcal{F}$, we have $\Box\varphi \notin L$. Since $L \Rightarrow R$ is $\mathcal{F}$-maximal, we obtain that $\Box\varphi \in R$. Hence, we have $v(\Box\varphi) \notin \mathcal{D}$.

$\Box$

Lemma 19. *Let $\mathcal{F} \subseteq \mathcal{L}$ and $m \geq 0$. Suppose that for every $k < m$ and $\mathcal{F}$-ω-sequent $L' \Rightarrow R'$, we have that $L' \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},k}} R'$ implies that $\Vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},k} L' \Rightarrow R'$. Let $L \Rightarrow R$ be a $\langle \mathsf{G}_{\mathsf{K}}, \mathcal{F}, m\rangle$-max-con ω-sequent. Then, $v(\mathcal{F}, L \Rightarrow R, m) \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F},m}$.*

Proof. Let $v \overset{\mathsf{def}}{=} v(\mathcal{F}, L \Rightarrow R, m)$. We show that $v \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F},k}$ for every $k \leq m$ by induction on k. For $k = 0$, this holds by Lemma 18. Let $k > 0$. We have $v \in \mathbb{V}_{\mathsf{K}}^{\mathcal{F},k-1}$ by the induction hypothesis. Let $\varphi \in \mathcal{F}$, and suppose that $v^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},k-1}} \varphi$. To show that $v(\varphi) \in \mathsf{TF}$, we prove that $\Vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m-1} \mathbb{B}_{\mathcal{F}}^{L} \Rightarrow \varphi$. By our assumption, $v^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{K}}^{\mathcal{F},k-1}} \varphi$ implies that $\Vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},k-1} v^{-1}[\mathsf{TF}] \Rightarrow \varphi$, which implies that $\Vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m-1} v^{-1}[\mathsf{TF}] \Rightarrow \varphi$ (since $k-1 \leq m-1$). Hence, there is a finite set $\{\varphi_1, \dots, \varphi_n\} \subseteq v^{-1}[\mathsf{TF}]$ such that $\vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m-1} \{\varphi_1, \dots, \varphi_n\} \Rightarrow \varphi$. For every $1 \leq i \leq n$, since $\varphi_i \in v^{-1}[\mathsf{TF}]$, we have that $\Vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m-1} \mathbb{B}_{\mathcal{F}}^{L} \Rightarrow \varphi_i$ and hence $\vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m-1} \Gamma_i \Rightarrow \varphi_i$ for some $\Gamma_i \subseteq \mathbb{B}_{\mathcal{F}}^{L}$. Using (CUT) on $\vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m-1} \Gamma_1 \Rightarrow \varphi_1$ and $\vdash_{\mathsf{G}_{\mathsf{K}}}^{\mathcal{F},m-1} \{\varphi_1, \dots, \varphi_n\} \Rightarrow \varphi$, we obtain that $\vdash^{\mathcal{F},m-1}_{\mathsf{G_K}} \Gamma_1, \{\varphi_2, \ldots, \varphi_n\} \Rightarrow \varphi$. Continuing this way with $n-1$ more cuts, we obtain that $\vdash^{\mathcal{F},m-1}_{\mathsf{G_K}} \Gamma_1, \ldots, \Gamma_n \Rightarrow \varphi$, and so $\Vdash^{\mathcal{F},m-1}_{\mathsf{G_K}} \mathbb{B}^{L}_{\mathcal{F}} \Rightarrow \varphi$. □

Proof (of Theorem 7). We prove that claim by induction on m. Suppose that $\nVdash^{\mathcal{F},m}_{\mathsf{G_K}} L \Rightarrow R$. Let $L^* = L_{MC(\mathsf{G_K},\mathcal{F},m,L\Rightarrow R)}$ and $R^* = \Delta_{MC(\mathsf{G_K},\mathcal{F},m,L\Rightarrow R)}$ from Lemma 16, and $v^* = v(\mathcal{F}, L^* \Rightarrow R^*, m)$ from Definition 10. Then, $L^* \Rightarrow R^*$ is $\langle \mathsf{G_K}, \mathcal{F}, m\rangle$-max-con, $L \subseteq L^* \subseteq \mathcal{F}$, and $R \subseteq R^* \subseteq \mathcal{F}$. By Lemma 16, $v^* \not\models_{\mathcal{D}} L^* \Rightarrow R^*$, and therefore $v^* \not\models_{\mathcal{D}} L \Rightarrow R$. Finally, by the induction hypothesis, we have that for every $k < m$ and $\mathcal{F}$-ω-sequent $L' \Rightarrow R'$, $L' \vdash^{\mathbb{V}^{\mathcal{F},k}_{\mathsf{K}}}_{\mathcal{D}} R'$ implies $\Vdash^{\mathcal{F},k}_{\mathsf{G_K}} L' \Rightarrow R'$, and so, $v^* \in \mathbb{V}^{\mathcal{F},m}_{\mathsf{K}}$ follows from Lemma 19. □

A.9 Proof of Theorem 8

Suppose that $\nVdash^{\mathcal{F}}_{\mathsf{G_K}} L \Rightarrow R$. Then, $\nVdash^{\mathcal{F},m}_{\mathsf{G_K}} L \Rightarrow R$ for every $m \geq 0$. By Theorem 7, we have $L \nvdash^{\mathbb{V}^{\mathcal{F},m}_{\mathsf{K}}}_{\mathcal{D}} R$ for every $m \geq 0$. For every $m \geq 0$, let $v_m \in \mathbb{V}^{\mathcal{F},m}_{\mathsf{K}}$ such that $v_m \not\models_{\mathcal{D}} L \Rightarrow R$. Then, $v_0, v_1, \ldots$ is a sequence of valuations all with the same domain $\mathcal{F}$ and $v_m \in \mathbb{V}^{\mathcal{F},m}_{\mathsf{K}}$. By Lemma 3, there exists $v \in \mathbb{V}^{\mathcal{F}}_{\mathsf{K}}$ such that for every $\varphi \in \mathcal{F}$, $v(\varphi) = v_m(\varphi)$ for some $m \geq 0$. We show that $v \not\models_{\mathcal{D}} L \Rightarrow R$. Let $\psi \in L$. Then, $v(\psi) = v_m(\psi)$ for some $m \geq 0$. Since $v_m \not\models_{\mathcal{D}} L \Rightarrow R$, $v(\psi) = v_m(\psi) \in \mathcal{D}$. Now let $\psi \in R$. Then, $v(\psi) = v_m(\psi)$ for some $m \geq 0$. Since $v_m \not\models_{\mathcal{D}} L \Rightarrow R$, $v(\psi) = v_m(\psi) \notin \mathcal{D}$. Hence $L \nvdash^{\mathbb{V}^{\mathcal{F}}_{\mathsf{K}}}_{\mathcal{D}} R$.

A.10 Proof of Theorem 1

We show the first item. The second item is analogous. Let $m \geq 0$ and suppose that $L \vdash^{\mathbb{V}^{\mathcal{F},m}_{\mathsf{K}}}_{\mathcal{D}} R$. By Theorem 7, we have $\Vdash^{\mathcal{F},m}_{\mathsf{G_K}} L \Rightarrow R$. By Definition 7, this means that there are some finite $\Gamma \subseteq L$ and $\Delta \subseteq R$ such that $\vdash^{\mathcal{F},m}_{\mathsf{G_K}} \Gamma \Rightarrow \Delta$. By Theorem 3, $\Gamma \vdash^{\mathbb{V}^{\mathcal{F},m}_{\mathsf{K}}}_{\mathcal{D}} \Delta$.

A.11 Proof of Lemma 11

We start with the following proposition.

Proposition 1. *For every formula ψ with $sub(\psi) \subseteq \mathsf{Dom}(v)$, we have* $\mathsf{pos\text{-}val}(\psi, \mathsf{M_K}, v) \cap \mathsf{TF} \neq \emptyset$ *and* $\mathsf{pos\text{-}val}(\psi, \mathsf{M_K}, v) \cap \mathsf{tf} \neq \emptyset$.

Proof. If ψ is atomic, $\mathsf{pos\text{-}val}(\psi, \mathsf{M_K}, v) \cap \mathsf{TF} = \mathcal{V}_4 \cap \mathsf{TF} = \{\mathsf{T}, \mathsf{F}\}$ and $\mathsf{pos\text{-}val}(\psi, \mathsf{M_K}, v) \cap \mathsf{tf} = \mathcal{V}_4 \cap \mathsf{tf} = \{\mathsf{t}, \mathsf{f}\}$. Otherwise, notice that $\mathsf{pos\text{-}val}(\psi, \mathsf{M_K}, v) \in \{\mathcal{D}, \overline{\mathcal{D}}\}$ and we have $\mathcal{D} \cap \mathsf{TF} = \{\mathsf{T}\}$, $\overline{\mathcal{D}} \cap \mathsf{TF} = \{\mathsf{F}\}$, $\mathcal{D} \cap \mathsf{tf} = \{\mathsf{t}\}$, and $\overline{\mathcal{D}} \cap \mathsf{tf} = \{\mathsf{f}\}$.

Lemma 11 is proven by induction on m. If $m = 0$, we can set $v'(\psi)$ to be some element of $\mathsf{pos\text{-}val}(\psi, \mathsf{M_K}, v)$, and obtain that $v' \in \mathbb{V}^{\mathcal{F}\cup\{\psi\},0}_{\mathsf{K}}$. For $m > 0$, we define v' as follows (well defined following Proposition 1):[6]

$$v' \stackrel{\mathsf{def}}{=} \lambda\varphi \in \mathcal{F}\cup\{\psi\}\,.\begin{cases} v(\varphi) & \varphi \in \mathcal{F} \\ \min(\mathsf{pos\text{-}val}(\psi, \mathsf{M}_{\mathtt{K}}, v) \cap \mathsf{TF}) & \varphi = \psi \wedge v^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F}\cup\{\psi\},m-1}} \psi \\ \min(\mathsf{pos\text{-}val}(\psi, \mathsf{M}_{\mathtt{K}}, v) \cap \mathsf{tf}) & \text{otherwise} \end{cases}$$

We show by (an inner) induction on k that $v' \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F}\cup\{\psi\},k}$ for every $k \leq m$. The base case $k = 0$ trivially follows from the definition of v'. Let $k > 0$. By the (inner) induction hypothesis, we have $v' \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F}\cup\{\psi\},k-1}$. Let $\varphi \in \mathcal{F} \cup \{\psi\}$ such that $v'^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F}\cup\{\psi\},k-1}} \varphi$. We show that $v'(\varphi) \in \mathsf{TF}$. Distinguish the following cases:

1. $v^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F}\cup\{\psi\},m-1}} \psi$: In this case, by definition, we have $v'(\psi) \in \mathsf{TF}$. Thus, if $\varphi = \psi$, then we are done. Otherwise, $v'(\varphi) = v(\varphi)$, and so we prove $v(\varphi) \in \mathsf{TF}$. For that, since $v \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F},m}$ and $\varphi \in \mathcal{F}$, it suffices to show that $v^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F},m-1}} \varphi$.
   We first claim that $v^{-1}[\mathsf{TF}] \uplus \{\psi\} \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F}\cup\{\psi\},m-1}} \varphi$. Indeed, since $v'^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F}\cup\{\psi\},k-1}} \varphi$ and $k-1 \leq m-1$, we have $v'^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F}\cup\{\psi\},m-1}} \varphi$. Now, since $v'(\psi) \in \mathsf{TF}$, we have $v'^{-1}[\mathsf{TF}] = v^{-1}[\mathsf{TF}] \uplus \{\psi\}$, from which our claim follows.
   Now, to prove that $v^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F},m-1}} \varphi$, let $u \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F},m-1}$ such that $u \models_{\mathcal{D}} v^{-1}[\mathsf{TF}]$. By the (outer) induction hypothesis, u can be extended to some $u' \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F}\cup\{\psi\},m-1}$, and since u' extends u, we also have $u' \models_{\mathcal{D}} v^{-1}[\mathsf{TF}]$. From the assumption of this case it follows that $u' \models_{\mathcal{D}} \psi$. By our claim above, it follows that $u' \models_{\mathcal{D}} \varphi$, and since $\varphi \neq \psi$, we have $u \models_{\mathcal{D}} \varphi$ as well.
2. $v^{-1}[\mathsf{TF}] \not\vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F}\cup\{\psi\},m-1}} \psi$: In this case, the definition of v' ensures that $v'(\psi) \notin \mathsf{TF}$, and so $v'^{-1}[\mathsf{TF}] = v^{-1}[\mathsf{TF}]$. Hence, we have $v^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F}\cup\{\psi\},k-1}} \varphi$. Since $k-1 \leq m-1$, it follows that $v^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F}\cup\{\psi\},m-1}} \varphi$. Thus, we must have $\varphi \neq \psi$, so $v'(\varphi) = v(\varphi)$, and it remains to show that $v(\varphi) \in \mathsf{TF}$. For that, again, since $v \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F},m}$ and $\varphi \in \mathcal{F}$, it suffices to show that $v^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F},m-1}} \varphi$. Let $u \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F},m-1}$ such that $u \models_{\mathcal{D}} v^{-1}[\mathsf{TF}]$. By the (outer) induction hypothesis, u can be extended to some $u' \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F}\cup\{\psi\},m-1}$, and since u' extends u, we also have $u' \models_{\mathcal{D}} v^{-1}[\mathsf{TF}]$. Since $v^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F}\cup\{\psi\},m-1}} \varphi$, we obtain that $u' \models_{\mathcal{D}} \varphi$, and since $\varphi \neq \psi$, we have $u \models_{\mathcal{D}} \varphi$ as well.

A.12 Proof of Lemma 12

By induction on m. If $m = 0$, then this amounts to Lemma 1. Now suppose $m > 0$. Our induction hypothesis is that for every set $\mathcal{F}'$ closed under subformulas,

[6] The use of min here assumes an arbitrary order on truth values. It is used here only to choose *some* element from a non-empty set of truth values. In other words, min serves here as one particular choice function.

$k < m$ and $u \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F}',k}$, u can be extended to some $u' \in \mathbb{V}_{\mathtt{K}}^{k}$. Let $v \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F},m}$. If $\mathcal{F} = \mathcal{L}$, then the claim is trivial. Otherwise, note that the fact that $\mathcal{F}$ is closed under subformulas entails that $\mathcal{L} \setminus \mathcal{F}$ must be infinite. We define v' as follows. Let $\psi_1, \psi_2, ...$ be an enumeration of $\mathcal{L} \setminus \mathcal{F}$ such that $i < j$ whenever ψ_i is a proper subformula of ψ_j. As a step towards defining v', we first define a sequence $v_0, v_1, ...$ as follows: $v_0 \stackrel{\text{def}}{=} v$. Let $\mathcal{F}_0 \stackrel{\text{def}}{=} \mathcal{F}$, and for every $i > 0$, let $\mathcal{F}_i \stackrel{\text{def}}{=} \mathcal{F}_{i-1} \cup \{\psi_i\}$. If $v_{i-1} \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F}_{i-1},m}$, then by Lemma 11 there exists $v'_{i-1} \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F}_i,m}$ that extends v_{i-1}. Then, we set $v_i \stackrel{\text{def}}{=} v'_{i-1}$.

We prove by induction on i that v_i is well-defined and is an element of $\mathbb{V}_{\mathtt{K}}^{\mathcal{F}_i,m}$ that extends v_j for every $j < i$. For the base case $v_0 = v \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F},m} = \mathbb{V}_{\mathtt{K}}^{\mathcal{F}_0,m}$, is well defined. For the induction step, $i > 0$ and the induction hypothesis gives us that v_{i-1} is well-defined and is an element of $\mathbb{V}_{\mathtt{K}}^{\mathcal{F}_{i-1},m}$ that extends v_j for every $j < i-1$. We prove v_i is well-defined and is an element of $\mathbb{V}_{\mathtt{K}}^{\mathcal{F}_i,m}$ that extends v_k for every $k < i$. By the induction hypothesis, $v_{i-1} \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F}_{i-1}}$. By construction, in this case, v_i is defined, $v_i = v'_{i-1} \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F}_i,m}$ from Lemma 11, and extends v_{i-1}, that by the induction hypothesis also extends v_j for every $j < i-1$.

Finally, we set $v' \stackrel{\text{def}}{=} \lambda\varphi \in \mathcal{L}.\, v_{min\{i \mid \varphi \in \mathcal{F}_i\}}(\varphi)$. We prove that $v' \in \mathbb{V}_{\mathtt{K}}^{m}$. Let $m \geq 0$. We prove that $v' \in \mathbb{V}_{\mathtt{K}}^{k}$ for every $0 \leq k \leq m$ by induction on k. For $k = 0$, let $\varphi_1, \dots, \varphi_n, \diamond(\varphi_1, \dots, \varphi_n) \in \mathcal{L}$. Let $i = min\{i \mid \varphi_1, \dots, \varphi_n, \diamond(\varphi_1, \dots, \varphi_n) \in \mathcal{F}_i\}$. Then, by construction, $v'(\diamond(\varphi_1, \dots, \varphi_n)) = v_i(\diamond(\varphi_1, \dots, \varphi_n))$. Now, $v_i \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F}_i} \subseteq \mathbb{V}_{\mathtt{K}}^{\mathcal{F}_i,0}$ and so $v_i(\diamond(\varphi_1, \dots, \varphi_n)) \in \tilde{\diamond}(v_i(\varphi_1), \dots, v_i(\varphi_n)) = \tilde{\diamond}(v'(\varphi_1), \dots, v'(\varphi_n))$. For $k > 0$, by the induction hypothesis, $v' \in \mathbb{V}_{\mathtt{K}}^{k-1}$. Let $\varphi \in \mathcal{L}$ and suppose $v'^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{k-1}} \varphi$. By Theorem 1, we get some finite $\Gamma \subseteq v'^{-1}[\mathsf{TF}]$ such that $\Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{k-1}} \varphi$. Let $i = min\{i \mid \Gamma \cup \{\varphi\} \subseteq \mathcal{F}_i\}$. Then, $\Gamma \subseteq v_i^{-1}[\mathsf{TF}]$. We prove that $v'(\varphi) = v_i(\varphi) \in \mathsf{TF}$. For that, it suffices to show that $\Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F}_i,k-1}} \varphi$, as this would entail $v_i^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F}_i,k-1}} \varphi$, and since $v_i \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F}_i,m} \subseteq \mathbb{V}_{\mathtt{K}}^{\mathcal{F}_i,k-1}$, we would then get $v_i(\varphi) \in \mathsf{TF}$. Let $u \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F}_i,k-1}$ and suppose $u \models_{\mathcal{D}} \Gamma$. By the outer induction hypothesis (regarding m), u can be extended to some $u' \in \mathbb{V}_{\mathtt{K}}^{k-1}$. Hence $u' \models_{\mathcal{D}} \Gamma$. Since $\Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{k-1}} \varphi$, $u' \models_{\mathcal{D}} \varphi$ and hence $u \models_{\mathcal{D}} \varphi$.

A.13 Proof of Lemma 13

Since $v \in \mathbb{V}_{\mathtt{K}}^{\mathcal{F}}$, for every $m \geq 0$, by Lemma 12, v can be extended to some $v_m \in \mathbb{V}_{\mathtt{K}}^{m}$. By Lemma 3, there exists some $v' \in \mathbb{V}_{\mathtt{K}}$ such that for every $\varphi \in \mathcal{F}$, $v'(\varphi) = v_m(\varphi)$ for some $m \geq 0$. The latter property ensures that v' extends v: for every $\varphi \in \mathcal{F}$, we have $v'(\varphi) = v_m(\varphi) = v(\varphi)$ for some m.

A.14 Proof of Lemma 14

The proof is similar to the proof of Lemma 13. The main difference is in the following helper lemma, that corresponds to Lemma 11, but requires special care in order to make sure that the constructed extension is three-valued.

Lemma 20. *Let $m \geq 0$, $\mathcal{F} \subseteq \mathcal{L}$, and $v \in \mathbb{V}_{\mathsf{KT}}^{\mathcal{F},m}$. Let $\psi \in \mathcal{L} \setminus \mathcal{F}$ such that $sub(\psi) \setminus \{\psi\} \subseteq \mathcal{F}$. Then, v can be extended to some $v' \in \mathbb{V}_{\mathsf{KT}}^{\mathcal{F}\cup\{\psi\},m}$.*

Proof. By induction on m. If $m = 0$, we can set $v'(\psi)$ to be some element of $\mathsf{pos\text{-}val}(\psi, \mathsf{M}_{\mathsf{KT}}, v)$, and obtain that $v' \in \mathbb{V}_{\mathsf{KT}}^{\mathcal{F}\cup\{\psi\},0}$. For $m > 0$, we define v' as follows (well defined following Proposition 1):[7]

$$v' \overset{\text{def}}{=} \lambda\varphi \in \mathcal{F}\cup\{\psi\} \,.\begin{cases} v(\varphi) & \varphi \in \mathcal{F} \\ \min(\mathsf{pos\text{-}val}(\psi, \mathsf{M}_{\mathsf{K}}, v) \cap \mathsf{TF}) & \varphi = \psi \wedge v^{-1}[\mathsf{T}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{KT}}^{\mathcal{F}\cup\{\psi\},m-1}} \psi \\ \min(\mathsf{pos\text{-}val}(\psi, \mathsf{M}_{\mathsf{K}}, v) \cap \mathsf{tf}) & \text{otherwise} \end{cases}$$

Notice that the construction above uses the M_{K} rather than M_{KT}. We show by (an inner) induction on k that $v' \in \mathbb{V}_{\mathsf{KT}}^{\mathcal{F}\cup\{\psi\},k}$ for every $k \leq m$.

For the base case $k = 0$, it suffices to show that $v'(\varphi) \neq \mathsf{F}$ for every $\varphi \in \mathcal{F} \cup \{\psi\}$. For $\varphi \in \mathcal{F}$, we have $v'(\varphi) = v(\varphi)$ and $v(\varphi) \neq \mathsf{F}$ has to hold since $v \in \mathbb{V}_{\mathsf{KT}}^{\mathcal{F},m}$. Now, suppose for contradiction that $v'(\psi) = \mathsf{F}$. Then, by the construction of v' we have that ψ is not an atomic formula (since $\mathsf{T} < \mathsf{F}$), $\mathsf{F} \in \mathsf{pos\text{-}val}(\psi, \mathsf{M}_{\mathsf{K}}, v)$, and $v^{-1}[\mathsf{T}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{KT}}^{\mathcal{F}\cup\{\psi\},m-1}} \psi$. By the (outer) induction hypothesis, v can be extended to some $v'' \in \mathbb{V}_{\mathsf{KT}}^{\mathcal{F}\cup\{\psi\},m-1}$, and since v'' extends v, we also have $v'' \models_{\mathcal{D}} v^{-1}[\mathsf{T}]$. Then, since $v^{-1}[\mathsf{T}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{KT}}^{\mathcal{F}\cup\{\psi\},m-1}} \psi$, we have $v'' \models_{\mathcal{D}} \psi$. Hence, $\mathcal{D} \cap \mathsf{pos\text{-}val}(\psi, \mathsf{M}_{\mathsf{KT}}, v) \neq \emptyset$, which implies (by inspecting all truth tables) that $\mathsf{pos\text{-}val}(\psi, \mathsf{M}_{\mathsf{KT}}, v) = \mathcal{D}$. It follows that $\mathsf{pos\text{-}val}(\psi, \mathsf{M}_{\mathsf{K}}, v) = \mathcal{D}$. This contradicts the fact that $\mathsf{F} \in \mathsf{pos\text{-}val}(\psi, \mathsf{M}_{\mathsf{K}}, v)$.

Now, for the induction step, let $k > 0$. By the (inner) induction hypothesis, we have $v' \in \mathbb{V}_{\mathsf{KT}}^{\mathcal{F}\cup\{\psi\},k-1}$. Let $\varphi \in \mathcal{F} \cup \{\psi\}$ such that $v'^{-1}[\mathsf{T}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{KT}}^{\mathcal{F}\cup\{\psi\},k-1}} \varphi$. We show that $v'(\varphi) = \mathsf{T}$. Distinguish the following cases:

1. $v^{-1}[\mathsf{T}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{KT}}^{\mathcal{F}\cup\{\psi\},m-1}} \psi$: In this case, by definition, we have $v'(\psi) = \mathsf{T}$. Thus, if $\varphi = \psi$, then we are done. Otherwise, $v'(\varphi) = v(\varphi)$, and so we prove $v(\varphi) = \mathsf{T}$. For that, since $v \in \mathbb{V}_{\mathsf{KT}}^{\mathcal{F},m}$ and $\varphi \in \mathcal{F}$, it suffices to show that $v^{-1}[\mathsf{T}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{KT}}^{\mathcal{F},m-1}} \varphi$.
   We first claim that $v^{-1}[\mathsf{T}] \uplus \{\psi\} \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{KT}}^{\mathcal{F}\cup\{\psi\},m-1}} \varphi$. Indeed, since $v'^{-1}[\mathsf{T}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{KT}}^{\mathcal{F}\cup\{\psi\},k-1}} \varphi$ and $k-1 \leq m-1$, we have $v'^{-1}[\mathsf{T}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{KT}}^{\mathcal{F}\cup\{\psi\},m-1}} \varphi$. Now, since $v'(\psi) = \mathsf{T}$, we have $v'^{-1}[\mathsf{T}] = v^{-1}[\mathsf{T}] \uplus \{\psi\}$, from which our claim follows.
   Now, to prove that $v^{-1}[\mathsf{T}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{KT}}^{\mathcal{F},m-1}} \varphi$, let $u \in \mathbb{V}_{\mathsf{KT}}^{\mathcal{F},m-1}$ such that $u \models_{\mathcal{D}} v^{-1}[\mathsf{T}]$. By the (outer) induction hypothesis, u can be extended to some $u' \in \mathbb{V}_{\mathsf{KT}}^{\mathcal{F}\cup\{\psi\},m-1}$, and since u' extends u, we also have $u' \models_{\mathcal{D}} v^{-1}[\mathsf{T}]$. From the assumption of this case it follows that $u' \models_{\mathcal{D}} \psi$. By our claim above, it follows that $u' \models_{\mathcal{D}} \varphi$, and since $\varphi \neq \psi$, we have $u \models_{\mathcal{D}} \varphi$ as well.
2. $v^{-1}[\mathsf{T}] \nvdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{KT}}^{\mathcal{F}\cup\{\psi\},m-1}} \psi$: In this case, the definition of v' ensures that $v'(\psi) \neq \mathsf{T}$, and so $v'^{-1}[\mathsf{T}] = v^{-1}[\mathsf{T}]$. Hence, we have $v^{-1}[\mathsf{T}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathsf{KT}}^{\mathcal{F}\cup\{\psi\},k-1}} \varphi$. Since

[7] The use of min here assumes $\mathsf{T} < \mathsf{t} < \mathsf{f} < \mathsf{F}$.

$k-1 \leq m-1$, it follows that $v^{-1}[\mathsf{T}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{KT}}^{\mathcal{F}\cup\{\psi\},m-1}} \varphi$. Thus, we must have $\varphi \neq \psi$, so $v'(\varphi) = v(\varphi)$, and it remains to show that $v(\varphi) = \mathsf{T}$. For that, again, since $v \in \mathbb{V}_{\mathtt{KT}}^{\mathcal{F},m}$ and $\varphi \in \mathcal{F}$, it suffices to show that $v^{-1}[\mathsf{T}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{KT}}^{\mathcal{F},m-1}} \varphi$. Let $u \in \mathbb{V}_{\mathtt{KT}}^{\mathcal{F},m-1}$ such that $u \models_{\mathcal{D}} v^{-1}[\mathsf{T}]$. By the (outer) induction hypothesis, u can be extended to some $u' \in \mathbb{V}_{\mathtt{KT}}^{\mathcal{F}\cup\{\psi\},m-1}$, and since u' extends u, we also have $u' \models_{\mathcal{D}} v^{-1}[\mathsf{T}]$. Since $v^{-1}[\mathsf{T}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{KT}}^{\mathcal{F}\cup\{\psi\},m-1}} \varphi$, we obtain that $u' \models_{\mathcal{D}} \varphi$, and since $\varphi \neq \psi$, we have $u \models_{\mathcal{D}} \varphi$ as well.