On the Construction of Analytic Sequent Calculi for Sub-classical Logics

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Motivation

- Sequent calculi reveal a wide variety of options to define sub-classical logics:

  - Begin with Gentzen's **LK**.
  - Discard some of its (logical) rules.
  - Add other (logical) rules, that are derivable in **LK**.

- The usefulness of the resulting calculus depends on its *analyticity*.

What general conditions guarantee the analyticity of the obtained calculus?

Examples

$$\xcancel{\frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}} \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B}$$

Intuitionistic logic

$$\xcancel{\frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}} \qquad \frac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}$$

Primal logic [Gurevich, Neeman '09]

$$\xcancel{\frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta}} \qquad \vdots$$

Paraconsistent logic

Pure Sequent Calculi

- *Pure sequent calculi* are propositional sequent calculi that include all usual structural rules (including (*cut*)), and a finite set of pure logical rules.
- *Pure logical rules* are logical rules that allow any context [Avron '91].

$$\frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta} \quad \text{but not} \quad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B}$$

- Rules vs. applications:

$$\underset{\text{Application Scheme}}{\frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}} \qquad \underset{\text{Rule}}{\frac{p_1 \Rightarrow p_2}{\Rightarrow p_1 \supset p_2}} \qquad \underset{\text{Application}}{\frac{p_1, p_3 \Rightarrow p_4, p_3}{p_1 \Rightarrow p_3 \supset p_4, p_3}}$$

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Application Scheme — Rule Application — Application Rule

Analyticity

Definition

A calculus is *analytic* if $\vdash \Gamma \Rightarrow \Delta$ implies that there is a derivation of $\Gamma \Rightarrow \Delta$ using only subformulas of $\Gamma \cup \Delta$.

- If a pure calculus is analytic then it is decidable.

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Definition (k-subformula)

$$\frac{}{A \preceq_k \neg A} \qquad \frac{}{\neg^k A_i \preceq_k A_1 \diamond A_2} \qquad \frac{}{A \preceq_k A} \qquad \frac{A \preceq_k B \quad B \preceq_k C}{A \preceq_k C}$$

Definition

A calculus is *k-analytic* if $\vdash \Gamma \Rightarrow \Delta$ implies that there is a derivation of $\Gamma \Rightarrow \Delta$ using only k-subformulas of $\Gamma \cup \Delta$.

Basic Criterion for k-Analyticity

- All rules are *k-closed*:
  - The conclusion has the form $\Rightarrow A$ or $A \Rightarrow$
  - The premises consist of k-subformulas of A
- Right and left rules play well together:

For any two contextless applications of the form

$$\frac{s_1 \quad \ldots \quad s_n}{\Rightarrow A} \qquad \frac{s'_1 \quad \ldots \quad s'_m}{A \Rightarrow}$$

we have

$$s_1, \ldots, s_n, s'_1, \ldots, s'_m \vdash^{(cut)} \Rightarrow$$

- Generalizes *coherence* (Avron, Lev '01,'05).

Basic Criterion for Analyticity

- The above criterion suffices for various calculi, e.g.:
  - The propositional fragment of **LK**.
  - The calculus for the logic of first degree entailment (FDE).
  - The calculus for Primal logic.
- However, this criterion is not necessary.

Example

$$\xcancel{\frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}} \qquad \frac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta} \qquad \frac{}{\Gamma \Rightarrow A \supset (B \supset A), \Delta}$$

$$\frac{}{\Gamma \Rightarrow A \supset (B \supset A), \Delta} \quad \text{and} \quad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta} \quad \text{do not “play well” together.}$$

Example: A 1-analytic Pure Calculus for da Costa's Paraconsistent Logic $\mathbf{C}_1$ [Avron, Konikowska, Zamansky '12]

$$\xcancel{\frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta}} \qquad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta} \qquad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \neg\neg A \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow \neg A, \Delta}{\Gamma, \neg(A \wedge \neg A) \Rightarrow \Delta} \qquad \frac{\Gamma, \neg A \Rightarrow \Delta \quad \Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \wedge B) \Rightarrow \Delta}$$

$$\frac{\Gamma, \neg A \Rightarrow \Delta \quad \Gamma, B, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \vee B) \Rightarrow \Delta} \qquad \frac{\Gamma, A, \neg A \Rightarrow \Delta \quad \Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \vee B) \Rightarrow \Delta}$$

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \supset B) \Rightarrow \Delta} \qquad \frac{\Gamma, A, \neg A \Rightarrow \Delta \quad \Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \supset B) \Rightarrow \Delta}$$

Analyticity by Construction

Definition (Safe Application)

An application of an **LK** rule is *k-safe* if the context formulas are all k-subformulas of the principal formula.

Example (Derivable rules that are 0-safe applications)

$$\frac{p_2, p_1 \Rightarrow p_2}{p_2 \Rightarrow p_1 \supset p_2} \qquad \frac{p_1 \Rightarrow p_1}{\Rightarrow p_1 \supset p_1} \qquad \frac{p_1 \Rightarrow p_2 \supset p_1}{\Rightarrow p_1 \supset (p_2 \supset p_1)} \qquad \frac{p_1 \wedge p_2 \Rightarrow p_1}{\Rightarrow (p_1 \wedge p_2) \supset p_1}$$

Theorem

A calculus whose rules are all k-safe applications is k-analytic.

- Simple "analyticity preserving transformations" are also useful.
- The same holds for any calculus admitting the basic criterion instead of **LK**.

Example: Paraconsistent Logic

$$\xcancel{\frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta}} \qquad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}$$

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$$

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta} \qquad \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}$$

Example: Paraconsistent Logic

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$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta} \qquad \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}$$

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow \neg A, \Delta}{\Gamma, \neg(A \wedge \neg A) \Rightarrow \Delta} \quad \simeq \quad \frac{\Gamma, A, \neg A \Rightarrow A \wedge \neg A, \Delta}{\Gamma, A, \neg A, \neg(A \wedge \neg A) \Rightarrow \Delta}$$

Semantics

- Pure calculi can be characterized by *two-valued valuations* [Béziau '01].
- Each pure rule is read as a semantic condition.
- By joining the semantic conditions of all rules in a calculus **G**, we obtain the set of **G**-*legal* valuations.

Soundness and Completeness

$\Gamma \Rightarrow \Delta$ is provable in **G** iff every **G**-legal valuation is a model of $\Gamma \Rightarrow \Delta$.

Stronger Version

$\Gamma \Rightarrow \Delta$ is provable in **G** using only formulas of $\mathcal{F}$ iff every **G**-legal valuation whose domain is $\mathcal{F}$ is a model of $\Gamma \Rightarrow \Delta$.

Semantic Analyticity

Definition

G is semantically k-analytic if every **G**-legal **partial** valuation whose domain is closed under k-subformulas can be extended to a full **G**-legal valuation.

Theorem

A calculus is k-analytic iff it is semantically k-analytic.

Proof Outline

Theorem

A calculus whose rules are all k-safe applications is k-analytic.

- Suppose that $\mathcal{F}$ is closed under k-subformulas.
- Enumerate all formulas that are not included in $\mathcal{F}$:

$$A_1 \preceq_k A_2 \preceq_k \ldots$$

- **LK** has a trivial "extension procedure" to assign values for all formulas.
- The same procedure applies for any collection of k-safe applications.

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Theorem

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- **LK** has a trivial "extension procedure" to assign values for all formulas.
- The same procedure applies for any collection of k-safe applications.

$$\frac{\Gamma \Rightarrow A \wedge B, \neg A, \neg B, \Delta}{\Gamma, \neg(A \wedge B) \Rightarrow \neg A, \neg B, \Delta}$$

If $\neg A$ and $\neg B$ are false, but $A \wedge B$ is true, then $\neg(A \wedge B)$ must be false.

(In the presence of $(\Rightarrow \wedge)$ and $(\Rightarrow \neg)$, this is equivalent to $\dfrac{\Gamma, \neg A \Rightarrow \Delta \quad \Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \wedge B) \Rightarrow \Delta}$)

Conclusions and Further Work

- We provided a general sufficient condition for k-analyticity in pure calculi.
- We identified a large family of classically derivable rules that form analytic calculi.
- This allows one to easily verify analyticity, introduce new analytic calculi, and augment analytic calculi with more useful rules.

- Further work:
  - Cut-elimination
  - Non-pure calculi (context restrictions)
  - First order logics

Thank you!