A Simple Cut-Free System for a Paraconsistent Logic Equivalent to S5

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Abstract

NS5 is a paraconsistent logic in the classical language, which is equivalent to the well-known modal logic **S5**. We provide a particularly simple hypersequential system for the propositional **NS5**, and prove a strong cut-admissibility theorem for it. Our system is obtained from the standard hypersequential system for classical logic by just weakening its two rules for negation, and without introducing any new structural rule. We also explain how to extend the results to the natural first-order extension of **NS5**. The latter is equivalent to the Constant Domain first-order **S5**.

Keywords: **S5**, modal logic, paraconsistent logic, hypersequents, cut-elimination

It is well-known that the cut-admissibility theorem fails for the standard Gentzen-type system for the propositional modal logic **S5** . Therefore various alternative Gentzen-type systems for this logic that do enjoy cut-admissibility have been proposed in the literature. Among those systems, the simplest are those that employ *hypersequents*. Several such systems for propositional **S5** have been presented over the years, e.g. in [12,18,2,19,16,6]. (See also [20], [17], and [10] for further examples and references.) However, we have not been able to find in the literature any reasonable sound and complete hypersequential calculus for *first-order* **S5** which is known to be cut-free.

The main goal of this paper is to present a particularly simple hypersequential system for first-order **S5**, for which even a *strong* cut-admissibility theorem holds. What is more, our system is obtained from the standard hypersequential system for *classical* logic using only slight changes in its two rules for *negation*, and without introducing any new structural rules. For this we extend to the first-order level the presentation which was investigated in [4] of the propositional **S5** as a paraconsistent logic **NS5** in the standard classical propositional

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language. This method also demonstrates the usefulness of viewing **S5** as a paraconsistent logic, and shows at the same time how close it is as such to classical logic.

The structure of the paper is as follows. In its first half (Sections 1–3) we provide our hypersequential version of the propositional **NS5**, and prove cut-admissibility for it using a *semantic* method. [3] The results and method are then extended to the first-order level in its second half (Section 4).

1 Preliminaries

We assume that all propositional languages share the same set $\{P_1, P_2, ...\}$ of propositional variables, and use p, q, r to vary over this set. The set of formulas of a propositional language $\mathcal{L}$ is denoted by $\mathcal{WFF}(\mathcal{L})$, and φ, ψ, σ will vary over it.

1.1 Calculi of Hypersequents

Definition 1.1 A *hypersequent* is a finite set of ordinary sequents. The elements of this set are called its *components*. We denote by $s_1 \mid \cdots \mid s_n$ the hypersequent whose components are $s_1, \ldots, s_n$, and use G, H as metavariables for (possibly empty) hypersequents. [4]

Note 1 Hypersequents are often taken to be *multisets* of ordinary sequents. In such a case it is necessary to add to the list of structural rules the rule [EC] of *external contraction*, which allows one to infer $H \mid s$ from $H \mid s \mid s$. This is a reasonable choice, since external contraction is the main source of problems for hypersequential calculi (both for proving cut-elimination and for producing efficient proof-search procedures), and this is hidden if it is built into the definition of a hypersequent (as we do here in order to get simpler and more economic systems).

Usually, most of the rules in hypersequential calculi are obtained from standard rules of ordinary sequential calculi by allowing also side components in applications of the rules (in addition to the presence of side formulas). A simple example is provided in Figure 1, which presents a hypersequential version LK^h of Gentzen's system LK for classical logic. The rules of LK^h are just the obvious hypersequential versions of the rules of LK. Therefore it is easy to see that a hypersequent is derivable in LK^h iff one of its components is classically valid.

1.2 The Propositional System NS5

The language of the propositional modal logic **S5** is usually taken to be the one induced by $\{\wedge, \vee, \supset, \mathsf{F}, \Box\}$. *Classical* negation $\sim$ can be defined by $\sim\varphi = \varphi \supset \mathsf{F}$. Following Béziau ([7]) and Batens ([5]), one may also define a *paraconsistent*

[3] Gentzen-style syntactic proofs of cut-elimination are notoriously error-prone and difficult to check. Accordingly, we continue in this paper our program of providing checkable semantic proofs for hypersequential calculi. (See e.g. [11,3])

[4] Semantically, the interpretation of '|' is usually taken to be disjunctive.

Structural rules:

[id] $\dfrac{}{\varphi \Rightarrow \varphi}$ [cut] $\dfrac{H \mid \Gamma \Rightarrow \Delta, \varphi \quad H \mid \varphi, \Gamma \Rightarrow \Delta}{H \mid \Gamma \Rightarrow \Delta}$

[IW] $\dfrac{H \mid \Gamma \Rightarrow \Delta}{H \mid \Gamma', \Gamma \Rightarrow \Delta, \Delta'}$ [EW] $\dfrac{H}{H \mid \Gamma \Rightarrow \Delta}$

Logical rules:

[$\wedge\Rightarrow$] $\dfrac{H \mid \Gamma, \varphi, \psi \Rightarrow \Delta}{H \mid \Gamma, \varphi \wedge \psi \Rightarrow \Delta}$ [$\Rightarrow\wedge$] $\dfrac{H \mid \Gamma \Rightarrow \Delta, \varphi \quad H \mid \Gamma \Rightarrow \Delta, \psi}{H \mid \Gamma \Rightarrow \Delta, \varphi \wedge \psi}$

[$\vee\Rightarrow$] $\dfrac{H \mid \Gamma, \varphi \Rightarrow \Delta \quad H \mid \Gamma, \psi \Rightarrow \Delta}{H \mid \Gamma, \varphi \vee \psi \Rightarrow \Delta}$ [$\Rightarrow\vee$] $\dfrac{H \mid \Gamma \Rightarrow \Delta, \varphi, \psi}{H \mid \Gamma \Rightarrow \Delta, \varphi \vee \psi}$

[$\supset\Rightarrow$] $\dfrac{H \mid \Gamma \Rightarrow \Delta, \varphi \quad H \mid \Gamma, \psi \Rightarrow \Delta}{H \mid \Gamma, \varphi \supset \psi \Rightarrow \Delta}$ [$\Rightarrow\supset$] $\dfrac{H \mid \Gamma, \varphi \Rightarrow \Delta, \psi}{H \mid \Gamma \Rightarrow \Delta, \varphi \supset \psi}$

[$\neg\Rightarrow$] $\dfrac{H \mid \Gamma \Rightarrow \Delta, \varphi}{H \mid \Gamma, \neg\varphi \Rightarrow \Delta}$ [$\Rightarrow\neg$] $\dfrac{H \mid \Gamma, \varphi \Rightarrow \Delta}{H \mid \Gamma \Rightarrow \Delta, \neg\varphi}$

[$\forall\Rightarrow$] $\dfrac{H \mid \Gamma, \varphi\{t/a\} \Rightarrow \Delta}{H \mid \Gamma, \forall x(\varphi\{x/a\}) \Rightarrow \Delta}$ [$\Rightarrow\forall$] $\dfrac{H \mid \Gamma \Rightarrow \Delta, \varphi}{H \mid \Gamma \Rightarrow \Delta, \forall x(\varphi\{x/a\})}$

[$\exists\Rightarrow$] $\dfrac{H \mid \Gamma, \varphi \Rightarrow \Delta}{H \mid \Gamma, \exists x(\varphi\{x/a\}) \Rightarrow \Delta}$ [$\Rightarrow\exists$] $\dfrac{H \mid \Gamma \Rightarrow \Delta, \varphi\{t/a\}}{H \mid \Gamma \Rightarrow \Delta, \exists x(\varphi\{x/a\})}$

The rules [$\Rightarrow\forall$] and [$\exists\Rightarrow$] must obey the eigenvariable condition: a must not occur in the lower hypersequent.

Fig. 1. The proof system LK^h

negation $\neg$ by $\neg\varphi = \Box\varphi \supset \mathsf{F}$, and in fact, we can take the language of **S5** to be the language of classical logic $\mathcal{L}_{CL} = \{\supset, \wedge, \vee, \neg\}$. The connectives F and $\Box$ of **S5** are definable in $\mathcal{L}_{CL}$: F is equivalent to $\neg(p \supset p)$, where p is some atomic formula; while $\Box\varphi$ is equivalent in **S5** to $\neg\neg\varphi$.[5] When formulated in this language, **S5** becomes a *paraconsistent* logic, which was called **NS5** in [4].[6] Its semantics is given in the next definitions.

Definition 1.2 A pair $\langle W, \nu \rangle$ is called an **NS5**-*frame* for $\mathcal{L}_{CL}$ if W is a nonempty (finite) set (of "worlds"), and $\nu : W \times \mathcal{WFF}(\mathcal{L}_{CL}) \to \{t, f\}$ satisfies the following conditions:[7]

[5] This implies that we could have restricted ourselves to the language of $\{\supset, \neg\}$, since the classical $\vee$ and $\wedge$ can of course be defined in terms of $\supset$ and F.

[6] What is called here **NS5** was independently introduced by Béziau ([7,8,9] and Batens ([5]). (**NS5** was called $\mathbb{Z}$ by Béziau, and **A** by Batens.) Further study of this system was done by Osorio, Carballido, Zepeda, and Castellanos in [14] and [15].

[7] For **S5**/**NS5** this suffices. For normal modal logics in general one should use triples

- $\nu(w, \psi \wedge \varphi) = t$ iff $\nu(w, \psi) = t$ and $\nu(w, \varphi) = t$.
- $\nu(w, \psi \vee \varphi) = t$ iff $\nu(w, \psi) = t$ or $\nu(w, \varphi) = t$.
- $\nu(w, \psi \supset \varphi) = t$ iff $\nu(w, \psi) = f$ or $\nu(w, \varphi) = t$.
- $\nu(w, \neg\psi) = t$ iff there exists $w' \in W$ such that $\nu(w', \psi) = f$.

Definition 1.3 Let $\langle W, \nu \rangle$ be an **NS5**-frame.

- A formula φ is *true* in a world $w \in W$ ($w \Vdash \varphi$) if $\nu(w, \varphi) = t$.
- Let $\mathcal{T} \cup \{\varphi\}$ be a set of formulas in $\mathcal{L}_{CL}$. φ *follows in* **NS5** from $\mathcal{T}$ ($\mathcal{T} \vdash_{\mathbf{NS5}} \varphi$) if for every **NS5**-frame $\langle W, \nu \rangle$ and every $w \in W$: if $w \Vdash \psi$ for every $\psi \in \mathcal{T}$ then $w \Vdash \varphi$.
- A sequent $s = \Gamma \Rightarrow \Delta$ is *true* in a world $w \in W$ ($w \Vdash s$) if $\nu(w, \varphi) = f$ for some $\varphi \in \Gamma$, or $\nu(w, \varphi) = t$ for some $\varphi \in \Delta$.
- A sequent s is *valid* in $\langle W, \nu \rangle$ ($\langle W, \nu \rangle \models s$) if it is true in every world $w \in W$.
- Let $S \cup \{s\}$ be a set of sequents in $\mathcal{L}_{CL}$. s *follows in* **NS5** from S ($S \vdash_{\mathbf{NS5}} s$) if for every **NS5**-frame $\mathcal{W}$, if $\mathcal{W} \models s'$ for every $s' \in S$, then $\mathcal{W} \models s$. s is **NS5**-*valid* if s follows in **NS5** from $\emptyset$ (that is, s is valid in every **NS5**-frame).
- A hypersequent H is *valid* in $\langle W, \nu \rangle$, or $\langle W, \nu \rangle$ is a *model* of H ($\langle W, \nu \rangle \models H$), if one of the components of H is valid in $\langle W, \nu \rangle$.
- Let $\mathcal{H} \cup \{H\}$ be a set of hypersequents in $\mathcal{L}_{CL}$. H follows from $\mathcal{H}$ in **NS5** ($\mathcal{H} \vdash_{\mathbf{NS5}} H$) if every model of $\mathcal{H}$ is also a model of H.

Note 2 Note that the consequence relation we use between formulas is the *local* one (or the "truth" consequence relation in the terminology of [1]), while those we use between sequents or hypersequents are the *global* ones (or the "validity" consequence relation in the terminology of [1]). This implies that if Γ is a finite set of formulas then $\Gamma \vdash_{\mathbf{NS5}} \varphi$ iff the sequent (which is also a hypersequent) $\Gamma \Rightarrow \varphi$ is **NS5**-valid.

2 The Hypersequential System $GNS5^h$

Our hypersequential system $GNS5^h$ for **NS5** differs from the propositional fragment of LK^h (given in Figure 1) only with respect to its two rules for $\neg$. Instead of the rules of LK^h, the system $GNS5^h$ employs the following two rules:

$$[\neg \Rightarrow]\ \frac{H \mid\ \Rightarrow \varphi}{H \mid \neg\varphi \Rightarrow} \qquad [\Rightarrow \neg]\ \frac{H \mid \Gamma, \varphi \Rightarrow \Delta}{H \mid \Gamma \Rightarrow \Delta \mid\ \Rightarrow \neg\varphi}$$

The rule $[\neg \Rightarrow\ ]$ of $GNS5^h$ is just a special case of the corresponding rule of LK^h. In contrast, the rule $[\ \Rightarrow \neg]$ of $GNS5^h$ is *stronger* than the corresponding rule of LK^h. Thus the latter is derivable from the former with the help of

$\langle W, R, \nu \rangle$, where R is a relation on W, which in the case of **NS5** should be an equivalence relation. Note also that in the literature on modal logics one usually means by a "frame" just the pair $\langle W, R \rangle$, while we find it convenient to follow [13], and use this technical term a little bit differently, so that the valuation ν is a part of it.

[IW]. On the other hand, the rule $[\Rightarrow \neg]$ of $GNS5^h$ allows the inference of $\Rightarrow \varphi \mid \ \Rightarrow \neg\varphi$ for every φ, while if p is atomic then $\Rightarrow p \mid \ \Rightarrow \neg p$ is not provable in LK^h (since neither $\Rightarrow p$ nor $\Rightarrow \neg p$ is classically valid).

Note 3 The rule $[\Rightarrow \neg]$ of $GNS5^h$ becomes derivable if we add to LK^h the following splitting rule: from $H \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ infer $H \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2$. It is easy to see that the extended system is again sound and complete for classical logic, but with a different semantic interpretation of hypersequents: A hypersequent H is provable in that system iff every classical valuation is a model of one of the components of H.

Example 2.1 Let *GNS*5 be the system which is obtained from the propositional fragment of LK (the ordinary, sequential Gentzen-type system for classical logic) by replacing its rule $[\neg\Rightarrow]$ by the rule:

$$[\neg\Rightarrow]_5 \quad \frac{\neg\Gamma \Rightarrow \psi, \neg\Delta}{\neg\Gamma, \neg\psi \Rightarrow \neg\Delta}$$

This system (which is the variant of the usual Gentzen-type system for $S5$ in which the present $\neg$ is used instead of $\Box$) is known to be sound and complete for $NS5$ ([4]). However, the cut-admissibility theorem fails for it. For example, the sequent $\neg\neg P_1 \Rightarrow P_1$ is derivable in *GNS*5 by applying the cut rule to the sequents $\Rightarrow P_1, \neg P_1$ and $\neg\neg P_1, \neg P_1 \Rightarrow$ (both of which have very short cut-free proofs in *GNS*5). It is easy to see that this cut cannot be eliminated. Here is a cut-free proof of this sequent in $GNS5^h$:

$$\cfrac{\cfrac{\cfrac{\cfrac{P_1 \Rightarrow P_1}{\Rightarrow P_1 \mid \ \Rightarrow \neg P_1}\,[\Rightarrow \neg]}{\Rightarrow P_1 \mid \neg\neg P_1 \Rightarrow}\,[\neg \Rightarrow]}{\neg\neg P_1 \Rightarrow P_1 \mid \neg\neg P_1 \Rightarrow}\,[IW]}{\neg\neg P_1 \Rightarrow P_1}\,[IW]$$

(Note that the last step includes a hidden application of [EC] — See Note 1.)

Note 4 Obviously, an equivalent hypersequential system $GS5^h$ for propositional $S5$ in its standard language $(\{\wedge, \vee, \supset, \mathsf{F}, \Box\})$ is obtained from $GNS5^h$ by adding all sequents of the form $H \mid \mathsf{F} \Rightarrow$ as axioms, and replacing the two rules for $\neg$ by their following counterparts for $\Box$:

$$[\Box \Rightarrow]\ \frac{H \mid \Gamma, \varphi \Rightarrow \Delta}{H \mid \Gamma \Rightarrow \Delta \mid \Box\varphi \Rightarrow} \qquad [\Rightarrow \Box]\ \frac{H \mid \ \Rightarrow \varphi}{H \mid \ \Rightarrow \Box\varphi}$$

The strong soundness and completeness theorem and the strong cut-admissibility theorem which are proved for $GNS5^h$ in the next section can be proved for $GS5^h$ by practically the same proofs. [8]

[8] The rules of the propositional $GS5^h$ are very close to those given by Restall in [19], which are themselves close to Poggiolesi's system in [16]. (We are indebted to an anonymous reviewer for bringing these papers, as well as [6], to our attention.)

3 Soundness, Completeness, Cut-Admissibility

In this section we prove the main properties of $GNS5^h$.

Proposition 3.1 (strong soundness of $GNS5^h$) *Let $\mathcal{H} \cup \{H\}$ be a set of hypersequents. If $\mathcal{H} \vdash_{GNS5^h} G$ then $\mathcal{H} \vdash_{\mathbf{NS5}} H$.*

Proof. It is easy to see that the axioms of $GNS5^h$, its structural rules, and its logical rules for the positive connectives, all preserve validity of hypersequents in frames. Now we show that the same applies to the two negation rules.

- Suppose that $H \mid \Rightarrow \varphi$ is valid in $\langle W, \nu \rangle$. Then either one of the components of H is valid in $\langle W, \nu \rangle$, or $\Rightarrow \varphi$ is. The second case holds iff φ is valid in $\langle W, \nu \rangle$, implying that $\neg\varphi \Rightarrow$ is valid in $\langle W, \nu \rangle$. Hence in both cases one of the components of $H \mid \neg\varphi \Rightarrow$ is valid in $\langle W, \nu \rangle$.
- Suppose that $H \mid \varphi, \Gamma \Rightarrow \Delta$ is valid in $\langle W, \nu \rangle$. Then either one of the components of H is valid in $\langle W, \nu \rangle$, or $\varphi, \Gamma \Rightarrow \Delta$ is. If the first case holds then obviously $H \mid \Gamma \Rightarrow \Delta \mid \Rightarrow \neg\varphi$ is valid in $\langle W, \nu \rangle$. So assume that $\varphi, \Gamma \Rightarrow \Delta$ is valid there. Then either $\nu(w, \varphi) = f$ for some $w \in W$, or $\Gamma \Rightarrow \Delta$ is valid in $\langle W, \nu \rangle$. In the first case $\Rightarrow \neg\varphi$ is valid in $\langle W, \nu \rangle$. Hence in either case $H \mid \Gamma \Rightarrow \Delta \mid \Rightarrow \neg\varphi$ is valid in $\langle W, \nu \rangle$.

□

We turn to the strong completeness of $GNS5^h$ and to the cut-admissibility theorem for it.

Notation. $\mathcal{H} \vdash^{cf}_{GNS5^h} H$ means that there is a proof in $GNS5^h$ of the hypersequent H from the set of hypersequents $\mathcal{H}$ in which each cut is on a formula φ such that $\varphi \in \Gamma \cup \Delta$ for some component $\Gamma \Rightarrow \Delta$ of some hypersequent in $\mathcal{H}$.

Proposition 3.2 (strong completeness of $GNS5^h$) *Let $\mathcal{H} \cup \{H\}$ be a finite set of hypersequents. If $\mathcal{H} \vdash_{\mathbf{NS5}} H$ then $\mathcal{H} \vdash^{cf}_{GNS5^h} H$.*

Proof. Suppose that $\mathcal{H} \nvdash^{cf}_{GNS5^h} H$. We construct a model of $\mathcal{H}$ which is not a model of H.

Let $\mathcal{F}$ be the set of subformulas of formulas in $\mathcal{H} \cup \{H\}$. We call a hypersequent H^* an *$\mathcal{F}$-hypersequent* if it has the following properties:

- Every formula which occurs in H^* belongs to $\mathcal{F}$.
- $\mathcal{H} \nvdash^{cf}_{GNS5^h} H^*$.
- If $\Gamma \cup \Delta \subseteq \mathcal{F}$ then either $\Gamma \Rightarrow \Delta \in H^*$, or $\mathcal{H} \vdash^{cf}_{GNS5^h} H^* \mid \Gamma \Rightarrow \Delta$.

Let $s_1, \ldots, s_n$ be an enumeration of all the sequents $\Gamma \Rightarrow \Delta$ such that $\Gamma \cup \Delta \subseteq \mathcal{F}$. ($n$ is finite because $\mathcal{H}$ is finite, and so $\mathcal{F}$ is finite.) Let $H_0 = H$. Define a sequence $H_1, \ldots, H_n$ of hypersequents by letting $H_i = H_{i-1} \mid s_i$ in case $\mathcal{H} \nvdash^{cf}_{GNS5^h} H_{i-1} \mid s_i$, and $H_i = H_{i-1}$ otherwise. Let $H^* = H_n$. Then H^* is an $\mathcal{F}$-hypersequent such that $H \subseteq H^*$. Call a component $\Gamma^* \Rightarrow \Delta^*$ of H^* *maximal* if it has no proper extension in H^* (i.e. if $\Gamma' \Rightarrow \Delta' \in H^*$, $\Gamma^* \subseteq \Gamma'$ and $\Delta^* \subseteq \Delta'$, then $\Gamma^* = \Gamma'$ and $\Delta^* = \Delta'$). Let W be the set of all maximal components of H^*. For a world $w \in W$, we denote by Γ_w and Δ_w the sets Γ^*

and Δ^* (respectively) such that $w = \Gamma^* \Rightarrow \Delta^*$. Let ν be the valuation defined by $\nu(w,p) = t$ iff $p \in \Gamma_w$ for every atomic variable p (its values for compound formulas are then uniquely determined following Definition 1.2).

We show by induction on the structure of formulas that the following hold for every $\varphi \in \mathcal{F}$ and every maximal component w of H^*:

(i) If $\varphi \in \Gamma_w$ then $\nu(w,\varphi) = t$.

(ii) If $\varphi \in \Delta_w$ then $\nu(w,\varphi) = f$.

- The case where φ is a propositional variable is immediate from the definition of ν, and the fact that if $\varphi \in \Delta_w$ then $\varphi \notin \Gamma_w$ (because $\mathcal{H} \nvdash^{cf}_{GNS5^h} H^*$).
- Suppose that $\varphi = \varphi_1 \wedge \varphi_2$.
  - Suppose $\varphi \in \Gamma_w$. Assume for contradiction that $\{\varphi_1, \varphi_2\} \not\subseteq \Gamma_w$. Then the maximality of w implies that $\varphi_1, \varphi_2, \Gamma_w \Rightarrow \Delta_w$ is not a component of H^*. Since H^* is an $\mathcal{F}$-hypersequent, it follows that $\mathcal{H} \vdash^{cf}_{GNS5^h} H^* \mid \varphi_1, \varphi_2, \Gamma_w \Rightarrow \Delta_w$. By applying $[\wedge \Rightarrow\,]$ to this hypersequent we get that $\mathcal{H} \vdash^{cf}_{GNS5^h} H^* \mid w$, and so that $\mathcal{H} \vdash^{cf}_{GNS5^h} H^*$ (implicitly using [EC]). A contradiction. Hence $\{\varphi_1, \varphi_2\} \subseteq \Gamma_w$, and so $\nu(w,\varphi_1) = \nu(w,\varphi_2) = t$ by the induction hypothesis for φ_1, φ_2. Hence $\nu(w,\varphi) = t$.
  - Suppose $\varphi \in \Delta_w$. Assume for contradiction that $\{\varphi_1, \varphi_2\} \cap \Delta_w = \emptyset$. Then the maximality of w and the fact that H^* is an $\mathcal{F}$-hypersequent imply that both $\mathcal{H} \vdash^{cf}_{GNS5^h} H^* \mid \Gamma_w \Rightarrow \Delta_w, \varphi_1$ and $\mathcal{H} \vdash^{cf}_{GNS5^h} H^* \mid \Gamma_w \Rightarrow \Delta_w, \varphi_2$. By applying $[\,\Rightarrow \wedge]$ to these two hypersequents (together with an implicit use of [EC]) we get that $\mathcal{H} \vdash^{cf}_{GNS5^h} H^*$. A contradiction. Hence either $\varphi_1 \in \Delta_w$ or $\varphi_2 \in \Delta_w$. It follows by the induction hypothesis for φ_1 and φ_2 that either $\nu(w,\varphi_1) = f$ or $\nu(w,\varphi_2) = f$. In both cases $\nu(w,\varphi) = f$.
- The cases $\varphi = \varphi_1 \vee \varphi_2$ and $\varphi = \varphi_1 \supset \varphi_2$ are similar to the case $\varphi = \varphi_1 \wedge \varphi_2$, and are left for the reader.
- Suppose $\varphi = \neg\psi$.
  - Suppose $\varphi \in \Gamma_w$. Assume for contradiction that $\Rightarrow \psi \notin H^*$. Then $\mathcal{H} \vdash^{cf}_{GNS5^h} H^* \mid\ \Rightarrow \psi$, because H^* is an $\mathcal{F}$-hypersequent. By applying $[\neg \Rightarrow\,]$ to this hypersequent followed by internal weakenings, we get that $\mathcal{H} \vdash^{cf}_{GNS5^h} H^* \mid w$, and so that $\mathcal{H} \vdash^{cf}_{GNS5^h} H^*$. A contradiction. Hence $\Rightarrow \psi \in H^*$. It follows that there is a maximal component w' of H^* that extends it, i.e. $\psi \in \Delta_{w'}$. Therefore the induction hypothesis for ψ implies that $\nu(w',\psi) = f$. Hence $\nu(w,\varphi) = t$.
  - Suppose $\varphi \in \Delta_w$. Let $w' \in W$. We show that $\psi \in \Gamma_{w'}$. Assume otherwise. Then $\mathcal{H} \vdash^{cf}_{GNS5^h} H^* \mid \psi, \Gamma_{w'} \Rightarrow \Delta_{w'}$ (by the maximality of w' and the fact that H^* is an $\mathcal{F}$-hypersequent). By applying $[\,\Rightarrow \neg]$ to $H^* \mid \psi, \Gamma_{w'} \Rightarrow \Delta_{w'}$, we get that $\mathcal{H} \vdash^{cf}_{GNS5^h} H^* \mid w' \mid\ \Rightarrow \varphi$, and so (implicitly using [EC]) $\mathcal{H} \vdash^{cf}_{GNS5^h} H^* \mid\ \Rightarrow \varphi$. Since $\varphi \in \Delta_w$, by applying [IW] to $H^* \mid\ \Rightarrow \varphi$ we get $\mathcal{H} \vdash^{cf}_{GNS5^h} H^* \mid w$, and so $\mathcal{H} \vdash^{cf}_{GNS5^h} H^*$. A contradiction. It follows by the induction hypothesis for ψ that $\nu(w,\psi) = t$ for every $w \in W$. Hence $\nu(w,\varphi) = f$.

Suppose now that $\Gamma \Rightarrow \Delta$ is some component of H. Since $H \subseteq H^*$, there is a maximal component w of H^* such that $\Gamma \subseteq \Gamma_w$ and $\Delta \subseteq \Delta_w$. Therefore properties 1 and 2 above imply that if $\varphi \in \Gamma$ then $\nu(w, \varphi) = t$, while if $\varphi \in \Delta$ then $\nu(w, \varphi) = f$. It follows that $\Gamma \Rightarrow \Delta$ is not true in the world w, and so $\Gamma \Rightarrow \Delta$ is not valid in $\langle W, \nu \rangle$. Hence $\langle W, \nu \rangle$ is not a model of H.

Finally, we prove that $\langle W, \nu \rangle$ is a model of $\mathcal{H}$. So let $H' \in \mathcal{H}$. It is impossible that every component of H' is a subsequent of some component of H^*, because otherwise H^* can be derived from H' (and so from $\mathcal{H}$) using just internal and external weakenings ([IW] and [EW]), and this contradicts the fact that $\mathcal{H} \nvdash^{cf}_{GNS5^h} H^*$. Therefore there is a component $\Gamma \Rightarrow \Delta$ of H' which is not a subsequent of any component of H^*. We show that $\Gamma \Rightarrow \Delta$ is valid in $\langle W, \nu \rangle$. So let $w \in W$. Then either $\Gamma \not\subseteq \Gamma_w$, or $\Delta \not\subseteq \Delta_w$. Assume e.g. the former. (The proof in the second case is similar). Then $\varphi \notin \Gamma_w$ for some $\varphi \in \Gamma$. Hence $\mathcal{H} \vdash^{cf}_{GNS5^h} H^* \mid \varphi, \Gamma_w \Rightarrow \Delta_w$ (because $\varphi \in \mathcal{F}$, w is maximal, and H^* is an $\mathcal{F}$-hypersequent). Assume for contradiction that $\varphi \notin \Delta_w$. Then $\mathcal{H} \vdash^{cf}_{GNS5^h} H^* \mid \Gamma_w \Rightarrow \Delta_w, \varphi$ as well. By applying a cut on φ to these two hypersequents, we get that $\mathcal{H} \vdash^{cf}_{GNS5^h} H^* \mid w$, and so that $\mathcal{H} \vdash^{cf}_{GNS5^h} H^*$. A contradiction. It follows that $\varphi \in \Delta_w$, and so $\nu(w, \varphi) = f$ by property 2 above of maximal components of H^*. Since $\varphi \in \Gamma$, this means that $\Gamma \Rightarrow \Delta$ is true in the world w. It follows that $\Gamma \Rightarrow \Delta$ is valid in $\langle W, \nu \rangle$, and so H' is valid $\langle W, \nu \rangle$. □

Theorem 3.3 (strong soundness and completeness) *Let $\mathcal{H} \cup \{H\}$ be a finite set of hypersequents. Then $\mathcal{H} \vdash_{\mathbf{NS5}} H$ iff $\mathcal{H} \vdash_{GNS5^h} H$.*

Proof. Immediate from Propositions 3.1 and 3.2. □

Note 5 Theorem 3.3 can be extended to the case in which $\mathcal{H}$ is an arbitrary set of hypersequents (not necessarily finite). Like in the first-order case which is discussed below, this is done by using infinite hypersequents, letting such an infinite hypersequent H^* follow from a set $\mathcal{H}$ of finite hypersequents iff there is finite subset H of H^* such that $\mathcal{H} \vdash^{cf}_{GNS5^h} H$. We omit the details.

Theorem 3.4 (cut-admissibility) *$GNS5^h$ admits strong cut-admissibility: If $\mathcal{H} \vdash_{GNS5^h} H$ then $\mathcal{H} \vdash^{cf}_{GNS5^h} H$. In particular: If $\vdash_{GNS5^h} H$ then H has a cut-free proof in $GNS5^h$.*

Proof. Immediate from Propositions 3.1 and 3.2. □

4 The First-order Case

In this section we explain how the results of the previous sections can be extended to the first-order level.

4.1 Preliminaries

Let $\mathcal{L}$ be the version of the classical first-order language in which the set of free variables and the set of bounded variables are disjoint (thus in a well-formed formula, the use of the bound variables is always in the scope of a quantification of the same variables). We use the metavariables a, b to range over the free variables, x to range over the bounded variables, p to range over the predicate symbols of $\mathcal{L}$, c to range over its constant symbols, and f to range over its function symbols. The sets of $\mathcal{L}$-terms and $\mathcal{L}$-formulas are defined as usual, and are denoted by $trm_{\mathcal{L}}$ and $frm_{\mathcal{L}}$, respectively. We mainly use t as a metavariable standing for $\mathcal{L}$-terms, φ, ψ for $\mathcal{L}$-formulas, Γ, Δ for finite sets of $\mathcal{L}$-formulas, and $\mathcal{T}, \mathcal{U}$ for (possibly infinite) sets of $\mathcal{L}$-formulas.

Given an $\mathcal{L}$-term t, a free variable a, and another $\mathcal{L}$-term t', we denote by $t\{t'/a\}$ the $\mathcal{L}$-term obtained from t by replacing all occurrences of a by t'. This notation is extended to formulas, sets of formulas, etc. in the obvious way.

To improve readability we use square parentheses in the meta-language, and reserve round parentheses to the first-order language.

4.2 The Hypersequential System $GQNS5^h$

Let $GQNS5^h$ denote the extension of $GNS5^h$ with the rules for the quantifiers of LK^h. (See Figure 1. Note that again, $GQNS5^h$ differs from the classical system LK^h only with respect to its two rules for $\neg$.) For a set $\mathcal{H}$ of hypersequents and a hypersequent H, we write $\mathcal{H} \vdash_{GQNS5^h} H$ if there is a proof in $GQNS5^h$ of H from the set $\mathcal{H}$, and $\mathcal{H} \vdash^{cf}_{GQNS5^h} H$ if there is such a proof in which each cut is on a formula that belongs to some component $\Gamma \Rightarrow \Delta$ of some hypersequent in $\mathcal{H}$.

4.3 The Constant Domain Semantics of NS5

Definition 4.1 An $\mathcal{L}$-*algebra* is a pair $\langle D, I\rangle$ where D is a non-empty domain and I is an interpretation of constant and function symbols of $\mathcal{L}$ such that $I[c] \in D$ for every constant symbol c of $\mathcal{L}$, and $I[f] \in D^n \to D$ for every n-ary function symbol f of $\mathcal{L}$.

Definition 4.2 Let $M = \langle D, I\rangle$ be an $\mathcal{L}$-algebra. An $\langle \mathcal{L}, M\rangle$-evaluation is a function assigning an element in D to every free variable of $\mathcal{L}$. An $\langle \mathcal{L}, M\rangle$-evaluation e is naturally extended to $trm_{\mathcal{L}}$ as follows: $e[c] = I[c]$ for every constant symbol c; and $e[f(t_1, \dots, t_n)] = I[f][e[t_1], \dots, e[t_n]]$ for every $f(t_1, \dots, t_n) \in trm_{\mathcal{L}}$.

Notation. Given an $\langle \mathcal{L}, M\rangle$-evaluation e, a free variable a, and $d \in D$, we denote by $e_{[a:=d]}$ the $\langle \mathcal{L}, M\rangle$-evaluation which is identical to e except that $e_{[a:=d]}[a] = d$.

Definition 4.3 An $\mathcal{L}$-*frame* is a tuple $\mathcal{W} = \langle W, M, \mathcal{I}\rangle$, where W is a set (of "worlds"), $M = \langle D, I\rangle$ is an $\mathcal{L}$-algebra, and $\mathcal{I} = \{I_w\}_{w\in W}$, where for every $w \in W$, I_w is an interpretation of predicate symbols, i.e., a function assigning a subset of D^n to every n-ary predicate symbol of $\mathcal{L}$.

Definition 4.4 Let $\mathcal{W} = \langle W, M, \mathcal{I}\rangle$ be an $\mathcal{L}$-frame, where $M = \langle D, I\rangle$ and $\mathcal{I} = \{I_w\}_{w\in W}$. Let e be an $\langle \mathcal{L}, M\rangle$-evaluation. The satisfaction relation $\models$ is recursively defined as follows:

(i) $\mathcal{W}, w, e \models p(t_1, \dots, t_n)$ iff $\langle e[t_1], \dots, e[t_n]\rangle \in I_w[p]$.

(ii) $\mathcal{W}, w, e \models \varphi_1 \wedge \varphi_2$ iff $\mathcal{W}, w, e \models \varphi_1$ and $\mathcal{W}, w, e \models \varphi_2$.

(iii) $\mathcal{W}, w, e \models \varphi_1 \vee \varphi_2$ iff $\mathcal{W}, w, e \models \varphi_1$ or $\mathcal{W}, w, e \models \varphi_2$.

(iv) $\mathcal{W}, w, e \models \varphi_1 \supset \varphi_2$ iff $\mathcal{W}, w, e \not\models \varphi_1$ or $\mathcal{W}, w, e \models \varphi_2$.

(v) $\mathcal{W}, w, e \models \neg\varphi$ iff $\mathcal{W}, w', e \not\models \varphi$ for some $w' \in W$.

(vi) $\mathcal{W}, w, e \models \forall x(\varphi\{x/a\})$ iff $\mathcal{W}, w, e_{[a:=d]} \models \varphi$ for every $d \in D$.

(vii) $\mathcal{W}, w, e \models \exists x(\varphi\{x/a\})$ iff $\mathcal{W}, w, e_{[a:=d]} \models \varphi$ for some $d \in D$.

It is easy to see that $\models$ is well-defined, and in particular in vi and vii, the exact choice of the free variable a is immaterial.

We now define the consequence relation of **NS5** in semantic terms.

Definition 4.5 Let $\mathcal{T} \cup \{\varphi\}$ be a set of $\mathcal{L}$-formulas. $T \vdash_{\mathbf{NS5}} \varphi$ if $\mathcal{W}, w, e \models \mathcal{T}$ implies $\mathcal{W}, w, e \models \varphi$ for every $\mathcal{L}$-frame $\mathcal{W} = \langle W, M, \mathcal{I}\rangle$, world $w \in W$, and $\langle \mathcal{L}, M\rangle$-evaluation e.

4.4 Soundness, Completeness and Cut-Admissibility

Notation. Given an $\mathcal{L}$-frame $\mathcal{W} = \langle W, M, \mathcal{I}\rangle$, an $\langle \mathcal{L}, M\rangle$-evaluation e, $w \in W$, and a sequent $\Gamma \Rightarrow \Delta$, we write $\mathcal{W}, w, e \models \Gamma \Rightarrow \Delta$ if either $\mathcal{W}, w, e \not\models \varphi$ for some $\varphi \in \Gamma$, or $\mathcal{W}, w, e \models \varphi$ for some $\varphi \in \Delta$.

Definition 4.6 Let $\mathcal{W} = \langle W, M, \mathcal{I}\rangle$ be an $\mathcal{L}$-frame. $\mathcal{W}$ is a *model* of a hypersequent H if for every $\langle \mathcal{L}, M\rangle$-evaluation e, there exists a component $s \in H$ such that $\mathcal{W}, w, e \models s$ for every $w \in W$. $\mathcal{W}$ is a model of a set $\mathcal{H}$ of hypersequents if it is a model of every $H \in \mathcal{H}$.

Definition 4.7 Let $\mathcal{H} \cup \{H\}$ be a set of hypersequents. $\mathcal{H} \vdash^{hs}_{\mathbf{NS5}} H$ iff every $\mathcal{L}$-frame which is a model of $\mathcal{H}$ is also a model of H.

The proof of the next theorem is not difficult:

Theorem 4.8 (strong soundness of $GQNS5^h$) *$GQNS5^h$ is strongly sound with respect to $\vdash^{hs}_{\mathbf{NS5}}$: If $\mathcal{H} \vdash_{GQNS5^h} H$ then $\mathcal{H} \vdash^{hs}_{\mathbf{NS5}} H$.*

The completeness proof is of course more complicated. Essentially, it combines the ideas of the proof of Theorem 3.2 with the method of [11]. In particular: we have to use in it the notions of *extended sequents* and *extended hypersequents.*

Definition 4.9 An *extended sequent* is an ordered pair of (possibly infinite) sets of $\mathcal{L}$-formulas. Given two extended sequents $\mu_1 = \langle \mathcal{T}_1, \mathcal{U}_1\rangle$ and $\mu_2 = \langle \mathcal{T}_2, \mathcal{U}_2\rangle$, we write $\mu_1 \sqsubseteq \mu_2$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2$ and $\mathcal{U}_1 \subseteq \mathcal{U}_2$. An extended sequent is called *finite* if it consists of finite sets of formulas.

Definition 4.10 An *extended hypersequent* is a (possibly infinite) set of extended sequents. Given two extended hypersequents Ω_1, Ω_2, we write $\Omega_1 \sqsubseteq \Omega_2$ (and say that Ω_2 *extends* Ω_1) if for every extended sequent $\mu_1 \in \Omega_1$, there exists $\mu_2 \in \Omega_2$ such that $\mu_1 \sqsubseteq \mu_2$. An extended hypersequent is called *finite* if it consists of finitely many finite extended sequents.

We use the same notations as above for extended sequents and extended hypersequents. For example, we write $\mathcal{T} \Rightarrow \mathcal{U}$ instead of $\langle \mathcal{T}, \mathcal{U}\rangle$, and $\Omega \mid \mathcal{T} \Rightarrow \mathcal{U}$ instead of $\Omega \cup \{\langle \mathcal{T}, \mathcal{U}\rangle\}$.

Definition 4.11 An extended sequent $\mathcal{T} \Rightarrow \mathcal{U}$ admits *the witness property* if the following hold:

(i) If $\forall x(\varphi\{x/a\}) \in \mathcal{U}$ then $\varphi\{b/a\} \in \mathcal{U}$ for some free variable b.

(ii) If $\exists x(\varphi\{x/a\}) \in \mathcal{T}$ then $\varphi\{b/a\} \in \mathcal{T}$ for some free variable b.

Definition 4.12 Let Ω be an extended hypersequent, and $\mathcal{H}$ be a set of (ordinary) hypersequents.

(i) Ω is called *$\mathcal{H}$-consistent* if $\mathcal{H} \not\vdash^{cf}_{GQNS5^h} H$ for every (ordinary) hypersequent $H \sqsubseteq \Omega$.

(ii) Ω is called *internally $\mathcal{H}$-maximal with respect to an $\mathcal{L}$-formula φ* if for every $\mathcal{T} \Rightarrow \mathcal{U} \in \Omega$:
   (a) If $\varphi \notin \mathcal{T}$ then $\Omega \mid \mathcal{T}, \varphi \Rightarrow \mathcal{U}$ is not $\mathcal{H}$-consistent.
   (b) If $\varphi \notin \mathcal{U}$ then $\Omega \mid \mathcal{T} \Rightarrow \mathcal{U}, \varphi$ is not $\mathcal{H}$-consistent.

(iii) Ω is called *internally $\mathcal{H}$-maximal* if it is internally $\mathcal{H}$-maximal with respect to any $\mathcal{L}$-formula.

(iv) Let s be a sequent. Ω is called *externally $\mathcal{H}$-maximal with respect to s* if either $\{s\} \sqsubseteq \Omega$, or $\Omega \mid s$ is not $\mathcal{H}$-consistent.

(v) Ω is called *externally $\mathcal{H}$-maximal* if it is externally $\mathcal{H}$-maximal with respect to any sequent of the form $\Rightarrow \varphi$.

(vi) Ω admits *the witness property* if every $\mu \in \Omega$ admits the witness property.

(vii) Ω is called *$\mathcal{H}$-maximal* if it is $\mathcal{H}$-consistent, internally $\mathcal{H}$-maximal, externally $\mathcal{H}$-maximal, and it admits the witness property.

Less formally, an extended hypersequent Ω is internally $\mathcal{H}$-maximal if every new formula added on some side of some component of Ω would make it $\mathcal{H}$-inconsistent. Similarly, Ω is externally $\mathcal{H}$-maximal if every new sequent of the form $\Rightarrow \varphi$ added to Ω would make it $\mathcal{H}$-inconsistent.

Obviously, every hypersequent is an extended hypersequent, and so all of these properties apply to (ordinary) hypersequents as well.

Next we list, without proofs, a sequence of lemmas which are needed for the proof of the completeness of $GQNS5^h$.

Lemma 4.13 *Let Ω be an extended hypersequent that is internally $\mathcal{H}$-maximal with respect to an $\mathcal{L}$-formula φ. For every $\mathcal{T} \Rightarrow \mathcal{U} \in \Omega$:*

(i) *If $\varphi \notin \mathcal{T}$, then $\mathcal{H} \vdash^{cf}_{GQNS5^h} H \mid \Gamma, \varphi \Rightarrow \Delta$ for some hypersequent $H \sqsubseteq \Omega$ and sequent $\Gamma \Rightarrow \Delta \sqsubseteq \mathcal{T} \Rightarrow \mathcal{U}$.*

(ii) *If $\varphi \notin \mathcal{U}$, then $\mathcal{H} \vdash^{cf}_{GQNS5^h} H \mid \Gamma \Rightarrow \Delta, \varphi$ for some hypersequent $H \sqsubseteq \Omega$ and sequent $\Gamma \Rightarrow \Delta \sqsubseteq \mathcal{T} \Rightarrow \mathcal{U}$.*

Lemma 4.14 *Let Ω be an extended hypersequent that is externally $\mathcal{H}$-maximal with respect to a sequent s. If $\{s\} \not\sqsubseteq \Omega$, then there exists a hypersequent $H \sqsubseteq \Omega$ such that $\mathcal{H} \vdash^{cf}_{GQNS5^h} H \mid s$.*

Lemma 4.15 *Let $\mathcal{H}$ be a set of hypersequents, and let $H = \Gamma_1 \Rightarrow \Delta_1 \mid ... \mid \Gamma_n \Rightarrow \Delta_n$ be a $\mathcal{H}$-consistent finite extended hypersequent. Then, there exists a $\mathcal{H}$-consistent finite extended hypersequent H' of the form $\Gamma'_1 \Rightarrow \Delta'_1 \mid ... \mid \Gamma'_n \Rightarrow \Delta'_n$, such that $\Gamma_i \subseteq \Gamma'_i$ and $\Delta_i \subseteq \Delta'_i$ for every $1 \leq i \leq n$, and H' admits the witness property.*

Lemma 4.16 *Let $\mathcal{H}$ be a set of hypersequents, and $H = \Gamma_1 \Rightarrow \Delta_1 \mid ... \mid \Gamma_n \Rightarrow \Delta_n$ be a $\mathcal{H}$-consistent finite extended hypersequent. Let φ be an $\mathcal{L}$-formula, and $\Gamma^* \Rightarrow \Delta^*$ be a sequent. Then, there exists a $\mathcal{H}$-consistent finite extended hypersequent H', such that:*

- *$H' = \Gamma'_1 \Rightarrow \Delta'_1 \mid ... \mid \Gamma'_{n'} \Rightarrow \Delta'_{n'}$, where $n' \in \{n, n+1\}$, $\Gamma_i \subseteq \Gamma'_i$ and $\Delta_i \subseteq \Delta'_i$ for every $1 \leq i \leq n$.*
- *H' is internally $\mathcal{H}$-maximal with respect to φ.*
- *H' is externally $\mathcal{H}$-maximal with respect to $\Gamma^* \Rightarrow \Delta^*$.*
- *H' admits the witness property.*

Lemma 4.17 *Let $\mathcal{H}$ be a set of hypersequents. Every $\mathcal{H}$-consistent hypersequent can be extended to a $\mathcal{H}$-maximal extended hypersequent Ω.*

Next we define the $\mathcal{L}$-algebra used in the completeness proof.

Definition 4.18 The *Herbrand $\mathcal{L}$-algebra* is an $\mathcal{L}$-algebra, $\langle D, I \rangle$, such that $D = trm_{\mathcal{L}}$ (the set of *all* $\mathcal{L}$-terms), $I[c] = c$ for every constant c, and $I[f][t_1, ..., t_n] = f(t_1, ..., t_n)$ for every n-ary function symbol f and $t_1, ..., t_n \in D$.

Note that the domain of the Herbrand $\mathcal{L}$-algebra contains also non-closed terms. However, recall that we assume that the set of free variables and the set of bounded variables are disjoint, so an $\mathcal{L}$-term cannot contain a bounded variable.

We are now ready to establish the main completeness theorem.

Theorem 4.19 (strong completeness of $GNS5^h$) *Let $\mathcal{H}_0$ be a set of hypersequents closed under substitutions, and H_0 be a hypersequent. If $\mathcal{H}_0 \vdash^{hs}_{\mathbf{NS5}} H_0$ then $\mathcal{H}_0 \vdash^{cf}_{GQNS5^h} H_0$.*

Outline of Proof: Assume that $\mathcal{H}_0 \not\vdash^{cf}_{GQNS5^h} H_0$. We construct an $\mathcal{L}$-frame $\mathcal{W}$ that is a model of $\mathcal{H}_0$ but not of H_0. The availability of external and internal weakenings ensures that H_0 is $\mathcal{H}_0$-consistent. Thus by Lemma 4.17, there exists a $\mathcal{H}_0$-maximal extended hypersequent Ω such that $H_0 \sqsubseteq \Omega$. Using Ω, $\mathcal{W} = \langle W, M, \mathcal{I} \rangle$ is defined as follows:

- $W = \Omega$.
- $M = \langle D, I \rangle$ is the Herbrand $\mathcal{L}$-algebra.
- $\mathcal{I} = \{I_w\}_{w \in W}$ where $\langle t_1, ..., t_n \rangle \in I_{\mathcal{T} \Rightarrow \mathcal{U}}[p]$ iff $p(t_1, ..., t_n) \in \mathcal{T}$.

Now, let e be the identity $\langle \mathcal{L}, M \rangle$-evaluation (defined by $e[a] = a$ for every free variable a). We prove that the following hold for every $w = \mathcal{T} \Rightarrow \mathcal{U} \in W$:

(a) If $\psi \in \mathcal{T}$ then $\mathcal{W}, w, e \models \psi$.

(b) If $\psi \in \mathcal{U}$ then $\mathcal{W}, w, e \not\models \psi$.

(a) and **(b)** are proved together by induction on the complexity of ψ.

Once **(a)** and **(b)** are established, one shows that $\mathcal{W}$ is a model of $\mathcal{H}_0$ but not of H_0. □

The following corollary establishes the link to our logic (cf. Definition 4.5).

Corollary 4.20 *Let $\Gamma \cup \{\varphi\}$ be a finite set of $\mathcal{L}$-formulas. Then, $\Gamma \vdash_{\mathbf{NS5}} \varphi$ iff $\vdash_{GQNS5^h} \Gamma \Rightarrow \varphi$.*

Taken together, Theorems 4.8 and 4.19 naturally entail the following strong cut-admissibility result.

Corollary 4.21 *$\mathcal{H} \vdash_{GQNS5^h} H$ implies $\mathcal{H} \vdash^{cf}_{GQNS5^h} H$, for every set $\mathcal{H}$ of hypersequents closed under substitutions, and a hypersequent H. In particular, for every hypersequent H, $\vdash_{GQNS5^h} H$ implies that there exists a cut-free derivation of H in $GQNS5^h$.*

Note that it is necessary to require that the set of assumptions $\mathcal{H}$ is closed under substitutions. Indeed, $\Rightarrow p(a) \vdash_{GQNS5^h} \Rightarrow p(b)$, but if $a \neq b$ there is no derivation of $\Rightarrow p(b)$ from $\Rightarrow p(a)$ in $GQNS5^h$ with cuts only on $p(a)$.

References

[1] Avron, A., *Natural 3-valued logics: Characterization and proof theory*, Journal of Symbolic Logic **56** (1991), pp. 276–294.

[2] Avron, A., *The method of hypersequents in proof theory of propositional non-classical logics*, in: W. Hodges, M. Hyland, C. Steinhorn and J. Truss, editors, *Logic: Foundations to Applications*, Oxford Science Publications, 1996 pp. 1–32.

[3] Avron, A., *Cut-elimination in* **RM** *proved semantically*, IFCoLog Journal of Logics and their Applications **4** (2017), pp. 605–621.

[4] Avron, A. and A. Zamansky, *Paraconsistency, self-extensionality, modality*, to appear in a special issue of The Logic Journal of the IGPL.

[5] Batens, D., *On some remarkable relations between paraconsistent logics, modal logics, and ambiguity logics*, in: W. A. Carnielli, M. E. Coniglio and I. D'Ottaviano, editors, *Paraconsistency: The Logical Way to the Inconsistent*, number 228 in Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, 2002 pp. 275–293.

[6] Bednarska, K. and A. Indrzejczak, *Hypersequent calculi for S5: The methods of cut elimination*, Logic and Logical Philosophy **24** (2015), pp. 277–311.

[7] Béziau, J. Y., *S5 is a paraconsistent logic and so is first-order classical logic*, Logical Investigations **8** (2002), pp. 301–309.

[8] Béziau, J. Y., *Paraconsistent logic from a modal viewpoint*, Journal of Applied Logic **3** (2005), pp. 6–14.

[9] Béziau, J. Y., *The paraconsistent logic Z. A possible solution to Jaśkowski's problem*, Logic and Logical Philosophy **15** (2006), pp. 99–111.

[10] Bimbó, K., "Proof Theory: Sequent Calculi and Related Formalisms," Discrete Mathematics and Its Applications, CRC Press, 2015.

[11] Lahav, O. and A. Avron, *A semantic proof of strong cut-admissibility for first-order Gödel logic*, Journal of Logic and Computation **23** (2013), pp. 59–86.

[12] Mints, G. E., *Some calculi of modal logic*, Proc. Steklov Inst. of Mathematics **98** (1968), pp. 97–122.

[13] Nerode, A. and R. A. Shore, "Logic for Applications," Springer, 1997.

[14] Osorio, M., J. L. Carballido and C. Zepeda, *Revisiting* $\mathbb{Z}$, Notre Dame Journal of Formal Logic **55** (2014), pp. 129–155.

[15] Osorio, M., J. L. Carballido, C. Zepeda and J. A. Castellanos, *Weakening and extending* $\mathbb{Z}$, Logica Universalis **9** (2015), pp. 383–409.

[16] Poggiolesi, F., *A cut-free simple sequent calculus for modal logic S5*, The Review of Symbolic Logic **1** (2008), p. 315.

[17] Poggiolesi, P., "Gentzen Calculi for Modal Propositional Logic," Trends in Logic **32**, Springer, 2010.

[18] Pottinger, G., *Uniform, cut-free formulations of T, S4 and S5 (abstract)*, Journal of Symbolic Logic **48** (1983), p. 900.

[19] Restall, G., *Proofnets for S5: Sequents and circuits for modal logic*, in: C. Dimitracopoulos, L. Newelski and D. Normann, editors, *Logic Colloquium 2005*, Cambridge: Cambridge University Press, 2007 pp. 151–172.

[20] Wansing, H., *Sequent systems for modal logics*, in: D. Gabbay and F. Guenther, editors, *Handbook of Philosophical Logic, 2nd edition, Vol. 8*, Kluwer, 2002 pp. 61–145.