Strict Canonical Constructive Systems*

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To Yuri, on his seventieth birthday.

Abstract. We define the notions of a canonical inference rule and a canonical constructive system in the framework of strict single-conclusion Gentzen-type systems (or, equivalently, natural deduction systems), and develop a corresponding general non-deterministic Kripke-style semantics. We show that every strict constructive canonical system induces a class of non-deterministic Kripke-style frames, for which it is strongly sound and complete. This non-deterministic semantics is used for proving a strong form of the cut-elimination theorem for such systems, and for providing a decision procedure for them. These results identify a large family of basic constructive connectives, including the standard intuitionistic connectives, together with many other independent connectives.

Keywords: sequent calculus, cut-elimination, non-classical logics, non-deterministic semantics, Kripke semantics.

1 Introduction

The standard intuitionistic connectives ($\supset, \wedge, \vee$, and $\bot$) are of great importance in theoretical computer science, especially in type theory, where they correspond to basic operations on types (via the formulas-as-types principle and Curry-Howard isomorphism). Now a natural question is: what is so special about these connectives? The standard answer is that they are all *constructive* connectives. But then what exactly is a constructive connective, and can we define other basic constructive connectives beyond the four intuitionistic ones? And what does the last question mean anyway: how do we "define" new (or old) connectives?

Concerning the last question there is a long tradition starting from [12] (see e.g. [16] for discussions and references) according to which the meaning of a connective is determined by the introduction and elimination rules which are associated with it. Here one usually has in mind natural deduction systems of an ideal type, where each connective has its own introduction and elimination rules, and these rules should meet the following conditions: in a rule for some connective this connective should be mentioned exactly once, and no other connective should be involved. The rule should also be pure in the sense of [1] (i.e. there should be no side conditions limiting its application), and its active formulas should be immediate

* This research was supported by The Israel Science Foundation (grant no. 809-06).

A. Blass, N. Dershowitz, and W. Reisig (Eds.): Gurevich Festschrift, LNCS 6300, pp. 75–94, 2010.

subformulas of its principal formula. Now an n-ary connective $\diamond$ that can be defined using such rules may be taken as constructive if in order to prove the logical validity of a sentence of the form $\diamond(\varphi_1, \ldots, \varphi_n)$, it is necessary to prove first the premises of one of its possible introduction rules (see [9]).

Unfortunately, already the handling of negation requires rules which are not ideal in the sense described above. For intuitionistic logic this problem is usually solved by not taking negation as a basic constructive connective, but defining it instead in terms of more basic connectives that can be characterized by "ideal" rules ($\neg\varphi$ is defined as $\varphi \rightarrow\perp$). In contrast, for classical logic the problem was solved by Gentzen himself by moving to what is now known as Gentzen-type systems or sequential calculi. These calculi employ single-conclusion sequents in their intuitionistic version, and multiple-conclusion sequents in their classical version. Instead of introduction and elimination rules they use left introduction rules and right introduction rules. The intuitive notions of an "ideal rule" can be adapted to such systems in a straightforward way, and it is well known that the usual classical connectives and the basic intuitionistic connectives can indeed be fully characterized by "ideal" Gentzen-type rules. Moreover: although this can be done in several ways, in all of them the cut-elimination theorem obtains. This immediately implies that the connectives of intuitionistic logic are constructive in the sense explained above, because without using cuts the only way to derive $\Rightarrow \diamond(\varphi_1, \ldots, \varphi_n)$ in single conclusion systems of this sort is to prove first the premises of one of its introduction rules (and then apply that introduction rule). Note that the only formulas that can occur in such premises are $\varphi_1, \ldots, \varphi_n$.

For the multiple-conclusion framework the above-mentioned facts about the classical connectives were considerably generalized in [6,7] by defining "multiple-conclusion canonical propositional Gentzen-type rules and systems" in precise terms. A constructive necessary and sufficient *coherence* criterion for the non-triviality of such systems was then provided, and it was shown that a system of this kind admits cut-elimination iff it is coherent. It was further proved that the semantics of such systems is provided by two-valued non-deterministic matrices (two-valued Nmatrices) – a natural generalization of the classical truth-tables. In fact, a characteristic two-valued Nmatrix was constructed for every coherent canonical propositional system. That work shows that there is a large family of what may be called semi-classical connectives (which includes all the classical connectives), each of which has both a proof-theoretical characterization in terms of a coherent set of canonical (= "ideal") rules, and a semantic characterization using two-valued Nmatrices.

In this paper we develop a similar theory for the constructive propositional framework. We define the notions of a canonical rule and a canonical system in the framework of strict single-conclusion Gentzen-type systems (or, equivalently, natural deduction systems). We prove that here too a canonical system is non-trivial iff it is coherent (where coherence is a constructive condition, defined like in the multiple-conclusion case). We develop a general non-deterministic Kripke-style semantics for such systems, and show that every constructive canonical system (i.e. coherent canonical single-conclusion system) induces a class of non-deterministic

Kripke-style frames for which it is strongly sound and complete. We use this non-deterministic semantics to show that all constructive canonical systems admit a strong form of the cut-elimination theorem. We also use it for providing decision procedures for all such systems. These results again identify a large family of basic constructive connectives, each having both a proof-theoretical characterization in terms of a coherent set of canonical rules, and a semantic characterization using non-deterministic frames. The family includes the standard intuitionistic connectives ($\supset, \wedge, \vee$, and $\perp$), as well as many other independent connectives, like the semi-implication which has been introduced and used by Gurevich and Neeman in [13].[1]

2 Strict Canonical Constructive Systems

In what follows $\mathcal{L}$ is a propositional language, $\mathcal{F}$ is its set of wffs, p, q, r denote atomic formulas, ψ, φ, θ denote arbitrary formulas (of $\mathcal{L}$), T, S denote subsets of $\mathcal{F}$, and $\Gamma, \Delta, \Sigma, \Pi$ denote finite subsets of $\mathcal{F}$. We assume that the atomic formulas of $\mathcal{L}$ are $p_1, p_2, \ldots$ (in particular: $\{p_1, p_2, \ldots, p_n\}$ are the first n atomic formulas of $\mathcal{L}$).

Definition 1. A *(Tarskian) consequence relation* for $\mathcal{L}$ is a binary relation $\vdash$ between sets of formulas of $\mathcal{L}$ and formulas of $\mathcal{L}$ that satisfies the following conditions:

strong reflexivity: if $\varphi \in T$ then $T \vdash \varphi$.
monotonicity: if $T \vdash \varphi$ and $T \subseteq T'$ then $T' \vdash \varphi$.
transitivity (*cut*): if $T \vdash \psi$ and $T, \psi \vdash \varphi$ then $T \vdash \varphi$.

Definition 2. A *substitution* in $\mathcal{L}$ is a function σ from the atomic formulas to the set of formulas of $\mathcal{L}$. A substitution σ is extended to formulas and sets of formulas in the obvious way.

Definition 3. A consequence relation $\vdash$ for $\mathcal{L}$ is *structural* if for every substitution σ and every T and φ, if $T \vdash \varphi$ then $\sigma(T) \vdash \sigma(\varphi)$. A consequence relation $\vdash$ is *finitary* if the following condition holds for all T and φ: if $T \vdash \varphi$ then there exists a finite $\Gamma \subseteq T$ such that $\Gamma \vdash \varphi$. A consequence relation $\vdash$ is *consistent* (or *non-trivial*) if $p_1 \nvdash p_2$.

It is easy to see (see [7]) that there are exactly two inconsistent structural consequence relations in any given language.[2] These consequence relations are obviously trivial, so we exclude them from our definition of a *logic*:

Definition 4. A propositional *logic* is a pair $\langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ is a propositional language, and $\vdash$ is a consequence relation for $\mathcal{L}$ which is structural, finitary, and consistent.

[1] The results of this paper were first stated in [5] without proofs.
[2] In one $T \vdash \varphi$ for every T and φ; in the other $T \vdash \varphi$ for every nonempty T and φ.

Since a finitary consequence relation $\vdash$ is determined by the set of pairs $\langle \Gamma, \varphi \rangle$ such that $\Gamma \vdash \varphi$, it is natural to base proof systems for logics on the use of such pairs. This is exactly what is done in natural deduction systems and in (strict) single-conclusion Gentzen-type systems (both introduced in [12]). Formally, such systems manipulate objects of the following type:

Definition 5. A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite sets of formulas, and Δ is either a singleton or empty. A sequent of the form $\Gamma \Rightarrow \{\varphi\}$ is called *definite*, and we shall denote it by $\Gamma \Rightarrow \varphi$. A sequent of the form $\Gamma \Rightarrow \{\}$ is called *negative*, and we shall denote it by $\Gamma \Rightarrow$. A Horn *clause* is a sequent which consists of atomic formulas only.

Note 1. Natural deduction systems and the strict single-conclusion Gentzen-type systems investigated in this paper manipulate only definite sequents in their derivations. However, negative sequents may be used in the formulations of their rules (in the form of negative Horn clauses).

The following definitions formulate in exact terms the idea of an "ideal rule" which was described in the introduction. We first formulate these definitions in terms of Gentzen-type systems. We consider natural deduction systems in a separate subsection.

Definition 6.

1. A *strict canonical introduction rule* for a connective $\diamond$ of arity n is an expression constructed from a set of premises and a conclusion sequent, in which $\diamond$ appears in the right side. Formally, it takes the form:
$$\{\Pi_i \Rightarrow \Sigma_i\}_{1 \le i \le m} / \ \Rightarrow \diamond(p_1, p_2, \ldots, p_n)$$
where m can be 0, and for all $1 \le i \le m$, $\Pi_i \Rightarrow \Sigma_i$ is a *definite* Horn clause such that $\Pi_i \cup \Sigma_i \subseteq \{p_1, p_2, \ldots, p_n\}$.
2. A *strict canonical elimination*[3] *rule* for a connective $\diamond$ of arity n is an expression constructed from a set of premises and a conclusion sequent, in which $\diamond$ appears in the left side. Formally, it takes the form:
$$\{\Pi_i \Rightarrow \Sigma_i\}_{1 \le i \le m} / \ \diamond(p_1, p_2, \ldots, p_n) \Rightarrow$$
where m can be 0, and for all $1 \le i \le m$, $\Pi_i \Rightarrow \Sigma_i$ is a Horn clause (either definite or negative) such that $\Pi_i \cup \Sigma_i \subseteq \{p_1, p_2, \ldots, p_n\}$.
3. An *application* of the rule $\{\Pi_i \Rightarrow \Sigma_i\}_{1 \le i \le m} / \ \Rightarrow \diamond(p_1, p_2, \ldots, p_n)$ is any inference step of the form:
$$\frac{\{\Gamma, \sigma(\Pi_i) \Rightarrow \sigma(\Sigma_i)\}_{1 \le i \le m}}{\Gamma \Rightarrow \diamond(\sigma(p_1), \ldots, \sigma(p_n))}$$
where Γ is a finite set of formulas and σ is a substitution in $\mathcal{L}$.

[3] The introduction/elimination terminology comes from the natural deduction context. For the Gentzen-type context the names "right introduction rule" and "left introduction rule" might be more appropriate, but we prefer to use a uniform terminology.

4. An *application* of the rule $\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/\ \diamond(p_1, p_2, \ldots, p_n) \Rightarrow$ is any inference step of the form:

$$\frac{\{\Gamma, \sigma(\Pi_i) \Rightarrow \sigma(\Sigma_i), E_i\}_{1\leq i\leq m}}{\Gamma, \diamond(\sigma(p_1), \ldots, \sigma(p_n)) \Rightarrow \theta}$$

where Γ is a finite set of formulas, σ is a substitution in $\mathcal{L}$, θ is a formula, and for all $1 \leq i \leq m$: $E_i = \{\theta\}$ in case Σ_i is empty, and E_i is empty otherwise.

Here are some examples of well-known strict canonical rules:

Example 1 (Conjunction). The two usual rules for conjunction are:

$$\{p_1, p_2 \Rightarrow\ \}\ /\ p_1 \wedge p_2 \Rightarrow \quad \text{and} \quad \{\ \Rightarrow p_1\ ,\ \ \Rightarrow p_2\}\ /\ \ \Rightarrow p_1 \wedge p_2.$$

Applications of these rules have the form:

$$\frac{\Gamma, \psi, \varphi \Rightarrow \theta}{\Gamma, \psi \wedge \varphi \Rightarrow \theta} \qquad \frac{\Gamma \Rightarrow \psi \quad \Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \psi \wedge \varphi}$$

The above elimination rule can easily be shown to be equivalent to the combination of the two, more usual, elimination rules for conjunction.

Example 2 (Disjunction). The two usual introduction rules for disjunction are:

$$\{\ \Rightarrow p_1\}\ /\ \ \Rightarrow p_1 \vee p_2 \quad \text{and} \quad \{\ \Rightarrow p_2\}\ /\ \ \Rightarrow p_1 \vee p_2.$$

Applications of these rules have then the form:

$$\frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \psi \vee \varphi} \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \psi \vee \varphi}$$

The usual elimination rule for disjunction is:

$$\{p_1 \Rightarrow\ ,\ \ p_2 \Rightarrow\}\ /\ p_1 \vee p_2 \Rightarrow\ .$$

Its applications have the form:

$$\frac{\Gamma, \psi \Rightarrow \theta \quad \Gamma, \varphi \Rightarrow \theta}{\Gamma, \psi \vee \varphi \Rightarrow \theta}$$

Example 3 (Implication). The two usual rules for implication are:

$$\{\Rightarrow p_1\ ,\ \ p_2 \Rightarrow\}\ /\ p_1 \supset p_2 \Rightarrow \quad \text{and} \quad \{p_1 \Rightarrow p_2\}\ /\ \ \Rightarrow p_1 \supset p_2.$$

Applications of these rules have the form:

$$\frac{\Gamma \Rightarrow \psi \quad \Gamma, \varphi \Rightarrow \theta}{\Gamma, \psi \supset \varphi \Rightarrow \theta} \qquad \frac{\Gamma, \psi \Rightarrow \varphi}{\Gamma \Rightarrow \psi \supset \varphi}$$

Example 4 (Absurdity). In intuitionistic logic there is no introduction rule for the absurdity constant $\perp$, and there is exactly one elimination rule for it: $\{\} \;/\; \perp\Rightarrow$. Applications of this rule provide new *axioms*: $\Gamma, \perp\Rightarrow \varphi$.

Example 5 (Semi-implication). In [13] a "semi-implication" $\rightsquigarrow$ was introduced using the following two rules:

$$\{\Rightarrow p_1 \;,\;\; p_2 \Rightarrow\} \;/\; p_1 \rightsquigarrow p_2 \Rightarrow \quad \text{and} \quad \{\Rightarrow p_2\} \;/\; \Rightarrow p_1 \rightsquigarrow p_2.$$

Applications of these rules have the form:

$$\frac{\Gamma \Rightarrow \psi \quad \Gamma, \varphi \Rightarrow \theta}{\Gamma, \psi \rightsquigarrow \varphi \Rightarrow \theta} \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \psi \rightsquigarrow \varphi}$$

Now we define the notion of *a strict canonical Gentzen-type system.*

Definition 7. A single-conclusion Gentzen-type system is called *a strict canonical Gentzen-type system* if the following hold:

- Its axioms are all sequents of the form $\varphi \Rightarrow \varphi$.
- Cut (from $\Gamma \Rightarrow \varphi$ and $\Delta, \varphi \Rightarrow \psi$ infer $\Gamma, \Delta \Rightarrow \psi$) and weakening (from $\Gamma \Rightarrow \psi$ infer $\Gamma, \Delta \Rightarrow \psi$) are among its rules.
- Each of its other rules is either a strict canonical introduction rule or a strict canonical elimination rule.

Definition 8. Let $\mathbf{G}$ be a strict canonical Gentzen-type system.

1. A derivation of a sequent s from a set of sequents $\mathcal{S}$ in $\mathbf{G}$ is a sequence of sequents, such that each sequent in it is either an axiom, belongs to $\mathcal{S}$, or follows from previous sequents by a canonical rule of $\mathbf{G}$. If such a derivation exists, we denote $\mathcal{S} \vdash^{seq}_{\mathbf{G}} s$.
2. The consequence relation $\vdash_{\mathbf{G}}$ between *formulas* which is induced by $\mathbf{G}$ is defined by: $T \vdash_{\mathbf{G}} \varphi$ iff there exists a finite $\Gamma \subseteq T$ such that $\vdash^{seq}_{\mathbf{G}} \Gamma \Rightarrow \varphi$.

Proposition 1. *$T \vdash_{\mathbf{G}} \varphi$ iff $\{\Rightarrow \psi \mid \psi \in T\} \vdash^{seq}_{\mathbf{G}} \Rightarrow \varphi$.*

Proposition 2. *If $\mathbf{G}$ is strict canonical then $\vdash_{\mathbf{G}}$ is a structural and finitary consequence relation.*

The last proposition does not guarantee that every strict canonical system induces a *logic* (see Definition 4). For this the system should satisfy one more condition:

Definition 9. A set $\mathcal{R}$ of strict canonical rules for an n-ary connective $\diamond$ is called *coherent* if $S_1 \cup S_2$ is classically inconsistent (and so the empty clause can be derived from it using cuts) whenever $\mathcal{R}$ contains both $S_1/\diamond(p_1, p_2, \ldots, p_n) \Rightarrow$ and $S_2/ \;\Rightarrow \diamond(p_1, p_2, \ldots, p_n)$.

Example 6. All the sets of rules for the connectives $\wedge, \vee, \supset, \bot$, and $\leadsto$ which were introduced in the examples above are coherent. For example, for the two rules for conjunction we have $S_1 = \{p_1, p_2 \Rightarrow\ \}$, $S_2 = \{\ \Rightarrow p_1\ ,\ \ \Rightarrow p_2\}$, and $S_1 \cup S_2$ is the classically inconsistent set $\{p_1, p_2 \Rightarrow\ ,\ \ \Rightarrow p_1\ ,\ \ \Rightarrow p_2\}$ (from which the empty sequent can be derived using two cuts).

Example 7. In [15] Prior introduced a "connective" T (which he called "Tonk") with the following rules: $\{p_1 \Rightarrow\ \}\ /\ p_1 T p_2 \Rightarrow$ and $\{\ \Rightarrow p_2\}\ /\ \ \Rightarrow p_1 T p_2$. Prior then used "Tonk" to infer everything from everything (trying to show by this that a set of rules might not define any connective). Now the union of the sets of premises of these two rules is $\{p_1 \Rightarrow\ ,\ \Rightarrow p_2\}$, and this is a classically consistent set of clauses. It follows that Prior's set of rules for Tonk is incoherent.

Definition 10. A strict canonical single-conclusion Gentzen-type system $\mathbf{G}$ is called *coherent* if every primitive connective of the language of $\mathbf{G}$ has a coherent set of rules in $\mathbf{G}$.

Theorem 1. *Let $\mathbf{G}$ be a strict canonical Gentzen-type system. $\langle \mathcal{L}, \vdash_{\mathbf{G}} \rangle$ is a logic (i.e. $\vdash_{\mathbf{G}}$ is structural, finitary and consistent) iff $\mathbf{G}$ is coherent.*

Proof. Proposition 2 ensures that $\vdash_{\mathbf{G}}$ is a structural and finitary consequence relation.

That the coherence of $\mathbf{G}$ implies the consistency of the *multiple* conclusion consequence relation which is naturally induced by $\mathbf{G}$ was shown in [6,7]. That consequence relation extends $\vdash_{\mathbf{G}}$, and therefore also the latter is consistent.

For the converse, assume that $\mathbf{G}$ is incoherent. This means that $\mathbf{G}$ includes two rules $S_1/ \diamond(p_1, \ldots, p_n) \Rightarrow$ and $S_2/\ \Rightarrow \diamond(p_1, \ldots, p_n)$, such that the set of clauses $S_1 \cup S_2$ is classically satisfiable. Let v be an assignment in $\{t, f\}$ that satisfies all the clauses in $S_1 \cup S_2$. Define a substitution σ by:

$$\sigma(p) = \begin{cases} p_{n+1} & v(p) = f \\ p & v(p) = t \end{cases}$$

Let $\Pi \Rightarrow q \in S_1 \cup S_2$. Then $\vdash_G^{seq} p_1, \ldots, p_n, \sigma(\Pi) \Rightarrow \sigma(q)$. This is trivial in case $v(q) = t$, since in this case $\sigma(q) = q \in \{p_1, \ldots, p_n\}$. On the other hand, if $v(q) = f$ then $v(p) = f$ for some $p \in \Pi$ (since v satisfies the clause $\Pi \Rightarrow q$). Therefore in this case $\sigma(p) = \sigma(q) = p_{n+1}$, and so again $p_1, \ldots, p_n, \sigma(\Pi) \Rightarrow \sigma(q)$ is trivially derived from an axiom. We can similarly prove that $\vdash_{\mathbf{G}}^{seq} p_1, \ldots, p_n, \sigma(\Pi) \Rightarrow p_{n+1}$ in case $\Pi \Rightarrow\ \in S_1 \cup S_2$. Now by applying the rules $S_1/ \diamond(p_1, \ldots, p_n) \Rightarrow$ and $S_2/\ \Rightarrow \diamond(p_1, \ldots, p_n)$ to these provable sequents we get proofs in $\mathbf{G}$ of the sequent $p_1, \ldots, p_n \Rightarrow \diamond(\sigma(p_1), \ldots, \sigma(p_n))$ and of $p_1, \ldots, p_n, \diamond(\sigma(p_1), \ldots, \sigma(p_n)) \Rightarrow p_{n+1}$. That $\vdash_{\mathbf{G}}^{seq} p_1, \ldots, p_n \Rightarrow p_{n+1}$ then follows using a cut. This easily entails that $p_1 \vdash_{\mathbf{G}} p_2$, and hence $\vdash_{\mathbf{G}}$ is not consistent. □

The last theorem implies that coherence is a minimal demand from any acceptable strict canonical Gentzen-type system $\mathbf{G}$. It follows that not every set of such rules is legitimate for defining constructive connectives – only coherent ones do (and this is what is wrong with "Tonk"). Accordingly we define:

Definition 11. A *strict canonical constructive system* is a coherent strict canonical single-conclusion Gentzen-type system.

The following definition will be needed in the sequel:

Definition 12. Let $\mathcal{S}$ be a set of sequents.

1. A cut is called an $\mathcal{S}$-cut if the cut formula occurs in $\mathcal{S}$.
2. We say that there exists in a system **G** an $\mathcal{S}$-(cut-free) proof of a sequent s from a set of sequents $\mathcal{S}$ iff there exists a proof of s from $\mathcal{S}$ in **G** where all cuts are $\mathcal{S}$-cuts.
3. ([2]) A system **G** admits strong cut-elimination iff whenever $\mathcal{S} \vdash^{seq}_{\mathbf{G}} s$, there exists an $\mathcal{S}$-(cut-free) proof of s from $\mathcal{S}$.[4]

2.1 Natural Deduction Version

We formulated the definitions above in terms of Gentzen-type systems. However, we could have formulated them instead in terms of natural deduction systems. The definition of canonical rules in this context is exactly as above. An application of an introduction rule is also defined exactly as above, while an application of an elimination rule of the form $\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m} / \ \diamond(p_1, p_2, \ldots, p_n) \Rightarrow$ is, in the context of natural deduction, any inference step of the form:

$$\frac{\{\Gamma, \sigma(\Pi_i) \Rightarrow \sigma(\Sigma_i), E_i\}_{1\leq i\leq m} \qquad \Gamma \Rightarrow \diamond(\sigma(p_1), \ldots, \sigma(p_n))}{\Gamma \Rightarrow \theta}$$

where Γ, σ, θ and E_i are as above: Γ is a finite set of formulas, σ is a substitution in $\mathcal{L}$, θ is a formula, and for all $1 \leq i \leq m$: $E_i = \{\theta\}$ in case Σ_i is empty, and E_i is empty otherwise. We present some examples of the natural deduction version of well-known strict canonical rules. Translating our other notions and results to natural deduction systems is easy.

Example 8 (Conjunction). Applications of the rule $\{p_1, p_2 \Rightarrow \} \ / \ p_1 \wedge p_2 \Rightarrow$ have here the form:

$$\frac{\Gamma, \psi, \varphi \Rightarrow \theta \qquad \Gamma \Rightarrow \psi \wedge \varphi}{\Gamma \Rightarrow \theta}$$

Example 9 (Disjunction). Applications of the rule $\{p_1 \Rightarrow \ , \quad p_2 \Rightarrow\} \ / \ p_1 \vee p_2 \Rightarrow$ have here the form:

$$\frac{\Gamma, \psi \Rightarrow \theta \qquad \Gamma, \varphi \Rightarrow \theta \qquad \Gamma \Rightarrow \psi \vee \varphi}{\Gamma \Rightarrow \theta}$$

Example 10 (Implication). Applications of the rule $\{\Rightarrow p_1 \ , \quad p_2 \Rightarrow\} \ / \ p_1 \supset p_2 \Rightarrow$ have here the form:

[4] By cut-elimination we mean here just the existence of proofs without (certain forms of) cuts, rather than an algorithm to transform a given proof to a cut-free one (for the assumption-free case the term "cut-admissibility" is sometimes used).

$$\frac{\Gamma \Rightarrow \psi \quad \Gamma, \varphi \Rightarrow \theta \quad \Gamma \Rightarrow \psi \supset \varphi}{\Gamma \Rightarrow \theta}$$

This form of the rule is obviously equivalent to the more usual one (from $\Gamma \Rightarrow \psi$ and $\Gamma \Rightarrow \psi \supset \varphi$ infer $\Gamma \Rightarrow \varphi$).

Example 11 (Absurdity). In natural-deduction systems applications of the rule $\{\} \ / \ \bot \Rightarrow$ for the absurdity constant allow us to infer $\Gamma \Rightarrow \varphi$ from $\Gamma \Rightarrow \bot$.

3 Semantics for Strict Canonical Constructive Systems

The most useful semantics for propositional intuitionistic logic (the paradigmatic constructive logic) is that of Kripke frames. In this section we generalize this semantics to arbitrary strict canonical constructive systems. For this we should introduce *non-deterministic* Kripke frames.[5]

Definition 13. A generalized $\mathcal{L}$-*frame* is a triple $\mathcal{W} = \langle W, \leq, v\rangle$ such that:

1. $\langle W, \leq\rangle$ is a nonempty partially ordered set.
2. v is a function from $\mathcal{F}$ to the set of persistent functions from W into $\{t, f\}$ A function $h : W \to \{t, f\}$ is *persistent* if $h(a) = t$ implies that $h(b) = t$ for every $b \in W$ such that $a \leq b$.

Notation: We shall usually write $v(a, \varphi)$ instead of $v(\varphi)(a)$.

Definition 14. A generalized $\mathcal{L}$-frame $\langle W, \leq, v\rangle$ is a *model* of a formula φ if $v(\varphi) = \lambda a \in W.t$ (i.e. $v(a, \varphi) = t$ for every $a \in W$). It is a model of a theory T if it is a model of every $\varphi \in T$.

Definition 15. Let $\mathcal{W} = \langle W, \leq, v\rangle$ be a generalized $\mathcal{L}$-frame, and let $a \in W$.

1. A sequent $\Gamma \Rightarrow \varphi$ is *locally true* in a if either $v(a, \psi) = f$ for some $\psi \in \Gamma$, or $v(a, \varphi) = t$.
2. A sequent $\Gamma \Rightarrow \varphi$ is *true* in a if it is locally true in every $b \geq a$.
3. A sequent $\Gamma \Rightarrow$ is *(locally) true* in a if $v(a, \psi) = f$ for some $\psi \in \Gamma$.
4. The generalized $\mathcal{L}$-frame $\mathcal{W}$ is a *model* of a sequent s (either of the form $\Gamma \Rightarrow \varphi$ or $\Gamma \Rightarrow$) if s is true in every $a \in W$ (iff s is locally true in every $a \in W$). It is a model of a set of sequents $\mathcal{S}$ if it is a model of every $s \in \mathcal{S}$.

Note 2. The generalized $\mathcal{L}$-frame $\mathcal{W}$ is a model of a formula φ iff it is a model of the sequent $\Rightarrow \varphi$.

Definition 16. Let $\langle W, \leq, v\rangle$ be a generalized $\mathcal{L}$-frame. A substitution σ in $\mathcal{L}$ satisfies a Horn clause $\Pi \Rightarrow \Sigma$ in $a \in W$ if $\sigma(\Pi) \Rightarrow \sigma(\Sigma)$ is true in a.

[5] Another type of non-deterministic (intuitionistic) Kripke frames, based on 3-valued and 4-valued non-deterministic matrices, was used in [3,4]. Non-deterministic modal Kripke frames were recently used in [11].

Note 3. Because of the persistence condition, a definite Horn clause of the form $\Rightarrow q$ is satisfied in a by σ iff $v(a, \sigma(q)) = t$.

Definition 17. Let $\mathcal{W} = \langle W, \leq, v \rangle$ be a generalized $\mathcal{L}$-frame, and let $\diamond$ be an n-ary connective of $\mathcal{L}$.

1. The frame $\mathcal{W}$ *respects* an introduction rule r for $\diamond$ if $v(a, \diamond(\psi_1, \ldots, \psi_n)) = t$ whenever all the premises of r are satisfied in a by a substitution σ such that $\sigma(p_i) = \psi_i$ for $1 \leq i \leq n$ (The values of $\sigma(q)$ for $q \notin \{p_1, \ldots, p_n\}$ are immaterial here).
2. The frame $\mathcal{W}$ *respects* an elimination rule r for $\diamond$ if $v(a, \diamond(\psi_1, \ldots, \psi_n)) = f$ whenever all the premises of r are satisfied in a by a substitution σ such that $\sigma(p_i) = \psi_i$ $(1 \leq i \leq n)$.
3. Let **G** be a strict canonical Gentzen-type system for $\mathcal{L}$. The generalized $\mathcal{L}$-frame $\mathcal{W}$ is **G**-legal if it respects all the rules of **G**.

Example 12. By definition, a generalized $\mathcal{L}$-frame $\mathcal{W} = \langle W, \leq, v \rangle$ respects the rule $(\supset\Rightarrow)$ iff for every $a \in W$, $v(a, \varphi \supset \psi) = f$ whenever $v(b, \varphi) = t$ for every $b \geq a$ and $v(a, \psi) = f$. Because of the persistence condition, this is equivalent to $v(a, \varphi \supset \psi) = f$ whenever $v(a, \varphi) = t$ and $v(a, \psi) = f$. Again by the persistence condition, $v(a, \varphi \supset \psi) = f$ iff $v(b, \varphi \supset \psi) = f$ for some $b \geq a$. Hence, we get: $v(a, \varphi \supset \psi) = f$ whenever there exists $b \geq a$ such that $v(b, \varphi) = t$ and $v(b, \psi) = f$. The frame $\mathcal{W}$ respects $(\Rightarrow\supset)$ iff for every $a \in W$, $v(a, \varphi \supset \psi) = t$ whenever for every $b \geq a$, either $v(b, \varphi) = f$ or $v(b, \psi) = t$. Hence the two rules together impose exactly the well-known Kripke semantics for intuitionistic implication ([14]).

Example 13. A generalized $\mathcal{L}$-frame $\mathcal{W} = \langle W, \leq, v \rangle$ respects the rule $(\leadsto\Rightarrow)$ under the same conditions under which it respects $(\supset\Rightarrow)$. The frame $\mathcal{W}$ respects $(\Rightarrow\leadsto)$ iff for every $a \in W$, $v(a, \varphi \leadsto \psi) = t$ whenever $v(a, \psi) = t$ (recall that this is equivalent to $v(b, \psi) = t$ for every $b \geq a$). Note that in this case the two rules for $\leadsto$ do not always determine the value assigned to $\varphi \leadsto \psi$: if $v(a, \psi) = f$, and there is no $b \geq a$ such that $v(b, \varphi) = t$ and $v(b, \psi) = f$, then $v(a, \varphi \leadsto \psi)$ is free to be either t or f. So the semantics of this connective is non-deterministic.

Example 14. A generalized $\mathcal{L}$-frame $\mathcal{W} = \langle W, \leq, v \rangle$ respects the rule $(T \Rightarrow)$ (see Example 7) if $v(a, \varphi T \psi) = f$ whenever $v(a, \varphi) = f$. It respects $(\Rightarrow T)$ if $v(a, \varphi T \psi) = t$ whenever $v(a, \psi) = t$. The two constraints contradict each other in case both $v(a, \varphi) = f$ and $v(a, \psi) = t$. This is a semantic explanation why Prior's "connective" T ("Tonk") is meaningless.

Definition 18. Let **G** be a strict canonical constructive system.

1. We denote $\mathcal{S} \models^{seq}_{\mathbf{G}} s$ (where $\mathcal{S}$ is a set of sequents and s is a sequent) iff every **G**-legal model of $\mathcal{S}$ is also a model of s.
2. The semantic consequence relation $\models_{\mathbf{G}}$ between *formulas* which is induced by **G** is defined by: $T \models_{\mathbf{G}} \varphi$ if every **G**-legal model of T is also a model of φ.

Again we have:

Proposition 3. *$T \models_{\mathbf{G}} \varphi$ iff $\{\Rightarrow \psi \mid \psi \in T\} \models^{seq}_{\mathbf{G}} \Rightarrow \varphi$.*

4 Soundness, Completeness, Cut-Elimination

In this section we show that the two logics induced by a strict canonical constructive system **G** ($\vdash_{\mathbf{G}}$ and $\models_{\mathbf{G}}$) are identical. Half of this identity is given in the following theorem:

Theorem 2. *Every strict canonical constructive system* **G** *is strongly sound with respect to the semantics of* **G***-legal generalized frames. In other words:*

1. *If* $T \vdash_{\mathbf{G}} \varphi$ *then* $T \models_{\mathbf{G}} \varphi$.
2. *If* $\mathcal{S} \vdash^{seq}_{\mathbf{G}} s$ *then* $\mathcal{S} \models^{seq}_{\mathbf{G}} s$.

Proof. We prove the second part first. Assume that $\mathcal{S} \vdash^{seq}_{\mathbf{G}} s$, and $\mathcal{W} = \langle W, \leq, v\rangle$ is a **G**-legal model of $\mathcal{S}$. We show that s is locally true in every $a \in W$. Since the axioms of G and the premises of $\mathcal{S}$ trivially have this property, and the cut and weakening rules obviously preserve it, it suffices to show that the property of being locally true is preserved also by applications of the logical rules of **G**.

- Suppose $\Gamma \Rightarrow \diamond(\psi_1, \dots, \psi_n)$ is derived from $\{\Gamma, \sigma(\Pi_i) \Rightarrow \sigma(q_i)\}_{1\leq i\leq m}$ using the introduction rule $r = \{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/ \ \Rightarrow \diamond(p_1, p_2, \dots, p_n)$ (σ is a substitution such that $\sigma(p_j) = \psi_j$ for $1 \leq j \leq n$). Assume that all the premises of this application have the required property. We show that so does its conclusion. Let $a \in W$. If $v(a, \psi) = f$ for some $\psi \in \Gamma$, then obviously $\Gamma \Rightarrow \diamond(\psi_1, \dots, \psi_n)$ is locally true in a. Assume otherwise. Then the persistence condition implies that $v(b, \psi) = t$ for every $\psi \in \Gamma$ and $b \geq a$. Hence our assumption concerning $\{\Gamma, \sigma(\Pi_i) \Rightarrow \sigma(q_i)\}_{1\leq i\leq m}$ entails that for every $b \geq a$ and $1 \leq i \leq m$, either $v(b, \psi) = f$ for some $\psi \in \sigma(\Pi_i)$, or $v(b, \sigma(q_i)) = t$. It follows that for $1 \leq i \leq m$, $\Pi_i \Rightarrow q_i$ is satisfied in a by σ. Since $\mathcal{W}$ respects r, it follows that $v(a, \diamond(\psi_1, \dots, \psi_n)) = t$, as required.
- Now we deal with the elimination rules of **G**. Suppose $\Gamma, \diamond(\psi_1, \dots, \psi_n) \Rightarrow \theta$ is derived from $\{\Gamma, \sigma(\Pi_i) \Rightarrow \sigma(\Sigma_i)\}_{1\leq i\leq m_1}$ and $\{\Gamma, \sigma(\Pi_i) \Rightarrow \theta\}_{m_1+1\leq i\leq m}$, using the elimination rule $r = \{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/ \ \diamond(p_1, p_2, \dots, p_n) \Rightarrow$ (where Σ_i is empty for $m_1 + 1 \leq i \leq m$, and σ is a substitution such that $\sigma(p_j) = \psi_j$ for $1 \leq j \leq n$). Assume that all the premises of this application have the required property. Let $a \in W$. If $v(a, \psi) = f$ for some $\psi \in \Gamma$ or $v(a, \theta) = t$, then we are done. Assume otherwise. Then $v(a, \theta) = f$, and (by the persistence condition) $v(b, \psi) = t$ for every $\psi \in \Gamma$ and $b \geq a$. Hence our assumption concerning $\{\Gamma, \sigma(\Pi_i) \Rightarrow \sigma(\Sigma_i)\}_{1\leq i\leq m_1}$ entails that for every $b \geq a$ and $1 \leq i \leq m_1$, either $v(b, \psi) = f$ for some $\psi \in \sigma(\Pi_i)$, or $v(b, \sigma(\Sigma_i)) = t$. This immediately implies that every definite premise of the rule is satisfied in a by σ. Since $v(a, \theta) = f$, our assumption concerning $\{\Gamma, \sigma(\Pi_i) \Rightarrow \theta\}_{m_1+1\leq i\leq m}$ entails that for every $m_1 + 1 \leq i \leq m$, $v(a, \psi) = f$ for some $\psi \in \sigma(\Pi_i)$. Hence the negative premises of the rule are also satisfied in a by σ. Since $\mathcal{W}$ respects r, it follows that $v(a, \diamond(\psi_1, \dots, \psi_n)) = f$, as required.

The first part follows from the second by Propositions 1 and 3. □

For the converse, we first prove the following key result.

Theorem 3. *Let* **G** *be a strict canonical constructive system in* $\mathcal{L}$*, and let* $\mathcal{S} \cup \{s\}$ *be a set of sequents in* $\mathcal{L}$*. Then either there is an* $\mathcal{S}$*-(cut-free) proof of* s *from* $\mathcal{S}$*, or there is a* **G***-legal model of* $\mathcal{S}$ *which is not a model of* s*.*

Proof. Assume that $s = \Gamma_0 \Rightarrow \varphi_0$ does not have an $\mathcal{S}$-(cut-free) proof in **G**. Let $\mathcal{F}'$ be the set of subformulas of $\mathcal{S} \cup \{s\}$. Given a formula $\varphi \in \mathcal{F}'$, call a theory $\mathcal{T} \subseteq \mathcal{F}'$ *φ-maximal* if there is no finite $\Gamma \subseteq \mathcal{T}$ such that $\Gamma \Rightarrow \varphi$ has an $\mathcal{S}$-(cut-free) proof from $\mathcal{S}$, but every proper extension $\mathcal{T}' \subseteq \mathcal{F}'$ of $\mathcal{T}$ contains such a finite subset Γ. Obviously, if $\Gamma \subseteq \mathcal{F}'$, $\varphi \in \mathcal{F}'$ and $\Gamma \Rightarrow \varphi$ has no $\mathcal{S}$-(cut-free) proof from $\mathcal{S}$, then Γ can be extended to a theory $\mathcal{T} \subseteq \mathcal{F}'$ which is φ-maximal. In particular: Γ_0 can be extended to a φ_0-maximal theory $\mathcal{T}_0$.

Now let $\mathcal{W} = \langle W, \subseteq, v\rangle$, where:

- W is the set of all extensions of $\mathcal{T}_0$ in $\mathcal{F}'$ which are φ-maximal for some $\varphi \in \mathcal{F}'$.
- v is defined inductively as follows. For atomic formulas:

$$v(\mathcal{T}, p) = \begin{cases} t & p \in \mathcal{T} \\ f & p \notin \mathcal{T} \end{cases}$$

  Suppose $v(\mathcal{T}, \psi_i)$ has been defined for all $\mathcal{T} \in W$ and $1 \le i \le n$. We let $v(\mathcal{T}, \diamond(\psi_1, \ldots, \psi_n)) = t$ iff at least one of the following holds:
  1. There exists an introduction rule for $\diamond$ whose set of premises is satisfied in $\mathcal{T}$ by a substitution σ such that $\sigma(p_i) = \psi_i$ $(1 \le i \le n)$.
  2. $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{T}$ and there does not exist $\mathcal{T}' \in W$, $\mathcal{T} \subseteq \mathcal{T}'$, and an elimination rule for $\diamond$ whose set of premises is satisfied in $\mathcal{T}'$ by a substitution σ such that $\sigma(p_i) = \psi_i$ $(1 \le i \le n)$.[6]

First we prove that $\mathcal{W}$ is a generalized $\mathcal{L}$-frame:

- W is not empty because $\mathcal{T}_0 \in W$.
- We prove by structural induction that v is persistent:
  For atomic formulas v is trivially persistent since the order is $\subseteq$.
  Assume that v is persistent for $\psi_1, \ldots, \psi_n$. We prove its persistence for $\diamond(\psi_1, \ldots, \psi_n)$. So assume that $v(\mathcal{T}, \diamond(\psi_1, \ldots, \psi_n)) = t$ and $\mathcal{T} \subseteq \mathcal{T}^*$. By v's definition there are two possibilities:
  1. There exists an introduction rule for $\diamond$ whose set of premises is satisfied in $\mathcal{T}$ by a substitution σ such that $\sigma(p_i) = \psi_i$ $(1 \le i \le n)$. In such a case, the premises are all definite Horn clauses. Hence by definition, σ satisfies the same rule's premises also in $\mathcal{T}^*$, and so $v(\mathcal{T}^*, \diamond(\psi_1, \ldots, \psi_n)) = t$.

[6] This inductive definition isn't absolutely formal, since satisfaction by a substitution is defined for a generalized $\mathcal{L}$-frame, which we are in the middle of constructing, but the intention should be clear.

2. $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{T}$ and there does not exist $\mathcal{T}' \in W$, $\mathcal{T} \subseteq \mathcal{T}'$, and an elimination rule for $\diamond$ whose set of premises is satisfied in $\mathcal{T}'$ by a substitution σ such that $\sigma(p_i) = \psi_i$ $(1 \leq i \leq n)$. Then $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{T}^*$ (since $\mathcal{T} \subseteq \mathcal{T}^*$), and there surely does not exist $\mathcal{T}' \in W$, $\mathcal{T}^* \subseteq \mathcal{T}'$, and an elimination rule for $\diamond$ whose set of premises is satisfied in $\mathcal{T}'$ by a substitution σ such that $\sigma(p_i) = \psi_i$ $(1 \leq i \leq n)$ (otherwise the same would hold for $\mathcal{T}$). It follows that $v(\mathcal{T}^*, \diamond(\psi_1, \ldots, \psi_n)) = t$ in this case too.

Next we prove that $\mathcal{W}$ is **G**-legal:

1. The introduction rules are directly respected by the first condition in v's definition.
2. Let r be an elimination rule for $\diamond$, and suppose all its premises are satisfied in some $\mathcal{T} \in W$ by a substitution σ such that $\sigma(p_i) = \psi_i$. Then neither of the conditions under which $v(\mathcal{T}, \diamond(\psi_1, \ldots, \psi_n)) = t$ can hold:
   (a) The second condition explicitly excludes the option that all the premises are satisfied (in any $\mathcal{T}' \in W$, $\mathcal{T} \subseteq \mathcal{T}'$, so also in $\mathcal{T}$ itself).
   (b) The first condition cannot be met because of **G**'s coherence, which does not allow the two sets of premises (of an introduction rule and an elimination rule) to be satisfied together. To see this, assume for contradiction that S_1 is the set of premises of an elimination rule for $\diamond$, S_2 is the set of premises of an introduction rule for $\diamond$, and there exists $\mathcal{T} \in W$ in which both sets of premises are satisfied by a substitution σ such that $\sigma(p_i) = \psi_i$ $(1 \leq i \leq n)$. Let u be an assignment in $\{t, f\}$ in which $u(p_i) = v(\mathcal{T}, \psi_i)$. Since σ satisfies in $\mathcal{T}$ both sets of premises, u classically satisfies S_1 and S_2. But, **G** is coherent, i.e. $S_1 \cup S_2$ is classically inconsistent. A contradiction.

   It follows that $v(\mathcal{T}, \diamond(\psi_1, \ldots, \psi_n)) = f$, as required.

It remains to prove that $\mathcal{W}$ is a model of $\mathcal{S}$ but not of s. For this we first prove that the following hold for every $\mathcal{T} \in W$ and every formula $\psi \in \mathcal{F}'$:

(a) If $\psi \in \mathcal{T}$ then $v(\mathcal{T}, \psi) = t$.
(b) If $\mathcal{T}$ is ψ-maximal then $v(\mathcal{T}, \psi) = f$.

We prove **(a)** and **(b)** together by a simultaneous induction on the complexity of ψ. For atomic formulas they easily follow from v's definition, and the fact that $p \Rightarrow p$ is an axiom. For the induction step, assume that **(a)** and **(b)** hold for $\psi_1, \ldots, \psi_n \in \mathcal{F}'$. We prove them for $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{F}'$.

– Assume that $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{T}$, but $v(\mathcal{T}, \diamond(\psi_1, \ldots, \psi_n)) = f$. By v's definition, since $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{T}$ there should exist $\mathcal{T}' \in W$, $\mathcal{T} \subseteq \mathcal{T}'$, and an elimination rule for $\diamond$, r, whose set of premises is satisfied in $\mathcal{T}'$ by a substitution σ such that $\sigma(p_i) = \psi_i$ $(1 \leq i \leq n)$. Let $\{\Pi_i \Rightarrow\}_{1 \leq i \leq m_1}$ be the negative premises of r, and $\{\Pi_i \Rightarrow q_i\}_{m_1+1 \leq i \leq m}$ – the definite ones. Since σ satisfies in $\mathcal{T}'$ every sequent in $\{\Pi_i \Rightarrow\}_{1 \leq i \leq m_1}$, then for all $1 \leq i \leq m_1$ there exists $\psi_{j_i} \in \sigma(\Pi_i)$ such that $v(\mathcal{T}', \psi_{j_i}) = f$. By the induction hypothesis this implies that for all $1 \leq i \leq m_1$, there exists $\psi_{j_i} \in \sigma(\Pi_i)$ such that $\psi_{j_i} \notin \mathcal{T}'$.

Let φ be the formula for which $\mathcal{T}'$ is maximal. Then for all $1 \leq i \leq m_1$ there is a finite $\Delta_i \subseteq \mathcal{T}'$ such that $\Delta_i, \psi_{j_i} \Rightarrow \varphi$ has an $\mathcal{S}$-(cut-free) proof from $\mathcal{S}$, and so $\Delta_i, \sigma(\Pi_i) \Rightarrow \varphi$ has such a proof. This in turn implies that there must exist $m_1 + 1 \leq i_0 \leq m$ such that $\Gamma, \sigma(\Pi_{i_0}) \Rightarrow \sigma(q_{i_0})$ has no $\mathcal{S}$-(cut-free) proof from $\mathcal{S}$ for any finite $\Gamma \subseteq \mathcal{T}'$. Indeed, if such a proof exists for every $m_1 + 1 \leq i \leq m$, we would use the m_1 proofs of $\Delta_i, \sigma(\Pi_i) \Rightarrow \varphi$ for $1 \leq i \leq m_1$, the $m - m_1$ proofs for $\Gamma'_i, \sigma(\Pi_i) \Rightarrow \sigma(q_i)$ for $m_1 + 1 \leq i \leq m$, some trivial weakenings, and the elimination rule r to get an $\mathcal{S}$-(cut-free) proof from $\mathcal{S}$ of the sequent $\cup_{i=1}^{i=m_1} \Delta_i, \cup_{i=m_1+1}^{i=m} \Gamma_i, \diamond(\psi_1, \ldots, \psi_n) \Rightarrow \varphi$. Since $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{T}$, this would contradict $\mathcal{T}'$'s φ-maximality. Using this i_0, we extend $\mathcal{T}' \cup \sigma(\Pi_{i_0})$ to a theory $\mathcal{T}''$ which is $\sigma(q_{i_0})$-maximal. By the induction hypothesis $v(\mathcal{T}'', \psi) = t$ for all $\psi \in \sigma(\Pi_{i_0})$ and $v(\mathcal{T}'', \sigma(q_{i_0})) = f$. Since $T' \subseteq \mathcal{T}''$, this contradicts the fact that σ satisfies $\Pi_{i_0} \Rightarrow q_{i_0}$ in $\mathcal{T}'$.

– Assume that $\mathcal{T}$ is $\diamond(\psi_1, \ldots, \psi_n)$-maximal, but $v(\mathcal{T}, \diamond(\psi_1, \ldots, \psi_n)) = t$. Obviously, $\diamond(\psi_1, \ldots, \psi_n) \notin \mathcal{T}$ (because $\diamond(\psi_1, \ldots, \psi_n) \Rightarrow \diamond(\psi_1, \ldots, \psi_n)$ is an axiom). Hence by v's definition there exists an introduction rule for $\diamond$, r, whose set of premises is satisfied in $\mathcal{T}$ by a substitution σ such that $\sigma(p_i) = \psi_i$ $(1 \leq i \leq n)$. Let $\{\Pi_i \Rightarrow q_i\}_{1 \leq i \leq m}$ be the premises of r. As in the previous case, there must exist $1 \leq i_0 \leq m$ such that $\Gamma, \sigma(\Pi_{i_0}) \Rightarrow \sigma(q_{i_0})$ has no $\mathcal{S}$-(cut-free) proof from $\mathcal{S}$ for any finite $\Gamma \subseteq \mathcal{T}$ (if such a proof exists for all $1 \leq i \leq m$ with finite $\Gamma_i \subseteq \mathcal{T}$ than we could have an $\mathcal{S}$-(cut-free) proof from $\mathcal{S}$ of $\cup_{i=1}^{i=m} \Gamma_i \Rightarrow \diamond(\psi_1, \ldots, \psi_n)$ using the m proofs of $\Gamma_i, \sigma(\Pi_i) \Rightarrow \sigma(q_i)$, some weakenings, and r). Using this i_0, we extend $\mathcal{T} \cup \sigma(\Pi_{i_0})$ to a theory $\mathcal{T}'$ which is $\sigma(q_{i_0})$-maximal. By the induction hypothesis, $v(\mathcal{T}', \psi) = t$ for all $\psi \in \sigma(\Pi_{i_0})$ and $v(\mathcal{T}', \sigma(q_{i_0})) = f$. Since $\mathcal{T} \subseteq \mathcal{T}'$, this contradicts the fact that σ satisfies $\Pi_{i_0} \Rightarrow q_{i_0}$ in $\mathcal{T}$.

Next we note that **(b)** can be strengthened as follows:

(c) If $\psi \in \mathcal{F}'$, $\mathcal{T} \in W$ and there is no finite $\Gamma \subseteq \mathcal{T}$ such that $\Gamma \Rightarrow \psi$ has an $\mathcal{S}$-(cut-free) proof from $\mathcal{S}$, then $v(\mathcal{T}, \psi) = f$.

Indeed, under these conditions $\mathcal{T}$ can be extended to a ψ-maximal theory $\mathcal{T}'$. Now $\mathcal{T}' \in W$, $\mathcal{T} \subseteq \mathcal{T}'$, and by **(b)**, $v(\mathcal{T}', \psi) = f$. Hence also $v(\mathcal{T}, \psi) = f$.

Now **(a)** and **(b)** together imply that $v(\mathcal{T}_0, \psi) = t$ for every $\psi \in \Gamma_0 \subseteq \mathcal{T}_0$, and $v(\mathcal{T}_0, \varphi_0) = f$. Hence $\mathcal{W}$ is not a model of s. We end the proof by showing that $\mathcal{W}$ is a model of $\mathcal{S}$. So let $\psi_1, \ldots, \psi_n \Rightarrow \theta \in \mathcal{S}$ and let $\mathcal{T} \in W$, where $\mathcal{T}$ is φ-maximal. Assume by way of contradiction that $v(\mathcal{T}, \psi_i) = t$ for $1 \leq i \leq n$, while $v(\mathcal{T}, \theta) = f$. By **(c)**, for every $1 \leq i \leq n$ there is a finite $\Gamma_i \subseteq \mathcal{T}$ such that $\Gamma_i \Rightarrow \psi_i$ has an $\mathcal{S}$-(cut-free) proof from $\mathcal{S}$. On the other hand $v(\mathcal{T}, \theta) = f$ implies (by **(a)**) that $\theta \notin \mathcal{T}$. Since $\mathcal{T}$ is φ-maximal, it follows that there is a finite $\Sigma \subseteq \mathcal{T}$ such that $\Sigma, \theta \Rightarrow \varphi$ has an $\mathcal{S}$-(cut-free) proof from $\mathcal{S}$. Now from $\Gamma_i \Rightarrow \psi_i$ $(1 \leq i \leq n)$, $\Sigma, \theta \Rightarrow \varphi$, and $\psi_1, \ldots, \psi_n \Rightarrow \theta$ one can infer $\Gamma_1, \ldots, \Gamma_n, \Sigma \Rightarrow \varphi$ by $n + 1$ $\mathcal{S}$-cuts (on $\psi_1, \ldots, \psi_n$ and θ). It follows that the last sequent has an $\mathcal{S}$-(cut-free) proof from $\mathcal{S}$. Since $\Gamma_1, \ldots, \Gamma_n, \Sigma \subseteq \mathcal{T}$, this contradicts the φ-maximality of $\mathcal{T}$. □

Theorem 4 (Soundness and Completeness). *Every strict canonical constructive system* **G** *is strongly sound and complete with respect to the semantics of* **G***-legal generalized frames. In other words:*

1. $T \vdash_{\mathbf{G}} \varphi$ *iff* $T \models_{\mathbf{G}} \varphi$.
2. $\mathcal{S} \vdash^{seq}_{\mathbf{G}} s$ *iff* $\mathcal{S} \models^{seq}_{\mathbf{G}} s$.

Proof. Immediate from Theorems 3 and 2, and Propositions 1, 3. □

Corollary 1. *If* **G** *is a strict canonical constructive system in* $\mathcal{L}$ *then* $\langle \mathcal{L}, \models_{\mathbf{G}} \rangle$ *is a logic.*

Corollary 2 (Compactness). *Let* **G** *be a strict canonical constructive system.*

1. *If* $\mathcal{S} \models^{seq}_{\mathbf{G}} s$ *then there exists a finite* $\mathcal{S}' \subseteq \mathcal{S}$ *such that* $\mathcal{S}' \models^{seq}_{\mathbf{G}} s$.
2. $\models_{\mathbf{G}}$ *is finitary.*

Theorem 5 (General Strong Cut Elimination Theorem).

1. *Every strict canonical constructive system* **G** *admits strong cut-elimination (see Definition 12).*
2. *in a strict canonical constructive system* **G** *iff it has a cut-free proof there.*

Proof. The first part follows from Theorems 4 and 3. The second part is a special case of the first, where the set $\mathcal{S}$ of premises is empty. □

Corollary 3. *The four following conditions are equivalent for a strict canonical single-conclusion Gentzen-type system* **G***:*

1. $\langle \mathcal{L}, \vdash_{\mathbf{G}} \rangle$ *is a logic (by Proposition 2, this means that* $\vdash_{\mathbf{G}}$ *is consistent).*
2. **G** *is coherent.*
3. **G** *admits strong cut-elimination.*
4. **G** *admits cut-elimination.*

Proof. Condition 1 implies condition 2 by Theorem 1. Condition 2 implies condition 3 by Theorem 5. Condition 3 trivially implies condition 4. Finally, without using cuts there is no way to derive $p_1 \Rightarrow p_2$ in a strict canonical Gentzen-type system. Hence condition 4 implies condition 1. □

5 Analycity and Decidability

In general, in order for a denotational semantics of a propositional logic to be useful and effective, it should be *analytic*. This means that to determine whether a formula φ follows from a theory $\mathcal{T}$, it suffices to consider *partial* valuations, defined on the set of all subformulas of the formulas in $\mathcal{T} \cup \{\varphi\}$. Now we show that the semantics of **G**-legal frames is analytic in this sense.

Definition 19. Let **G** be a strict canonical constructive system for $\mathcal{L}$. A **G**-legal *semiframe* is a triple $\mathcal{W}' = \langle W, \leq, v' \rangle$ such that:

1. $\langle W, \leq \rangle$ is a nonempty partially ordered set.
2. v' is a partial function from the set of formulas of $\mathcal{L}$ into the set of persistent functions from W into $\{t, f\}$ such that:
   – $\mathcal{F}'$, the domain of v', is closed under subformulas.
   – v' respects the rules of **G** on $\mathcal{F}'$ (e.g. if r is an introduction rule for an n-ary connective $\diamond$, and $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{F}'$, then $v(a, \diamond(\psi_1, \ldots, \psi_n)) = t$ whenever all the premises of r are satisfied in a by a substitution σ such that $\sigma(p_i) = \psi_i$ $(1 \leq i \leq n)$).

Theorem 6. *Let* **G** *be a strict canonical constructive system for* $\mathcal{L}$*. Then the semantics of* **G***-legal frames is* analytic *in the following sense:*
If $\mathcal{W}' = \langle W, \leq, v' \rangle$ *is a* **G***-legal* semiframe*, then* v' *can be extended to a function* v *so that* $\mathcal{W} = \langle W, \leq, v \rangle$ *is a* **G***-legal frame.*

Proof. Let $\mathcal{W}' = \langle W, \leq, v' \rangle$ be a **G**-legal semiframe. We recursively extend v' to a total function v. For atomic p we let $v(p) = v'(p)$ if $v'(p)$ is defined, and $v(p) = \lambda a \in W.t$ (say) otherwise. For $\varphi = \diamond(\psi_1, \ldots, \psi_n)$ we let $v(\varphi) = v'(\varphi)$ whenever $v'(\varphi)$ is defined, and otherwise we define $v(\varphi, a) = f$ iff there exists an elimination rule r with $\diamond(p_1, \ldots, p_n) \Rightarrow$ as its conclusion, and an element $b \geq a$ of W, such that all premises of r are satisfied in b (with respect to $\langle W, \leq, v \rangle$) by a substitution σ such that $\sigma(p_j) = \psi_j$ $(1 \leq j \leq n)$. Note that the satisfaction of the premises of r by σ in elements of W depends only on the values assigned by v to $\psi_1, \ldots, \psi_n$, so the recursion works, and v is well defined. From the definition of v and the assumption that $\mathcal{W}'$ is a **G**-legal semiframe, it immediately follows that v is an extension of v', that $v(\varphi)$ is a persistent function for every φ (so $\mathcal{W} = \langle W, \leq, v \rangle$ is a generalized $\mathcal{L}$-frame), and that $\mathcal{W}$ respects all the elimination rules of **G**. Hence it only remains to prove that it respects also the introduction rules of **G**. Let $r = \{\Pi_i \Rightarrow q_i\}_{1 \leq i \leq m} / \ \Rightarrow \diamond(p_1, p_2, \ldots, p_n)$ be such a rule, and assume that for every $1 \leq i \leq m$, $\sigma(\Pi_i) \Rightarrow \sigma(q_i)$ is true in a with respect to $\langle W, \leq, v \rangle$. We should show that $v(a, \diamond(\psi_1, \ldots, \psi_n)) = t$.

If $v'(a, \diamond(\psi_1, \ldots, \psi_n))$ is defined, then since its domain is closed under subformulas, for every $1 \leq i \leq n$ and every $b \in W$ $v'(b, \psi_i)$ is defined. In this case, our construction ensures that for every $1 \leq i \leq n$ and every $b \in W$ we have $v'(b, \psi_i) = v(b, \psi_i)$. Therefore, since for every $1 \leq i \leq m$, $\sigma(\Pi_i) \Rightarrow \sigma(q_i)$ is locally true in every $b \geq a$ with respect to $\langle W, \leq, v \rangle$, it is also locally true with respect to $\langle W, \leq, v' \rangle$. Since v' respects r, $v'(a, \diamond(\psi_1, \ldots, \psi_n)) = t$, so $v(a, \diamond(\psi_1, \ldots, \psi_n)) = t$ as well, as required.

Now, assume $v'(a, \diamond(\psi_1, \ldots, \psi_n))$ is not defined, and assume by way of contradiction that $v(a, \diamond(\psi_1, \ldots, \psi_n)) = f$. So, there exists $b \geq a$ and an elimination rule $\{\Delta_j \Rightarrow \Sigma_j\}_{1 \leq j \leq k} / \ \diamond(p_1, p_2, \ldots, p_n) \Rightarrow$ such that $\sigma(\Delta_j) \Rightarrow \sigma(\Sigma_j)$ is locally true in b for $1 \leq j \leq k$. Since $b \geq a$, our assumption about a implies that $\sigma(\Pi_i) \Rightarrow \sigma(q_i)$ is locally true in b for $1 \leq i \leq m$. It follows that by defining $u(p) = v(b, \sigma(p))$ we get a valuation u in $\{t, f\}$ which satisfies all the clauses in the union of $\{\Pi_i \Rightarrow q_i \mid 1 \leq i \leq m\}$ and $\{\Delta_j \Rightarrow \Sigma_j \mid 1 \leq j \leq k\}$. This contradicts the coherence of **G**. □

The following two theorems are now easy consequences of Theorem 6 and the soundness and completeness theorems of the previous section:[7]

Theorem 7. *Let* $\mathbf{G}$ *be a strict canonical constructive system. Then* $\mathbf{G}$ *is strongly decidable: Given a finite set* $\mathcal{S}$ *of sequents, and a sequent* s*, it is decidable whether* $\mathcal{S} \vdash^{seq}_{\mathbf{G}} s$ *or not. In particular: it is decidable whether* $\Gamma \vdash_{\mathbf{G}} \varphi$*, where* φ *is formula and* Γ *is a finite set of formulas.*

Proof. Let $\mathcal{F}'$ be the set of subformulas of the formulas in $\mathcal{S} \cup \{s\}$. From Theorem 6 and the proof of Theorem 3 it easily follows that in order to decide whether $\mathcal{S} \vdash^{seq}_{\mathbf{G}} s$ it suffices to check all triples of the form $\langle W, \subseteq, v' \rangle$ where $W \subseteq 2^{\mathcal{F}'}$ and $v' : \mathcal{F}' \to (W \to \{t, f\})$, and see if any of them is a $\mathbf{G}$-legal semiframe which is a model of $\mathcal{S}$ but not a model of s. □

Theorem 8. *Let* $\mathbf{G_1}$ *be a strict canonical constructive system in a language* $\mathcal{L}_1$*, and let* $\mathbf{G_2}$ *be a strict canonical constructive system in a language* $\mathcal{L}_2$*. Assume that* $\mathcal{L}_2$ *is an extension of* $\mathcal{L}_1$ *by some set of connectives, and that* $\mathbf{G_2}$ *is obtained from* $\mathbf{G_1}$ *by adding to the latter strict canonical rules for connectives in* $\mathcal{L}_2 - \mathcal{L}_1$*. Then* $\mathbf{G_2}$ *is a conservative extension of* $\mathbf{G_1}$ *(i.e. if all formulas in* $\mathcal{T} \cup \{\varphi\}$ *are in* $\mathcal{L}_1$ *then* $\mathcal{T} \vdash_{\mathbf{G_1}} \varphi$ *iff* $\mathcal{T} \vdash_{\mathbf{G_2}} \varphi$*).*

Proof. Suppose that $\mathcal{T} \not\vdash_{\mathbf{G_1}} \varphi$. Then there is $\mathbf{G_1}$-legal model $\mathcal{W}$ of $\mathcal{T}$ which is not a model of φ. Since the set of formulas of $\mathcal{L}_1$ is a subset of the set of formulas of $\mathcal{L}_2$ which is closed under subformulas, Theorem 6 implies that $\mathcal{W}$ can be extended to a $\mathbf{G_2}$-legal model of $\mathcal{T}$ which is not a model of φ. Hence $\mathcal{T} \not\vdash_{\mathbf{G_2}} \varphi$. □

Note 4. Prior's "connective" Tonk ([15]) has made it clear that not every combination of "ideal" introduction and elimination rules can be used for defining a connective. Some constraints should be imposed on the set of rules. Such a constraint was indeed suggested by Belnap in his famous [8]: the rules for a connective $\diamond$ should be *conservative*, in the sense that if $\mathcal{T} \vdash \varphi$ is derivable using them, and $\diamond$ does not occur in $\mathcal{T} \cup \varphi$, then $\mathcal{T} \vdash \varphi$ can also be derived without using the rules for $\diamond$. This solution to the problem has two problematic aspects:

1. Belnap did not provide any effective necessary and sufficient criterion for checking whether a given set of rules is conservative in the above sense. Without such criterion every connective defined by inference rules (without an independent denotational semantics) is suspected of being a Tonk-like connective, and should not be used until a proof is given that it is "innocent".
2. Belnap formulated the condition of conservativity only with respect to the basic deduction framework, in which no connectives are assumed. But nothing in what he wrote excludes the possibility of a system G having two connectives, each of them "defined" by a set of rules which is conservative

[7] The two theorems can also be proved directly from the cut-elimination theorem for strict canonical constructive systems.

over the basic system B, while G itself is not conservative over B. If this happens then it will follow from Belnap's thesis that each of the two connectives is well-defined and meaningful, but they cannot exist together. Such a situation is almost as paradoxical as that described by Prior.

Now the first of these two objections is met, of course, by our coherence criterion for strict canonical systems, since coherence of a finite set of strict canonical rules can effectively be checked. The second is met by Theorem 8. That theorem shows that a very strong form of Belnap's conservativity criterion is valid for strict canonical constructive systems, and so what a set of strict canonical rules defines in such systems is independent of the system in which it is included.

6 Related Works and Further Research

There have been several works in the past on conditions for cut-elimination. Except for [7], the closest to the present one is [10]. The range of systems dealt with there is in fact broader than ours, since it deals with various types of structural rules, while in this paper we assume the standard structural rules of minimal logic. On the other hand the results and characterization given in [10] are less satisfactory than those given here. First, in the framework of [10] any connective has essentially infinitely many introduction (and elimination) rules, while our framework makes it possible to convert these infinite sets of rules into a finite set. Second, our coherence criterion (for non-triviality and cut-elimination) is simple and constructive. In contrast, its counterpart in [10] (called *reductivity*) is not constructive. Third, our notion of *strong cut-elimination* simply limits the set of possible cut formulas used in a derivation of a sequent from other sequents to those that occur in the premises of that derivation. In contrast, *reductive cut-elimination*, its counterpart in [10], imposes conditions on *applications* of the cut rule in proofs which involve examining the whole proofs of the two premises of that application. Finally, both works use non-deterministic semantic frameworks (in [10] this is only implicit!). However, while we use the concrete framework of intuitionistic-like Kripke frames, variants of the significantly more abstract and complicated phase semantics are used in [10]. This leads to the following crucial difference: our semantics leads to decision procedures for all the systems we consider. This does not seem to be the case for the semantics used in [10].

It should be noted that unlike the present work, [10] treats *non-strict* systems (as is done in most presentations of intuitionistic logic as well as in Gentzen's original one), i.e. single-conclusion sequential systems which allow the use of negative sequents in derivations. In addition to being widely used, the non-strict framework makes it possible to define negation as a basic connective. It is natural to try to extend our methods and results to the non-strict framework. However, as the next observations show, doing it is not a straightforward matter:

– Consider a non-strict canonical Gentzen-type system **G** containing only the following rules for an unary connective, denoted by $\circ$:

$$\{p_1 \Rightarrow\} \ / \ \circ p_1 \Rightarrow \quad \text{and} \quad \{p_1 \Rightarrow\} \ / \ \Rightarrow \circ p_1$$

Applications of these rules have the following form (where E is either empty or a singleton):

$$\frac{\Gamma, \varphi \Rightarrow E}{\Gamma, \circ\varphi \Rightarrow E} \qquad \frac{\Gamma, \varphi \Rightarrow}{\Gamma \Rightarrow \circ\varphi}$$

Obviously, **G** is not coherent. However, in **G** there is no way to derive a negative sequent from no assumptions (this is proved by simple induction). Hence, the introduction rule for $\circ$ can never be used in proofs without assumptions. For this trivial reason, **G** is consistent. Hence, in this framework coherence is no longer equivalent to consistency.

- For the same reason, **G** from the previous example admits cut-elimination but does not admit strong cut-elimination. Hence, strong cut-elimination and cut-elimination are also no longer equivalent.
- Consider the well-known rules for intuitionistic negation:

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma, \neg\varphi \Rightarrow} \qquad \frac{\Gamma, \varphi \Rightarrow}{\Gamma, \Rightarrow \neg\varphi}$$

If we naively extend our semantic definition to apply to this kind of rule, we will obtain that in every legal frame $v(a, \neg\varphi) = t$ iff $v(a, \varphi) = f$ (since a negative sequent is true iff it is locally true). This is not the well-known Kripke-style semantics for negation. Moreover, $\varphi \vee \neg\varphi$, which is obviously not provable in intuitionistic logic, is true in this semantics. Hence some changes must be done in our semantic framework if we wish to *directly* handle negation (and other negation-like connectives) in an adequate way.

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