Local And Global Colorability of Graphs

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Abstract

It is shown that for any fixed $c \geq 3$ and r, the maximum possible chromatic number of a graph on n vertices in which every subgraph of radius at most r is c-colorable is $\tilde{\Theta}\left(n^{\frac{1}{r+1}}\right)$: it is $O\left((n/\log n)^{\frac{1}{r+1}}\right)$ and $\Omega\left(n^{\frac{1}{r+1}}/\log n\right)$. The proof is based on a careful analysis of the local and global colorability of random graphs and implies, in particular, that a random n-vertex graph with the right edge probability has typically a chromatic number as above and yet most balls of radius r in it are 2-degenerate.

1 Introduction

1.1 Notation and Definitions

For a simple undirected graph $G = (V, E)$ denote by $d(u, v)$ the *distance* between the vertices $u, v \in V$. The *degree* of a vertex $v \in V$, denoted by $\deg(v)$, is the number of its neighbours in G. A subset $V' \subseteq V$ is *independent* if no edge of G has both of its endpoints in V'. The *chromatic number* of G, denoted by $\chi(G)$, is the minimal number of independent subsets of V whose union covers V. A graph is k-*degenerate* if the minimum degree of every subgraph of it is at most k. In particular, a k-degenerate graph is $k+1$-colorable. We will work with random graphs $G_{n,p}$ in the Erdős-Rényi model, where there are n labelled vertices and each edge is included in the graph with probability p, independently of all other edges. We say that a property of G holds *with high probability (w.h.p.)* if this property holds with probability that tends to 1 as n tends to ∞. In this paper we are only interested in graphs with large chromatic number ℓ. It will be therefore equivalent to say that a property holds w.h.p. if its probability tends to 1 as ℓ tends to ∞.

Consider the following definition of r-local colorability:

Definition 1.1. Let r be a positive integer. Let $U_r(v, G)$ be the ball with radius r around $v \in V$ in G (i.e. the induced subgraph on all vertices in V

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whose distance from v is $\leq r$). Let

$$\ell\chi_r(G) = \max_{v \in V} \chi(U_r(v, G)) \tag{1.1}$$

denote the *r-local chromatic number* of G.

We also say that $U_r(v, G)$ is the r-ball around v in G. Finally, we define the main quantity discussed in this paper.

Definition 1.2. For $\ell \geq c \geq 2$ and $r > 0$ let $f_c(\ell, r)$ be the greatest integer n such that every graph on n vertices whose r-local chromatic number is $\leq c$ is ℓ-colorable.

In other words, $f_c(\ell, r) + 1$ is the minimal number of vertices in a non-ℓ-colorable graph in which every r-ball is c-colorable. Note that $f_{c_1}(\ell, r) \leq f_{c_2}(\ell, r)$ for $c_1 \geq c_2$.

Definitions 1.1 and 1.2 appear explicitly in the paper of Bogdanov [5], but the quantity $f_c(\ell, r)$ itself has been investigated well before (see Sections 1.2, 8 for more details).

The main goal of this paper is to estimate $f_c(\ell, r)$ for fixed c, r as ℓ tends to ∞. The main result is an upper bound tight up to a polylogarithmic factor for $f_c(\ell, r)$ for all fixed $c \geq 3$ and r.

1.2 Background and our contribution

Fix an $r > 0$. Somewhat surprisingly, the gap between $f_2(\ell, r)$ and $f_3(\ell, r)$ might be much bigger than the gap between $f_3(\ell, r)$ and $f_c(\ell, r)$ for any other fixed $c \geq 3$. Here is a short background on previous results regarding $f_c(\ell, r)$ for fixed c and r and large ℓ and our contributions to these problems.

Known upper bounds for $f_c(\ell, r)$ with fixed c, r, large ℓ

Erdős [7] showed that for sufficiently large m there exists a graph G with $m^{1+1/2k}$ vertices, that neither contains a cycle of length $\leq k$ nor an independent set of size m. As an easy consequence, G is not $m^{1/2k}$-colorable. Put $k = 2r+1, \ell = m^{1/2k}$ and note that G has $n = m^{1+1/2k} = \ell^{2k+1} = \ell^{4r+3}$ vertices and $\ell\chi_r(G) \leq 2$ but is not ℓ-colorable. Hence

$$f_2(\ell, r) < \ell^{4r+3}.$$

A better estimate follows from the results of Krivelevich in [11]. Indeed, Theorem 1 in his paper implies that there exists an absolute positive constant c so that

$$f_2(\ell, r) < (c\ell \log \ell)^{2r} \tag{1.2}$$

An upper bound for $f_3(\ell, r)$ can be derived from another result by Erdős [8]. Erdős worked with random graphs in the $G_{n,m}$ model, in which we consider random graphs with n vertices and exactly m edges. He showed that with probability > 0.8 and for $k \leq O(n^{1/3})$ large enough, $G_{n,kn}$ is not $\frac{k}{\log k}$-colorable but every subgraph spanned by $O(nk^{-3})$ vertices is 3-colorable.

It is easy to show that with high probability every r-ball in $G_{n,kn}$ has $O(k)^r$ vertices (later we prove and apply a similar result for graphs in the $G_{n,p}$ model). Combining the above results and taking $k = 2\ell \log \ell$, $n = O(k)^{r+3} =$

$O(\ell \log \ell)^{r+3}$, it follows that with positive probability the graph $G_{n,kn}$ is not ℓ-colorable but every r-ball (and in fact every subgraph on $O(nk^{-3}) = O(k)^r$ vertices) is 3-colorable. Hence there exists $\beta > 0$ such that:

$$f_c(\ell, r) \leq f_3(\ell, r) \leq (\beta \ell \log \ell)^{r+3} \tag{1.3}$$

for large ℓ, fixed $r \geq 3$ and for $c \geq 3$.

Lower bounds for $f_c(\ell, r)$ with fixed c, r, large ℓ

Bogdanov [5] showed that for all $r > 0$ and $\ell \geq c \geq 2$:

$$f_c(\ell, r) \geq \frac{(\ell/c + r/2)(\ell/c + r/2 + 1) \dots (\ell/c + 3r/2)}{(r+1)^{r+1}} \geq \left(\frac{\ell/c + r/2}{r+1} \right)^{r+1} \tag{1.4}$$

When c and r are fixed, (1.4) implies that $f_c(\ell, r) = \Omega(\ell^{r+1})$. In Subsection 7.2, we improve the lower bound in this domain by a logarithmic factor: it is shown that for fixed $c \geq 2$ and r, $f_c(\ell, r) = \Omega\left(\ell^{r+1} \log \ell\right)$.

A special case - $f_c(\ell, 1)$ for fixed c, large ℓ

It is not hard to prove that $f_2(\ell, 1) = \Theta(\ell^2 \log \ell)$, using the known fact that the Ramsey number $R(t, 3)$ is $\Theta(t^2 / \log t)$ (see [1], [10]). In Section 7 we extend this result to every fixed $c \geq 2$, showing that $f_c(\ell, 1) = \Theta(\ell^2 \log \ell)$ for fixed $c \geq 2$.

The main contribution

The main result in this paper is an improved upper bound for $f_3(\ell, r)$. We show that for fixed $r > 0$:

$$f_3(\ell, r) \leq (10 \ell \log \ell)^{r+1} \tag{1.5}$$

Fix r and $c \geq 3$. By the result above (together with 1.4) it follows that there exists a constant $\delta = \delta(r, c)$ such that

$$\delta \ell^{r+1} \log \ell \leq f_c(\ell, r) \leq f_3(\ell, r) \leq (10 \ell \log \ell)^{r+1} \tag{1.6}$$

The last result determines, up to a sub-logarithmic factor, the maximum possible chromatic number $M_{c,r}(n)$ of a graph on n vertices in which every r-ball is c-colorable:

$$a \frac{n^{\frac{1}{r+1}}}{\log n} \leq M_{c,r}(n) \leq b_{c,r} \left(\frac{n}{\log n} \right)^{\frac{1}{r+1}} \tag{1.7}$$

for suitable positive constants a, $b_{c,r}$.
Note that for $c = 2$ the best known estimates are weaker, namely it is only known that

$$\Omega\left(\frac{n^{1/(2r)}}{\log n} \right) \leq M_{2,r}(n) \leq O\left(\left(\frac{n}{\log n} \right)^{1/(r+1)} \right).$$

1.3 Paper Structure

The rest of the paper is organized as follows:

- In Section 2 we present the basic approach of gradually revealing information on a random graph. Two examples of this are given. Both will be useful in subsequent sections.

- In Section 3 we give an upper bound for $f_5(\ell, r)$ for fixed r and large ℓ using the random graph $G_{n,p}$ with $n = (10\ell \log \ell)^{r+1}$ and $p = \frac{3}{10}(10\ell \log \ell)^{-r}$. It is shown that with high probability, all r-balls in the graph are 4-degenerate.

- In Section 4, the same upper bound is obtained for $f_4(\ell, r)$. It is shown that most r-balls in the above graph are 4-colorable. Deleting the center of every non-4-colorable r-ball results in a graph with r-local chromatic number ≤ 4 and chromatic number $> \ell$ with positive probability.

- Section 5 includes the proof of the main result of the paper. It is shown that typically most r-balls in the above graph are 2-degenerate. This proof is much harder than the previous one. Again we delete the center of every non-2-degenerate r-ball to obtain a graph with r-local chromatic number at most 3 and chromatic number $> \ell$ with positive probability.

  Note that the result in this section is stronger than those in the previous two sections. Still, we prefer to include all three as each of the results has its merits: indeed, to get local 5-colorability it suffices to consider random graphs with no changes. Getting local 4-colorability requires some modifications in the random graph, but the proof is very short.

  Getting local 3-colorability is significantly more complicated, and is proved by a delicate exposure of the information about the edges of the random graph considered.

- In Section 6 we extend the result from Section 5 to large values of c.

- In Section 7 we give a new lower bound for $f_c(\ell, r)$ when $c \geq 2$ and r are fixed. This improves the known lower bound by a multiplicative logarithmic factor. In particular, it implies that $f_c(\ell, 1) = \Theta(\ell^2 \log \ell)$ for any fixed $c \geq 2$.

- The final Section 8 contains some concluding remarks including a discussion of what can be proved about the behaviour of $f_c(\ell, r)$ for non-constant values of r.

2 Gradually Revealing the Random Graph

In random graphs of the $G_{n,p}$ model the edges can be examined (that is, accepted to the graph or rejected from it) in any order. This fact can be used to reveal some of the information regarding the graph, while preserving the randomness of other information. Two examples of this basic approach are shown below, both will be used later in this paper.

2.1 Spanning tree with root

Let $r > 0$. This model first determines the vertices of $U_r(v, G)$ while also revealing a spanning tree for this subgraph, and only then continues to reveal all other edges of the graph.

Choose a root vertex v. Let $L_i = L_i(v, G)$ denote *the i-th level* with respect to v in G - that is, the set of all vertices of distance i from v. Trivially, $L_0(v, G) = \{v\}$. Also define $L_{\leq i} = L_{\leq i}(v, G) = \bigcup_{j=0}^{i} L_j(v, G)$.

Assuming L_i is already known and T is constructed up to the i-th level, reveal L_{i+1} and expand T as follows: for every $u \in V$ not in the tree, examine the possible edges from u to L_i one by one. Stop either when an examined edge from u to L_i is accepted to the graph (in this case, $u \in L_{i+1}$ and the accepted edge is added to the tree) or when all possible edges from u to L_i are rejected (here $u \notin L_{i+1}$). An easy induction shows that the newly added vertices are exactly all vertices of L_{i+1}.

Stop this process after L_r is revealed. The remaining unexamined edges can later be examined in any order. Let $T = T(v)$ be the spanning tree of $U_r(v, G)$ and let $R = R(v) = U_r(v, G) \setminus T(v)$ (i.e. R is the subgraph of $U_r(v, G)$ whose edges are those of $U_r(v, G)$ not in $T(v)$). Note that R only consists of unexamined (at this point) edges and rejected edges.

This model with R and T defined as above will be used in Sections 3 and 4.

2.2 Reveal vertices, then connect them

Let $r > 0$ and $v \in V$. This model consists of two phases: the creation phase determines the vertices of $U_r(v, G)$ while the connection phase gradually reveals all edges of $U_r(v, G)$, separating it to a spanning tree T and a subgraph R containing all other edges.

Creation phase This phase constructs L_{i+1} given L_i (starting at $i = 0$ and ending at $i = r - 1$) in the following manner: for every $u \notin L_{\leq i}$, flip a coin with probability p a total of $|L_i|$ times or until the first "yes" answer, whichever comes first. In case of "yes" add u to L_{i+1}.

Connection phase Connect L_i to L_{i-1}, starting at $i = r$ and ending at $i = 1$. The connection of L_i to L_{i-1} consists of two steps:

- **Inner step** Connect every couple of vertices in L_i randomly and independently with probability p.
- **Counting step** For every $u \in L_i$, let $k_u \leq |L_{i-1}|$ be the number of coin flips taken until the first "yes" determined that u is in L_i in the creation phase. Flip the coin $|L_{i-1}| - k_u$ more times. Let $t_u \geq 0$ be the number of additional "yes" answers obtained.
- **Linkage step** For every $u \in L_i$, reveal the neighbours of u in L_{i-1}: choose a vertex in L_{i-1} randomly. Connect it to u and add this edge to T. Now choose (randomly and independently) t_u more vertices from L_{i-1}, connect each of them to u and add the resulting edges to R.

All other possible edges can be later examined in an arbitrary order. This model will be used in Section 5.

3 4-Degeneracy and Upper Bound For $f_5(\ell, r)$

Theorem 3.1. *Let $r > 0$. There exists $\ell_0 = \ell_0(r)$ such that for every $\ell > \ell_0$:*

$$f_5(\ell, r) < (10\ell \log \ell)^{r+1} \tag{3.1}$$

Proof. Define $d(\ell) := 3\ell \log \ell$. Our choice of a random graph for the proof is based on the following proposition.

Proposition 3.2. *Any random graph $G_{n,p}$ with $np = d(\ell)$ satisfies w.h.p.*

$$\chi(G) > \ell \tag{3.2}$$

Proof. By a standard first moment argument (see [6]), w.h.p. there is no independent set of size $(1 + o(1))\frac{2 \log np}{p} = (1 + o(1))\frac{2 \log \ell}{p}$ in G. Consequently,

$$\chi(G) \geq (1 - o(1))\frac{n}{\frac{2 \log \ell}{p}} = (1 - o(1))\frac{d}{2 \log \ell} = (1 - o(1))\frac{3\ell \log \ell}{2 \log \ell} > \ell \tag{3.3}$$

for ℓ large enough. □

Take the random graph $G = (V, E) = G_{n,p}$ with $n = (10\ell \log \ell)^{r+1}$ and $p = \frac{3}{10}(10\ell \log \ell)^{-r}$. G is not ℓ-colorable with high probability since $np = d(\ell)$. We will show that w.h.p. every r-ball in G is 4-degenerate (and hence 5-colorable).

Lemma 3.3. *Fix $r > 0$ and let $\epsilon > 0$ be an arbitrary constant. The maximum degree of a vertex in the random graph $G_{n,p}$ with $n = (10\ell \log \ell)^{r+1}$ and $p = \frac{3}{10}(10\ell \log \ell)^{-r}$ is w.h.p. no more than $(1 + \epsilon)d$.*

Proof. Let $v \in V$. We have $deg(v) \sim Bin(n - 1, p)$ and $\mu = E[deg(v)] = d - p$. We use the following known Chernoff bound (see A.1.12 in [3]): For a binomial random variable X with expectation μ, and for all $\epsilon > 0$ (including $\epsilon > 1$):

$$\Pr(X > (1 + \epsilon)\mu) < \left(\frac{e^\epsilon}{(1 + \epsilon)^{(1+\epsilon)}}\right)^\mu \tag{3.4}$$

Noting that $(1 + \epsilon)d > (1 + \epsilon)\mu$, this bound in our case implies

$$\Pr[deg(v) \geq (1 + \epsilon)d] < \left[\frac{e^\epsilon}{(1 + \epsilon)^{(1+\epsilon)}}\right]^\mu = {\gamma_\epsilon}^{d-p} \tag{3.5}$$

Where $\gamma_\epsilon = e^\epsilon(1 + \epsilon)^{-(1+\epsilon)} < 1$ is a positive constant. Therefore, the probability that there exists a vertex with degree $\geq (1 + \epsilon)d$ is no more than

$$\begin{aligned} n{\gamma_\epsilon}^{d-p} =& e^{\log n + (d-p)\log \gamma_\epsilon} = e^{(1+o(1))(r+1)\log \ell - (1+o(1))\log(1/\gamma_\epsilon)\cdot 3\ell \log \ell} \\ \leq& \ell^{(2+o(1))r - (3+o(1))\log(1/\gamma_\epsilon)\ell} \xrightarrow{\ell \to \infty} 0 \end{aligned} \tag{3.6}$$

Hence with high probability the maximum degree is $< (1 + \epsilon)d$. □

Lemma 3.4. *Fix r and let $\epsilon > 0$. Then with high probability all r-balls in $G_{n,p}$ (with n, p as before) contain at most $(1 + \epsilon)^r d^r$ vertices.*

Proof. The max degree in the graph is w.h.p. $< (1+\epsilon)d$. In this case, an easy induction shows that every i-ball in the graph has at most $(1+\epsilon)^i d^i$ vertices. Setting $i = r$ gives the desired result. $\square$

We are now ready to prove the main result of this section.

Theorem 3.5. *Fix r and let $n = (10\ell \log \ell)^{r+1}$, $p = \frac{3}{10}(10\ell \log \ell)^{-r}$. Then with high probability, every r-ball in $G_{n,p}$ is 4-degenerate.*

To prove this, note that the probability that not every r-ball is 4-degenerate is no more than

$$\Pr\left[\exists v : U_r(v,G) \text{ not 4-degenerate and } \forall u \in V : \deg(u) < (1+\epsilon)d\right] + \\ + \Pr\left[\exists u \in V : \deg(u) \geq (1+\epsilon)d\right] \leq$$

$$\Pr\left[\exists v : U_r(v,G) \text{ not 4-degenerate} \middle| \forall u \in V : \deg(u) < (1+\epsilon)d\right] + o(1) \leq$$

$$n \Pr\left[U_r(v_0,G) \text{ not 4-degenerate} \middle| \forall u \in V : \deg(u) < (1+\epsilon)d\right] + o(1)$$

Where $v_0 \in V$ is an arbitrary vertex. It is therefore enough to show that for fixed $r > 0, v \in V$ and suitable $\epsilon > 0$:

$$\lim_{\ell \to \infty} n \Pr\left[U_r(v,G) \text{ not 4-degenerate} \middle| \forall u \in V : \deg(u) < (1+\epsilon)d\right] = 0 \qquad (3.7)$$

For the rest of the proof, assume that the maximum degree of G is less than $(1+\epsilon)d$. Fix $v \in V$. Then $|U_r(v,G)| \leq (1+\epsilon)^r d^r$ as in the proof of Lemma 3.4.

A non-4-degenerate r-ball contains a subgraph with average degree at least 5, hence it is enough to show that with probability high enough, every subgraph $S = (V_S, E_S) \subseteq U_r(v,G)$ satisfies $|E_S| < 5|V_S|/2$.

Construct a spanning tree T with root $v \in V$ for $U_r(v,G)$ in the spanning tree model described in Subsection 2.1.

Let $S = (V_S, E_S) \subseteq U_r(v,G)$ be an induced subgraph and put $s = |V_S| \leq (1+\epsilon)^r d^r$. Assume that $s \geq 6$ (as every subgraph on < 6 vertices has minimal degree ≤ 4).

The possible edges of S are either in T or rejected from the graph or not examined yet. $S \cap T$ is a forest and contains at most $s-1$ edges. $S \setminus T$ contains at most $\binom{s}{2}$ unexamined possible edges (all other edges are rejected). The probability that an unexamined edge is accepted to the graph is no more than p. Note that here we ignore the conditioning on the maximum degree. By the FKG Inequality (c.f., e.g., [3], Chapter 6) this conditioning can only reduce the probability that we are bounding. Let X be the random variable that counts the number of edges in $S \setminus T$. Then X is dominated by $Bin\left(\binom{s}{2}, p\right)$. That is, for a random variable $Y \sim Bin\left(\binom{s}{2}, p\right)$ we have $\Pr(X > k) \leq \Pr(Y > k)$ for every k. Hence

$$\Pr\left(|E_S| \geq \frac{5s}{2}\right) \leq \Pr\left(X > \frac{3s}{2}\right) \leq \Pr\left(Y > \frac{3s}{2}\right) \qquad (3.8)$$

The expectation of Y is $\mu = \binom{s}{2}p$. An easy consequence on the Chernoff bound in (3.4) implies that

$$\Pr\left(Y > (1+\tau)\mu\right) < \left(\frac{e}{1+\tau}\right)^{(1+\tau)\mu} \qquad (3.9)$$

Pick $\epsilon = \frac{1}{9}$ and put $1+\tau = \frac{3s}{2\mu} = \frac{3}{p(s-1)}$ (where, in particular, $1+\tau > \frac{3}{ps} \geq \frac{3}{3/10(10\ell\log\ell)^{-r}(1+\epsilon)^r d^r} = 10 \cdot 3^r \geq 30$) to get

$$\Pr\left(Y > \frac{3s}{2}\right) < \left(\frac{ep(s-1)}{3}\right)^{3s/2} < (ps)^{3s/2} \tag{3.10}$$

The number of induced subgraphs $S \subseteq U_r(u, G)$ on s vertices is

$$\binom{|U_r(u,G)|}{s} \leq \binom{(1+\epsilon)^r d^r}{s} \leq e^s[(1+\epsilon)d]^{rs} s^{-s} = e^s \left[\frac{10d}{9}\right]^{rs} s^{-s} \tag{3.11}$$

The probability that $U_r(v,G)$ is not 4-degenerate is therefore no more than

$$\sum_{s=6}^{[(1+\epsilon)d]^r} e^s \left[\frac{10d}{9}\right]^{rs} s^{-s}(ps)^{3s/2} = \sum_{s=6}^{[(1+\epsilon)d]^r} \left[ep\left(\frac{10}{9}d\right)^r\right]^s [ps]^{s/2} \tag{3.12}$$

but $ep(\frac{10d}{9})^r = \frac{3e}{10}(\frac{10d}{3})^{-r}(\frac{10d}{9})^r = \frac{3e}{10}3^{-r} < 1/3$, and the last expression is

$$\leq \sum_{s=6}^{[(1+\epsilon)d]^r} 3^{-s}(ps)^{s/2} \leq \sum_{s=6}^{d^{1/10}} (ps)^{s/2} + \sum_{s=d^{1/10}+1}^{\infty} 3^{-s} \tag{3.13}$$

$$\leq d^{1/10}(pd^{1/10})^3 + 3^{-d^{1/10}} \leq d^{-26r/10} + 3^{-d^{1/10}} \tag{3.14}$$

Since $n = (10\ell\log\ell)^{r+1} \leq (4d)^{r+1} \leq (4d)^{2r}$, we conclude that

$$n\Pr\left[U_r(v,G) \text{ not 4-degenerate} \middle| \forall u \in V : \deg(u) < (1+\epsilon)d\right] \leq \tag{3.15}$$

$$\left[d^{-26r/10} + 3^{-d^{1/10}}\right](4d)^{2r} \leq O(1)\left[d^{-r/2} + e^{-d^{1/10}+2r\log d}\right] \xrightarrow{\ell\to\infty} 0 \tag{3.16}$$

This proves (3.7) and completes the proof of the Theorem.

Theorem 3.1 follows from 3.2 and the last Theorem. □

4 Upper Bound For $f_4(\ell, r)$

Theorem 4.1. *Let $r > 0$. There exists $\ell_0(r)$ such that for every $\ell > \ell_0$:*

$$f_4(\ell, r) < (10\ell\log\ell)^{r+1} \tag{4.1}$$

Proof. Once again we take the random graph $G_{n,p}$ with $n = (10\ell\log\ell)^{r+1}, p = \frac{3}{10}(10\ell\log\ell)^{-r}$ and assume that the maximum degree in G is less than $(1+\epsilon)d = 10d/9$ (taking $\epsilon = 1/9$).

Let $v \in V$ and construct a spanning tree $T(v)$ for $U_r(v,G)$ as in Subsection 2.1. Let $R(v) = U_r(v,G) \setminus T(v)$ be the subgraph of all other edges of $U_r(v,G)$. At this point, the possible edges of R are either rejected or unexamined.

Suppose that R is 2-colorable. T is a tree and is thus 2-colorable. The cartesian multiple of a 2-coloring of T and a 2-coloring of R is a valid 4-coloring of $U_r(v,G) = T \cup R$.

To make R 2-colorable, it is enough to get rid of all cycles of odd length in it. This can be done by deleting a vertex (or an edge) from each such cycle. The expected number of cycles of length k in $R(v)$ is no more than

$$\binom{(1+\epsilon)^r d^r}{k}\frac{(k-1)!}{2}p^k \leq \frac{(1+\epsilon)^{rk} d^{rk} p^k}{2k}$$

$$\leq \frac{1}{2k}\left(\frac{10d}{9}\right)^{rk}\left(\frac{10d}{3}\right)^{-rk} \leq \frac{1}{2k}3^{-k} \tag{4.2}$$

Consequently, the expected number of cycles (in particular, of odd cycles) in $R(v)$ is bounded by

$$\sum_{i=1}^{\infty}\frac{1}{2(2i+1)}3^{-2i-1} < \frac{1}{100} \tag{4.3}$$

And so the probability that $R(v)$ is not 2-colorable is less then 1/100.

Let G' be a graph obtained from G by removing every v for which $R(v)$ contains an odd cycle (that is, the center of each r-ball for which R is not 2-colorable). Observe that $\ell\chi_r(G') \leq 4$. By (4.3), the expected number of vertices that need to be removed to obtain G' is less than $\frac{n}{100}$. By Markov's inequality, with probability at least 1/2 the number of vertices to be removed is less than $\frac{n}{50}$ (note that this computation is without the conditioning on the maximum degree, but by the FKG inequality the same estimate holds also after this conditioning).

On the other side, w.h.p. there is no independent set of size $(1+o(1))\frac{2\log(d)}{p}$ in G (as was discussed in the proof of 3.2). Consequently there is no independent set of such size in G'. We conclude that with probability $\geq \frac{1}{2} - o(1)$, the chromatic number of G' is at least

$$\frac{n - \frac{n}{50}}{(1+o(1))\frac{2\log(d)}{p}} = (1-o(1))\frac{49d}{100\log d} = (1-o(1))\frac{49\cdot 3\ell\log\ell}{100\log\ell} > \ell \tag{4.4}$$

For ℓ large enough. Recall that these estimates are only true assuming the maximum degree is $< (1+\epsilon)d$, but this property holds with high probability.

Thus, the process described above generates with probability $\frac{1}{2} - o(1)$ a graph G' on at most $(10\ell\log\ell)^{r+1}$ vertices which is not ℓ-colorable, but with r-local chromatic number ≤ 4. This completes the proof. □

5 2-Degeneracy And Upper bound For $f_3(\ell, r)$

The main result proved in this section is

Theorem 5.1. *Let $r > 0$. There exists $\ell_0(r)$ such that for every $\ell > \ell_0$:*

$$f_3(\ell, r) < (10\ell\log\ell)^{r+1} \tag{5.1}$$

To prove this, we show the following.

Theorem 5.2. *Let $r > 0$, $v \in V$ where $G = G_{n,p} = (V, E)$, $n = (10\ell\log\ell)^{r+1}$, $p = \frac{3}{10}(10\ell\log\ell)^{-r}$. Then $U_r(v, G)$ is 2-degenerate with probability at least $0.99 - o(1)$.*

The rest of this section is designed as follows. First it is shown that Theorem 5.1 follows easily from Theorem 5.2. To prove 5.2, we consider an algorithm that checks if $U_r(v, G)$ is 2-degenerate while revealing it as in Subsection 2.2. The algorithm is shown to be valid (that is, a "yes" answer implies that $U_r(v, G)$ is indeed 2-degenerate). The last part of this section shows that a "yes" answer is returned with probability $> 0.99 - o(1)$.

To see why 5.1 follows from 5.2, note that the expected number of non-2-degenerate r-balls in $G_{n,p}$ is no more than $(\frac{1}{100} + o(1))n$. Taking $G = G_{n,p}$ and deleting the centers of all non-2-degenerate r-balls generates a graph G' with $\ell\chi_r(G') \leq 3$. Markov's inequality implies, as in Section 4, that with probability at least $1/2 - o(1)$ we do not delete more than $\frac{2}{100}n$ centres, thus $\chi(G') > \ell$ holds with probability $> \frac{1}{2} - o(1)$. This completes the proof of Theorem 5.1.

The rest of this section is dedicated to proving Theorem 5.2. Let $v \in V$. For the (more complicated) analysis of this problem, we use the model of revealing $U_r(v, G)$ presented in subsection 2.2.

We start with some definitions. First, recall the definition of a level with respect to a vertex.

Definition 5.3. For a subgraph $F = (V_F, E_F) \subseteq U_r(v, G)$, let

$$L_i(v, F) = \{u \in V_F : d_G(u, v) = i\}$$

denote the *i-th level* (with respect to v in F). Moreover, define

$$L_{\geq i}(v, F) = \bigcup_{j=i}^{r} L_j(v, F) \ ; \ L_{\leq i}(v, F) = \bigcup_{j=0}^{i} L_j(v, F)$$

Note that the distance $d_G(u, v)$ here denotes distance in G, not in F.

The notation L_i (without specifying v and F) refers to $L_i(v, G)$. The same holds for $L_{\geq i} = L_{\geq i}(v, G)$ and $L_{\leq i} = L_{\leq i}(v, G)$. For convenience we will also sometimes use these notations to describe the induced subgraph of F on the relevant set of vertices.

The next definition presents a few special types of paths and cycles, to be used later when describing and analyzing the algorithm.

Definition 5.4. Let $F \subseteq U_r(v, G)$.

- An *i-path* in F is a simple path in $L_{\geq i}(v, F)$ whose endpoints belong to $L_i(v, F)$.
- An i-cycle in F is a simple cycle in $L_{\geq i}(v, F)$ with at least one vertex in $L_i(v, F)$.
- An *i-horseshoe* in F is a path of the form

  $$uw_1 \dots w_k z$$

  where $u, z \in L_{i-1}(v, F)$, $k \geq 1$, $uw_1, w_k z \in R$ and $w_1 \dots w_k$ is an i-path in F. Specifically in the case $k = 1$ we also require $u \neq z$.
- An *i-sub-horseshoe* in F is a path of the form

  $$u' w_1 \dots w_k z' \tag{5.2}$$

where $u', z' \in L_{i-1}(v, F)$, $k \geq 1$, $u'w_1, w_kz' \in F$ and $w_1 \dots w_k$ is included in the interior of some i-horseshoe. Specifically in the case $k = 1$ we also require $u' \neq z'$.

Note that every i-horseshoe is also an i-sub-horseshoe, but the other direction is not true in general. Here the interior of a path denotes the induced subpath on all vertices except for the endpoints.

5.1 Algorithm for checking if $U_r(v, G)$ is 2-degenerate

Consider the following algorithm to check if $U_r(v, G)$ is 2-degenerate. This algorithm always returns "no" if the ball is not 2-degenerate, but is not assured to return "yes" for a 2-degenerate ball. We will show that the probability of a "yes" answer is high enough, implying that the r-ball is 2-degenerate with high enough probability.

Our algorithm (applied while revealing $U_r(v, G)$ as described in Subsection 2.2) maintains a subgraph F which initially consists of all vertices of $U_r(v, G)$ where the edges are not yet revealed. It then gradually reveals information about the edges of $U_r(v, G)$ and adds these edges to F while deleting vertices whose neighbours in F are revealed but their degree is at most 2. Some conditions might lead to a "no" answer returned by the algorithm, but if it succeeds to delete all vertices of F, it returns "yes".

It can be seen as a pessimistic version of the naive approach of trying to remove vertices of degree ≤ 2 from the graph until all the vertices are removed (a "yes" answer) or until a subgraph with minimum degree ≥ 3 is revealed (a "no" answer). Our algorithm is less accurate but easier to analyze than the naive approach.

Algorithm 5.1 - detailed description

1. Creation phase
   (a) Reveal the levels L_i of $U_r(v, G)$.
      i. If for some $1 \leq i \leq r$ it holds that $|L_i| > (1 + \epsilon)d|L_{i-1}|$ with $\epsilon = 1/9$, return "no".
      ii. Initialize a subgraph F with all vertices of $U_r(v, G)$ and no edges.
2. Connection phase: For every level L_i from $i = r$ to $i = 1$ do:
   (a) Inner step: reveal all inner edges of L_i, i.e. edges in G of the form $\{u, u'\}$ where $u \neq u' \in L_i$. Add them to F.
      i. At this point all edges of $L_{\geq i}(v, F)$ are revealed. If there exists an i-cycle in F, return "no".
   (b) Counting step: for every $u \in L_i$, determine how many neighbours it has in L_{i-1}.
      i. At this point we know the degree (in F) of all vertices in $L_{\geq i}$. If there exists $u \in L_{\geq i}(v, F)$ with degree ≤ 2 in F – delete u. Repeat until all vertices of $L_{\geq i}(v, F)$ are of degree > 2 in F.

ii. The number of i-sub-horseshoes in F is also known now. If this number is bigger than b_i (to be determined later), return "no". Moreover, the structures of the i-(sub-)horseshoes are known aside from the identities of their endpoints in L_{i-1}.

(c) Linkage step: For every $u \in L_i$, reveal the neighbours of u in L_{i-1}, adding one of the new edges to T and the others to R. Add all new edges to F.

i. At this point, all the i-horseshoes and i-sub-horseshoes are revealed.

Finally, if the connection phase ends without returning "no", the algorithm returns "yes".

Lemma 5.5 (validity of the algorithm). *If algorithm 5.1 returns "yes", then $U_r(v, G)$ is 2-degenerate.*

Proof. Assume that the algorithm returned "yes". In the end of the iteration $i = 1$, $L_{\geq 1}(v, F)$ does not contain cycles – since a "no" has not been returned before then. Therefore, $L_{\geq 1}(v, F) = F \setminus \{v\}$ is a forest and thus 1-degenerate, implying that F is 2-degenerate at that point. Note that the algorithm does not need to inspect the edges between v and L_1, since the 1-degeneracy of $F \setminus \{v\}$ suffices.

Observe that if a vertex v has degree ≤ 2 in a graph H, then H is 2-degenerate if and only if $H \setminus \{v\}$ is 2-degenerate.

Let $v_1, \ldots, v_m$ be the ordered sequence of vertices that were deleted from F during the algorithm. Let $F_i = U_r(v, G) \setminus \{v_1, \ldots, v_i\}$ for $i = 0, \ldots, m$. Clearly, v_{i+1} is of degree at most 2 in F_i (since we only delete a vertex if it is of degree at most 2 in F at that point). The previous observation implies that F_i is 2-degenerate if and only if F_{i+1} is 2-degenerate. Moreover, the first argument states that F_m is 2-degenerate. Therefore, by induction F_i is 2-degenerate for every i. Noting that $F_0 = U_r(v, G)$ finishes the proof. □

5.2 Analysis of the algorithm

We first present notation that is used throughout the analysis. Afterwards we characterize the set of vertices in $L_{\geq j}$ that survive iteration $i = j$. We use this characterization to give bounds (valid with high probability) on the number of j-sub-horseshoes revealed in a given iteration as well as the probability to reveal a j-cycle. This gives us the desired lower bound on the probability that the algorithm returns "yes", which implies (along with Lemma 5.5) that an r-ball in $G_{n,p}$ is 2-degenerate with sufficiently high probability.

Notation The following quantities are of interest for analysing Algorithm 5.1:

n_j number of vertices in $L_j(v, G)$.

c_j number of j-cycles in F at the end of the inner step (2a) in iteration $i = j$ of the connection phase of Algorithm 5.1.

h_j number of j-sub-horseshoes in F at the end of the counting step (2b) in iteration $i = j$ of the connection phase.

The next group of notations refers to the probability to get a "no" answer at some point of the algorithm assuming a "no" has not been returned before then.

q^l probability that step (1(a)i) reveals that $n_{j+1} > (1+\epsilon)dn_j$ for some j.

q^c_j probability that $c_j > 0$ assuming the algorithm has not returned "no" before iteration $i = j$ of the connection phase.

q^h_j probability that $h_j > b_j$ (b_j will be determined later) assuming the algorithm has not returned "no" before iteration $i = j$ of the connection phase.

Note that $h_1 = 0$ and these three conditions are the only ones that lead to a "no" answer, implying the following lemma.

Lemma 5.6. *The probability that algorithm 5.1 returns "no" is no more than*

$$q^l + \sum_{j=1}^{r} q^c_j + \sum_{j=2}^{r} q^h_j \tag{5.3}$$

The proof of Theorem 5.2 follows from the next Theorem, along with Lemmas 5.5 and 5.6.

Theorem 5.7. *The following holds with respect to algorithm 5.1 on $G_{n,p}$ and v defined as above:*

$$q^l = o(1) \tag{5.4}$$

$$q^c_r < \frac{1}{100} \tag{5.5}$$

$$\sum_{j=1}^{r-1} q^c_j + \sum_{j=2}^{r} q^h_j = o(1) \tag{5.6}$$

Proof. (5.4) is immediate from Lemma 3.3.

As in (4.2) and (4.3) and since L_r is of size at most $(1+\epsilon)^r d^r$, the expected number of cycles in L_r is no more than

$$\sum_{k=3}^{\infty} \frac{1}{2k} 3^{-k} < \frac{1}{100} \tag{5.7}$$

which proves (5.5). In the rest of the proof we establish (5.6).

Horseshoes and sub-horseshoes We start by explaining why horseshoes and sub-horseshoes are important for the analysis of this problem.

Lemma 5.8. *Observe F after step 2(b)i of iteration $i = j$ of the algorithm. The following holds for F at that point:*

- *A vertex in $L_{\geq j}$ might remain in F only if it lies in a j-horseshoe of F.*
- *A path in $L_{\geq j}$ might remain in F only if it is contained in a j-horseshoe of F.*
- *An edge of $U_r(v, G)$ that has an endpoint (two endpoints) in $L_{\geq j}$ might remain in F only if it lies in some j-sub-horseshoe (j-horseshoe) of F.*

Proof. We will prove the first part of the lemma, and then show how the second part follows. The third part is an easy consequence of the first two parts.

To prove the first part, observe F at the end of step 2b in iteration $i = j$ of the algorithm. Let $w \in L_{\geq j}(v, F)$ be a vertex that is not contained in any j-horseshoe at this point. Then there is at most one edge e touching w that is the first edge of a path P from w to L_{j-1} whose interior is in $L_{\geq j}$ and last edge is in R (note that this interior might also be empty if P is a single edge). Otherwise, let $e_1 \neq e_2$ be such edges and let P_1, P_2 be the corresponding paths. Since $L_{\geq j}(v, F)$ does not contain cycles at this point, the interiors of P_1 and P_2 are disjoint. Thus w lies in the horseshoe $P_1 \cup P_2$, a contradiction. Hence there exists at most one edge e of this type. We can assume that there exists exactly one.

Let S_w be the connected component of w in $L_{\geq j}(v, F) \setminus \{e\}$ at this point. Any vertex aside from w has at most one neighbour in F outside S_w (that is its parent in T). Moreover, S_w is a forest and thus contains a leaf $z \neq w$. z has degree ≤ 2 in F and can be removed from it.
This process ends when all vertices of $S_w \setminus \{w\}$ are removed from F, leaving w with at most two neighbours: its parent in T and the other endpoint of e. At this point, w can be removed from F, completing the proof of the first part.

To show the second part of the lemma, let P be a path in $L_{\geq j}(F)$ whose endpoints are $u \neq w$. By the first part of the lemma, u (w) is contained in a j-horseshoe, thus there are at least two internally-disjoint paths from u (from w) to L_{j-1} in F whose last edge is in R, and at least one of them does not contain any other vertex of P – otherwise F would contain a j-cycle. Denote this path by P_u (P_w). P_u and P_w are internally-disjoint (again, since there are no j-cycles in F). Thus, $P_u \cup P \cup P_w$ is a horseshoe in F containing P. □

Recall that the bounds b_j in step 2(b)ii have not been defined yet. Take $b_1 = 0$ since there are no 1-horseshoes. For $1 < j \leq r$ take $b_j = \frac{n_{j-1}}{\ell}$. The reasoning for these choices will be clearer later.

r-horseshoes and q_r^h A r-horseshoe of length $k+1$ is a path in R with both endpoints in L_{r-1} and $k > 0$ interior points in L_r. The number of candidates to be r-horseshoes of length $k+1$ is $\leq n_{r-1}^2 n_r^k$. FKG inequality implies that each candidate is indeed a r-horseshoe in $U_r(v, G)$ with probability at most p^{k+1}. Such a horseshoe, if exists, forms no more than $1 + 2k + \binom{k}{2} \leq 3k^2$ r-sub-horseshoes. Combining everything we get

$$E[h_r|\text{algorithm did not return "no" before sampling } h_r] \leq \tag{5.8}$$

$$\sum_{k=1}^{\infty} 3k^2 p^{k+1} n_{r-1}^2 n_r^k = 3n_{r-1}(pn_{r-1}) \sum_{k=1}^{\infty} k^2 (pn_r)^k \leq \tag{5.9}$$

$$3n_{r-1} \frac{3^{-r}}{d} \sum_{k=1}^{\infty} k^2 3^{-rk} \leq O(1) \frac{n_{r-1}}{d} \tag{5.10}$$

the inequality in (5.9) is true since

$$pn_j \leq \frac{3}{10} \left(\frac{10}{3} d \right)^{-r} (10/9)^j d^j < 3^{-r} d^{j-r} \tag{5.11}$$

Applying Markov's inequality to (5.8) we get:

$$q_r^h \le \frac{O(1)\frac{n_{r-1}}{d}}{b_r} = O\left(\frac{\ell}{d}\right) = O(1/\log \ell) = o(1) \tag{5.12}$$

j-cycles and j-horseshoes for $j < r$

Assume that the algorithm has not returned "no" in step 1(a)i or in iterations $r, r-1, \dots, j+1$ of the connection phase. In particular, the number of $(j+1)$-sub-horseshoes in F is at most b_{j+1} and there are no $(j+1)$-cycles in F. At this point in the algorithm, the inner structures of the $(j+1)$-horseshoes are known, but their endpoints are not yet determined (as the last possible "no" answer of iteration $j+1$ of the connection phase comes after the inner structures are determined but before step 2c is taken).

A j-cycle has parameters m, k (with $1 \le m \le n_j$, $0 \le k \le m$) if it consists of exactly m vertices in L_j, k internally-disjoint $(j+1)$-sub-horseshoes (the interiors are disjoint since a j-cycle is simple) and $m-k$ inner edges of L_j. Any j-cycle in F can be presented in such a form: indeed, in a j-cycle of F, a path between two vertices of L_j whose interior does not contain vertices in L_j is either an inner edge of L_j or a path whose interior is in $L_{\ge j+1}$. In this case, by the second part of Lemma 5.8, the interior is contained in a $(j+1)$-horseshoe of F, and so the path is a $(j+1)$-sub-horseshoe.

A j-horseshoe with parameters m, k ($1 \le m \le n_j$, $0 \le k \le m-1$) is defined similarly: it consists of m vertices in L_j, k internally-disjoint $(j+1)$-sub-horseshoes, $m-1-k$ inner edges of L_j and two edges down to L_{j-1} that are in R. Again, any j-horseshoe can be presented in this form.

We now bound the expected number of j-cycles and j-horseshoes. We do so by estimating the number of such objects with parameters m, k for all possible values of m, k.

Fix $a_1, \dots, a_k, b_1, \dots, b_k \in V$ (not necessarily distinct) and internally-disjoint $(j+1)$-sub-horseshoes $H_1, \dots, H_k$. The probability that a specific H_i has endpoints a_i, b_i is at most $\frac{1}{\binom{n_j}{2}} \le \frac{4}{n_j^2}$ (this is true for $n_j \ge 2$; if $n_j = 1$ then there are no $(j+1)$-sub-horseshoes anyway). These k events are independent (as per step 2c in the connection phase), and the probability that all of them occur together is at most $\frac{1}{\binom{n_j}{2}^k} \le \frac{4^k}{n_j^{2k}}$.

There are no more than b_{j+1}^k possible ordered choices of $(H_1, \dots, H_k)$. Therefore, the expected number of ordered sets of k internally-disjoint $(j+1)$-sub-horseshoes with endpoints $(a_1, b_1), \dots, (a_k, b_k)$ is no more than

$$\frac{4^k b_{j+1}^k}{n_j^{2k}} \le \frac{4^k}{\ell^k n_j^k} \tag{5.13}$$

j-cycles First we bound the expected number of j-cycles with parameters m, k in F after step 2a in iteration $i = j$ of the connection phase. Fix m vertices $(v_1, v_2, \dots, v_m) \in L_j$ and order them cyclically (there are at most n_j^m such orderings). Now fix k couples of neighbouring vertices a_i, b_i in the chosen cyclic order (there are $\binom{m}{k} \le 2^m$ possible choices of k-tuples). The expected number of k-tuples of internally-disjoint $(j+1)$-sub-horseshoes with endpoints a_i, b_i is no more than $\frac{4^k}{\ell^k n_j^k}$. The probability for any other couple of neighbours in the cyclic ordering to have an edge between them is p independently of everything else. Since the expected product of independent random variables is the product of their expectations, we get that the expected number of j-cycles with parameters m, k is no more than

$$n_j^m 2^m \frac{4^k}{\ell^k n_j^k} p^{m-k} = (2pn_j)^{m-k} \left(\frac{8}{\ell}\right)^k \leq \left(\frac{1}{d}\right)^{m-k} \left(\frac{8}{\ell}\right)^k \leq \left(\frac{8}{\ell}\right)^m \tag{5.14}$$

And the total expected number of j-cycles is no more than

$$\sum_{m=1}^{n_j} \sum_{k=0}^{m} \left(\frac{1}{d}\right)^{m-k} \left(\frac{8}{\ell}\right)^k = \sum_{m=1}^{n_j} \left(\frac{8}{\ell}\right)^m (1+o(1)) = \frac{8}{\ell}(1+o(1)) = o(1) \tag{5.15}$$

In particular we get

$$q_j^c = O(1/\ell) = o(1) \tag{5.16}$$

j-horseshoes We bound the expected number of j-(sub-)horseshoes with parameters m, k in F after step 2b in iteration $i = j$ of the connection phase. Fix m linearly ordered vertices $(v_1, v_2, \ldots, v_m) \in L_j$ (there are at most n_j^m such orderings). Now fix k couples of neighbouring vertices a_i, b_i in the chosen linear ordering (there are $\binom{m-1}{k} \leq 2^{m-1}$ possible choices). The expected number of k-tuples of internally-disjoint $(j+1)$-sub-horseshoes with endpoints a_i, b_i is no more than $\frac{4^k}{\ell^k n_j^k}$. The probability for any other couple of neighbours in the linear ordering to have an edge between them is p independently of everything else. For a vertex in L_j, the expected number of neighbours via R it has in L_{j-1} is no more than $n_{j-1}p$. Combining all of the above, the expected number of j-horseshoes with parameters m, k is no more than

$$n_j^m 2^{m-1} \frac{4^k}{\ell^k n_j^k} p^{m-1-k} (n_{j-1}p)^2 = \tag{5.17}$$

$$(2n_j p)^{m-1-k} (n_j p)(n_{j-1}p)(8/\ell)^k n_{j-1} \leq \tag{5.18}$$

$$d^{(j-r)(m-1-k)} d^{j-r} d^{j-1-r} (8/\ell)^k n_{j-1} \leq \tag{5.19}$$

$$d^{-(m-1-k)} d^{-3} (8/\ell)^k n_{j-1} = \Theta(\log \ell)^k d^{-m-2} n_{j-1} \tag{5.20}$$

Each j-horseshoe with such parameters contributes no more than $1+2m+\binom{m}{2} \leq 3m^2$ j-sub-horseshoes, and the total expected number of j-sub-horseshoes in F is at most

$$3 \sum_{m=1}^{n_j} \sum_{k=0}^{m-1} \Theta(\log \ell)^k d^{-m-2} n_{j-1} m^2 \leq \tag{5.21}$$

$$\sum_{m=1}^{n_j} o(1) \Theta(\log \ell)^{m+2} d^{-m-2} m^2 n_{j-1} \leq \Theta(\ell)^{-3} n_{j-1} \tag{5.22}$$

By Markov's inequality,

$$q_j^h \leq \frac{\Theta(\ell)^{-3} n_{j-1}}{b_j} \leq \Theta(\ell)^{-2} = o(1) \tag{5.23}$$

The proof of (5.6) is now complete by (5.16), (5.23) and since r is fixed. □

Remark 5.9. Special care should be taken in proofs of this type to ensure that no source of randomness is used more than once (that is, to prevent the case when some information is revealed at some point of the algorithm but is assumed to be random later on). In particular, note that the information needed to determine how many j-sub-horseshoes there are does not interfere with the information needed to know, given all the interiors of j-sub-horseshoes without knowing their endpoints yet, what is the probability that specific k internally-disjoint j-sub-horseshoes have specific k couples of endpoints.

6 $f_c(\ell, r)$ For Non-Constant c

In the previous sections, $f_c(\ell, r)$ with small fixed c values was considered. In this section our results are extended to large values of c. Take G on n vertices, $0.98(10\ell \log \ell)^{r+1} \leq n \leq (10\ell \log \ell)^{r+1}$ with $\ell\chi_r(G) = 3$ and with no independent set of size $(1 + o(1))\frac{2\log(d)}{p}$, where, as before, $d = 3\ell \log \ell$ and $p = \frac{3}{10}(10\ell \log \ell)^{-r}$. Such G exists for any $\ell > \ell_0(r)$ by the results in Section 5.

Construct the following graph G_k: every vertex in the original G is expanded to a k-clique. Two vertices in G_k are connected if they lie in the same clique or if the cliques in which they lie were neighbours in G. Every independent set in G_k contains at most one vertex from each clique, and thus the maximal independent set in G_k is of size $< (1 + o(1))\frac{2\log(d)}{p}$. There are kn vertices in G_k and thus its chromatic number is (for ℓ large enough)

$$\chi(G_k) \geq \frac{kn}{(1 + o(1))\frac{2\log(d)}{p}} > k\ell \tag{6.1}$$

Every r-ball in G_k is contained in an expanded r-ball from G. Thus

$$\ell\chi_r(G_k) \leq 3k \tag{6.2}$$

We conclude that for ℓ^* large enough

$$f_{3k}(k\ell^*, r) < kn \leq k(10\ell^* \log \ell^*)^{r+1} \tag{6.3}$$

Taking $c = 3k$, $\ell = k\ell^*$ the last result implies that

$$f_c(\ell, r) < \frac{c}{3}\left(10\frac{\ell}{c/3}\log\left(\frac{\ell}{c/3}\right)\right)^{r+1} < \frac{(30\ell \log \ell)^{r+1}}{c^r} \tag{6.4}$$

When c and ℓ that are not of this form, we need to replace them by $3\lfloor c/3 \rfloor \leq c$ and $\lfloor c/3 \rfloor \lceil \frac{\ell}{\lfloor c/3 \rfloor} \rceil \geq \ell$ respectively. The following Theorem summarizes the discussion.

Theorem 6.1. *Let $r > 0$. There exists $\ell_0(r)$ such that for every positive c divisible by 3 and $\ell \geq max(c, \ell_0)$ divisible by $c/3$:*

$$f_c(\ell, r) < \frac{(30\ell \log \ell)^{r+1}}{c^r} \tag{6.5}$$

Thus there exists $\alpha_r > 0$ such that for every $\ell \geq c \geq 3$:

$$f_c(\ell, r) < \frac{[\alpha_r \ell \log \ell]^{r+1}}{c^r} \tag{6.6}$$

Remark 6.2. The contribution of c in this upper bound is c^{-r}, whereas this contribution in the corresponding lower bound by Bogdanov in (1.4) is c^{-r-1}.

7 Lower bounds for $f_c(\ell, r)$

In Subsection 7.1, a tight lower bound for $f_c(\ell, 1)$ is shown. In Subsection 7.2, this lower bound is generalized to any fixed $c \geq 2$ and r.

7.1 $f_c(\ell, 1)$

As stated in Section 1.2 it is known that $f_2(\ell, 1) = \Theta\left(\ell^2 \log \ell\right)$. In this section it is shown that $f_c(\ell, 1) = \Theta(\ell^2 \log \ell)$ for any fixed $c \geq 2$. Since $f_c(\ell, 1) \leq f_2(\ell, 1)$, we only need to show that $f_c(\ell, 1) = \Omega(\ell^2 \log \ell)$ for fixed $c \geq 2$.

Theorem 7.1. *There exists $\alpha > 0$ such that for every $\ell \geq c \geq 2$*

$$f_c(\ell, 1) \geq \alpha \frac{\ell^2 \log \ell}{c \log c} \tag{7.1}$$

In particular, for any fixed $c \geq 2$:

$$f_c(\ell, 1) = \Theta(\ell^2 \log \ell) \tag{7.2}$$

Proof. Let $G = (V, E)$ be a graph on $n = f_c(\ell, 1) + 1$ vertices with $\ell\chi_1(G) \leq c$ but $\chi(G) > \ell$. Our goal is to show[1] that $n \geq \alpha \frac{\ell^2 \log \ell}{c \log c}$ for a suitable choice of α. By taking a critical subgraph of G we can assume that the minimum degree of G is at least ℓ and clearly we can also assume that $n \leq \zeta \ell^2 \log \ell$ for some absolute constant $\zeta > 0$. By these assumptions, the average degree d in G satisfies $\ell \leq d < n \leq \zeta \ell^2 \log \ell$.

Large independent set in G Observe that

- There exists $v \in V$ with $\deg(v) \geq d$. The neighbourhood of v is c-colorable, and thus contains an independent set of size at least $d/c \geq \ell/c$.
- The first author [2] showed that there exists $\beta > 0$ such that any graph G on n vertices with average degree $d \geq 1$ and $\ell\chi_1(G) \leq c$ contains an independent set of size
$$\frac{\beta}{\log c} \frac{n}{d} \log d$$

Lemma 7.2. *There exists an independent set of size $\geq \delta \sqrt{\frac{n \log n}{c \log c}}$ in G where $\delta > 0$ is a suitable global constant.*

Proof. There exists an independent set of size

$$\max \left\{ \frac{d}{c}, \frac{\beta}{\log c} \frac{n}{d} \log d \right\} \geq \sqrt{\frac{d}{c} \frac{\beta}{\log c} \frac{n}{d} \log d} \geq \sqrt{\frac{\beta n \log \ell}{c \log c}} \geq \delta \sqrt{\frac{n \log n}{c \log c}} \tag{7.3}$$

as needed. ☐

[1] In fact we need to show this for $n - 1$ instead of n but it is clearly equivalent.

Removing an independent set of size $\delta\sqrt{\frac{f_c(\ell,1)\log f_c(\ell,1)}{c\log c}} \leq \delta\sqrt{\frac{n\log n}{c\log c}}$ from G results in a non-$(\ell-1)$-colorable graph. Hence

$$f_c(\ell-1,1) \leq f_c(\ell,1) - \delta\sqrt{\frac{f_c(\ell,1)\log f_c(\ell,1)}{c\log c}} \tag{7.4}$$

For δ small enough and $c \geq 2$, the function

$$h(x) := x - \delta\sqrt{\frac{x\log x}{c\log c}} \tag{7.5}$$

is increasing in the domain $[2,\infty)$. Now take $\alpha = \min(1, \delta^2/9)$ and fix $c \geq 2$. We will show that $f_c(\ell,1) \geq \alpha\frac{\ell^2\log\ell}{c\log c}$ for every $\ell \geq c$ by induction on ℓ. The base case $\ell = c$ satisfies

$$f_c(c,1) = c = \frac{\ell^2\log\ell}{c\log c} \geq \alpha\frac{\ell^2\log\ell}{c\log c} \tag{7.6}$$

Assuming that $f_c(\ell,1) \geq \alpha\frac{\ell^2\log\ell}{c\log c}$ and using (7.4) we get

$$\alpha\frac{\ell^2\log\ell}{c\log c} \leq f_c(\ell+1,1) - \delta\sqrt{\frac{f_c(\ell+1,1)\log f_c(\ell+1,1)}{c\log c}} = h\left(f_c(\ell+1,1)\right) \tag{7.7}$$

Note that $f_c(\ell+1,1) \geq \ell+1$. If $\alpha\frac{(\ell+1)^2\log(\ell+1)}{c\log c} \leq \ell+1$ then we are finished. Otherwise, take $x = \alpha\frac{(\ell+1)^2\log(\ell+1)}{c\log c} \geq \ell+1 \geq 3$. Then

$$\begin{aligned} x - \alpha\frac{\ell^2\log\ell}{c\log c} &= \alpha\left[((\ell+1)^2-\ell^2)\frac{\log(\ell+1)}{c\log c} + (\log(\ell+1)-\log\ell)\frac{\ell^2}{c\log c}\right] \\ &\leq \alpha\frac{(2\ell+1)\log(\ell+1)+\ell}{c\log c} \leq 3\alpha\frac{(\ell+1)\log(\ell+1)}{c\log c} \\ &= 3\alpha\sqrt{\frac{\frac{(\ell+1)^2\log(\ell+1)}{c\log c}\log(\ell+1)}{c\log c}} \leq 3\sqrt{\alpha}\sqrt{\frac{x\log x}{c\log c}} \leq \delta\sqrt{\frac{x\log x}{c\log c}} \end{aligned}$$

The last inequality and (7.7) imply that

$$h(x) \leq \alpha\frac{\ell^2\log\ell}{c\log c} \leq h(f_c(\ell+1,1)) \tag{7.8}$$

By the monotonicity of h,

$$\alpha\frac{(\ell+1)^2\log(\ell+1)}{c\log c} = x \leq f_c(\ell+1,1) \tag{7.9}$$

finishing the induction step and completing the proof. □

7.2 The general case

Here we generalize the lower bound from Subsection 7.1 to any fixed $c \geq 2$ and r. In this domain, our lower bound improves the known lower bound, due to Bogdanov [5], by a multiplicative factor logarithmic in ℓ.

The following theorem was proved together with Michael Krivelevich.

Theorem 7.3. *Let $c \geq 2$ and r. There exists $\beta_{c,r}$ such that for every $\ell \geq c$:*

$$f_c(\ell, r) \geq \beta_{c,r} \ell^{r+1} \log \ell \tag{7.10}$$

We will not try to determine the exact value of $\beta_{c,r}$ with respect to c and r. The main ingredient of the proof is the following lemma.

Lemma 7.4. *Let $c \geq 2$ and r. There exists $\alpha_{c,r} < 1$ such that any graph G on n vertices with $\ell\chi_r(G) \leq c$ contains a c-colorable induced subgraph on $\alpha n^{\frac{r}{r+1}} (\log n)^{\frac{1}{r+1}}$ vertices.*

Proof. Let $d = n^{\frac{1}{r+1}} (\log n)^{\frac{r}{r+1}}$. Let G be such a graph. If there exists an induced subgraph on $n/2$ vertices with maximum degree smaller than d, then by a result of the first author [2], there exists an independent set of size $\Omega_c\left(\frac{n \log d}{d}\right) = \Omega_{c,r}\left(n^{\frac{r}{r+1}} (\log n)^{\frac{1}{r+1}}\right)$ in this subgraph, and thus, in G.

Otherwise, we run the method presented in Bogdanov's proof with a minor modification, to construct a large c-colorable induced subgraph: start with an empty graph H and $R = G$. While R has at least $n/2$ vertices, take a vertex v with degree at least d in R. Since

$$\prod_{i=1}^{r} \frac{|U_{i+1}(v, R)|}{|U_i(v, R)|} = \frac{|U_{r+1}(v, R)|}{|U_1(v, R)|} \leq \frac{n}{d+1} < \frac{n}{d} \tag{7.11}$$

There exists $i \in \{1, \ldots, r\}$ with $\frac{|U_{i+1}(v,R)|}{|U_i(v,R)|} < (\frac{n}{d})^{1/r}$. Update H, R as follows:

$$H := H \cup U_i(v, R) \ , \ R := R \setminus U_{i+1}(v, R)$$

Repeat until R has less than $n/2$ vertices. The resulting H has at least $\frac{n}{2}(\frac{d}{n})^{1/r} = \frac{1}{2} n^{\frac{r}{r+1}} (\log n)^{\frac{1}{r+1}}$ vertices, and is c-colorable – in each step, a c-colorable ball in R is added to H, and the balls are the different connected components of H.

In both cases G contains an induced subgraph on $\Omega_{c,r}\left(n^{\frac{r}{r+1}} (\log n)^{\frac{1}{r+1}}\right)$ vertices that is c-colorable, completing the proof of the lemma. □

Now we prove the main result, namely that there exists $\beta_{c,r}$ such that for every $\ell \geq c$, $f_c(\ell, r) \geq \beta \ell^{r+1} \log \ell$. It is sufficient to show the same bound for $g_c(\ell, r) := f_c(\ell, r) + 1$, the smallest number of vertices in a graph G with $\ell\chi_r(G) \leq c$ and $\chi(G) > \ell$.

Let $L \geq c$. For every $c \leq \ell \leq L$, we have:

$$g_c(\ell, r) \geq \left(\frac{\ell/c + r/2}{r+1}\right)^{r+1} \geq \left(\frac{\ell}{2cr}\right)^{r+1} \geq \frac{\ell^{r+1} \log \ell}{(2cr)^{r+1} \log L} = \beta_{c,r,L} \ell^{r+1} \log \ell \tag{7.12}$$

Where $\beta_{c,r,L} := \left((2cr)^{r+1} \log L\right)^{-1}$. The first inequality follows from Bogdanov's lower bound. It remains to find a suitable L for which Inequality (7.12) (with the corresponding $\beta_{c,r,L}$) is true also when $\ell > L$. We will show that for $L_{c,r} \geq 2c$ large enough, the assumption

$$g_c(\ell - c, r) \geq \beta_{c,r,L} (\ell - c)^{r+1} \log(\ell - c) \tag{7.13}$$

For some $\ell > L$, implies that $g_c(\ell, r) \geq \beta_{c,r,L} \ell^{r+1} \log \ell$. By induction, this will finish the proof of the theorem.

Observe by Lemma 7.4 that when $\ell \geq 2c$,

$$g_c(\ell, r) - \alpha_{c,r}(g_c(\ell, r))^{\frac{r}{r+1}}(\log g_c(\ell, r))^{\frac{1}{r+1}} \geq g_c(\ell - c, r) \tag{7.14}$$

The function $h(n) := n - \alpha_{c,r} n^{r/r+1}(\log n)^{1/r+1}$ is increasing on $\mathbb{N}$ as $\alpha_{c,r} < 1$. Let $\ell > L$. Assuming (7.13) and using (7.14), it is enough to show that

$$h(\beta_{c,r,L}\ell^{r+1}\log \ell) \leq \beta_{c,r,L}(\ell - c)^{r+1}\log(\ell - c) \tag{7.15}$$

Since $\log(\ell - c) > \log \ell - \frac{c}{l-c}$, the RHS satisfies

$$\beta(\ell - c)^{r+1}\log(\ell - c) \geq \beta(\ell - c)^{r+1}\left(\log \ell - \frac{c}{\ell - c}\right) = \beta\ell^{r+1}\log \ell - \beta O_{c,r}(\ell^r \log \ell) \tag{7.16}$$

To bound the LHS of (7.15), note that for $\ell > L$:

$$\begin{aligned}\left[\log\left(\beta_{c,r,L}\ell^{r+1}\log \ell\right)\right]^{1/r+1} &\geq \left[(r+1)\left(\log \ell - \log(2cr)\right)\right]^{1/r+1} \\ &\geq (r+1)^{1/r+1}\left[(\log \ell)^{1/r+1} - (\log(2cr))^{1/r+1}\right] \\ &= \Omega_{c,r}\left((\log \ell)^{1/r+1}\right)\end{aligned}$$

The second inequality is true since $k(x) = x^{1/r+1}$ is concave for $x > 0$. Therefore

$$h(\beta\ell^{r+1}\log \ell) \leq \beta\ell^{r+1}\log \ell - \beta^{r/r+1}\Omega_{c,r}\left(\ell^r \log \ell\right) \tag{7.17}$$

Since $\lim_{L\to\infty}\beta_{c,r,L} = 0$, taking a large enough $L_{c,r}$ (and the corresponding $\beta_{c,r,L}$) implies that indeed for every $\ell > L_{c,r}$:

$$\beta\ell^{r+1}\log \ell - \beta^{r/r+1}\Omega_{c,r}\left(\ell^r \log \ell\right) \leq \beta\ell^{r+1}\log \ell - \beta O_{c,r}(\ell^r \log \ell) \tag{7.18}$$

Combining (7.16), (7.17), (7.18) gives (7.15), completing the proof of the theorem.

8 Final Remarks

8.1 Non-constant r

Our bounds for $f_c(\ell, r)$ are valid for fixed values of r. These bounds still hold if we require that $r \leq \gamma\ell$ for a suitable global constant $\gamma > 0$ instead of requiring r to be fixed. The following amendments of the proof need to be made:

- In Lemma 3.3 we need to make sure that $\ell^{(2+o(1))r-(3+o(1))\log(1/\gamma_\epsilon)\ell} \xrightarrow{\ell\to\infty} 0$ where $\epsilon = 1/9$ and $\gamma_\epsilon = e^\epsilon(1+\epsilon)^{-(1+\epsilon)} < 1$. For ℓ large enough, this expression indeed tends to 0 for every $r \leq \log(1/\gamma_\epsilon)\ell$. Take a suitable $\gamma \leq \log(1/\gamma_\epsilon)$ that is good for every $\ell \geq 2$.
- In Section 5 take

$$b_j = \begin{cases} \frac{n_{r-1}}{\ell} & j = r \\ \frac{n_{j-1}}{d} & 1 < j < r \\ 0 & j = 1 \end{cases} \tag{8.1}$$

  It can be shown that now $q_j^c, q_j^h \leq O(1/d)$ for any $j < r$. This proves (5.6) in Theorem 5.7 and completes the proof of Theorem 5.1.

Note also that in Theorem 3.5 a slightly different analysis is needed for large r, but the stated result remains valid.

8.2 More on $f_c(\ell, r)$

Our general upper bound for $f_c(\ell, r)$ is

$$f_c(\ell, r) < \frac{[\alpha_r \ell \log \ell]^{r+1}}{c^r} \tag{8.2}$$

We have already seen that this bound is tight up to a polylogarithmic factor for fixed $c \geq 3$ and r. For other range of the parameters and in particular when r is very large there is a result of Kierstead, Szemerédi and Trotter [9] providing a lower bound for $f_c(\ell, r)$, which is close to being tight in this range. See also [4]. In some cases, however, the gap between the known upper and lower bounds is large. In particular, it will be interesting to understand better the behaviour of $f_c(r, r)$, and of $f_2(\ell, r)$.

The question of obtaining a better estimation of $f_c(\ell, r)$ in the general case (as well as for fixed $c \geq 3$ and fixed r) is left as an open problem.

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