Very fast construction of bounded-degree spanning graphs via the semi-random graph process

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Abstract

Semi-random processes involve an adaptive decision-maker, whose goal is to achieve some predetermined objective in an online randomized environment. They have algorithmic implications in various areas of computer science, as well as connections to biological processes involving decision making. In this paper, we consider a recently proposed semi-random graph process, described as follows: we start with an empty graph on n vertices, and in each round, the decision-maker, called Builder, receives a uniformly random vertex v, and must immediately (in an online manner) choose another vertex u, adding the edge $\{u, v\}$ to the graph. Builder's end goal is to make the constructed graph satisfy some predetermined monotone graph property.

We consider the property $\mathcal{P}_H$ of containing a spanning graph H as a subgraph. It was asked by N. Alon whether for any bounded-degree H, Builder can construct a graph satisfying $\mathcal{P}_H$ with high probability in $O(n)$ rounds. We answer this question positively in a strong sense, showing that any graph with maximum degree Δ can be constructed with high probability in $(3\Delta/2 + o(\Delta))n$ rounds, where the $o(\Delta)$ term tends to zero as $\Delta \to \infty$. This is tight (even for the offline case) up to a multiplicative factor of 3. Furthermore, for the special case where H is a spanning forest of maximum degree Δ, we show that H can be constructed with high probability in $O(n \log \Delta)$ rounds. This is tight up to a multiplicative constant, even for the offline setting. Finally, we show a separation between *adaptive* and *non-adaptive* strategies, proving a lower bound of $\Omega(n\sqrt{\log n})$ on the number of rounds necessary to eliminate all isolated vertices w.h.p. using a non-adaptive strategy. This bound is tight, and in fact $O(n\sqrt{\log n})$ rounds are sufficient to construct a K_r-factor w.h.p. using a non-adaptive strategy.

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1 Introduction

Recently, the following *semi-random graph process* was proposed by P. Michaeli, and analyzed by Ben-Eliezer, Hefetz, Kronenberg, Parczyk, Shikhelman, and Stojaković [4]. A single adaptive player, called Builder, starts with an empty graph G on a set V of n vertices. The process then proceeds in *rounds*, where in each round Builder is offered a uniformly random vertex u, and chooses an edge of the form $\{u, v\}$ to add to the graph G. Builder's objective is typically to construct a graph that satisfies some predetermined monotone graph property; for example, to make G an expander with certain parameters, or to have G contain a Hamilton cycle. The natural question arising in this context is the following:

Given a monotone graph property $\mathcal{P}$, how many rounds of the semi-random graph process are required for Builder to construct (with high probability[1]) a graph which satisfies $\mathcal{P}$?

Semi-random problems of this type, involving both randomness and intelligent choices made by a "decision-maker", have been widely studied in the algorithmic literature. One of the first (and most famous) results on such processes, established by Azar et al. [2], concerns sequential allocation of n balls into n bins, where the goal is to minimize the number of balls in the fullest bin. It is well-known that if each ball is simply assigned to a bin uniformly at random, then w.h.p., the fullest bin will contain $\Theta(\ln n/ \ln \ln n)$ balls at the end of the process. However, as was shown in [2], very limited "intelligent intervention" substantially improves the above bound: if, instead of the random assignment, for any ball we are given *two (random) choices* of bins to pick from, then the trivial strategy of always choosing the least loaded bin out of the two offered, results w.h.p. in the maximum bin load dropping to $\Theta(\ln \ln n)$ – an exponential improvement. This idea has inspired many subsequent theoretical and practical results in various contexts within computer science, see e.g. [10, 26, 29] for a small sample of these.

In a sense, semi-random processes can be viewed as a situation where an *online algorithm* aims to achieve a predetermined objective in a *randomized environment*. As opposed to the "standard" setting where online algorithms are measured in terms of their worst-case performance, in the semi-random setting the task is to design online algorithms that achieve the goal with high (or at least constant) probability, and require as few rounds as possible. Further discussion of related semi-random graph models, such as the so-called Achlioptas model, can be found in Subsection 1.1.1.

Aside from their role within the algorithmic literature, semi-random processes seem to be somewhat related to *decision making in nature*, and in particular to *memory formation in the brain*, an intriguing connection that has not yet been thoroughly investigated, to the best of our knowledge. See Subsection 1.1.2 for more details.

In this paper we continue the investigation into the semi-random graph process. The first work on this topic [4] proved upper and lower bounds on the number of rounds required to w.h.p. satisfy various properties of interest. Among the upper bounds were an $O(n^{1-\varepsilon})$ bound for the property of containing a copy of any fixed graph H (here ε depends on H), an $O(n)$ bound for containing a perfect matching or a Hamilton cycle, and an $O(\Delta n)$ bound for the property of having minimum degree Δ, as well as for the property of Δ-vertex-connectivity. These results prompted N. Alon to ask whether it is the case that any given (spanning) graph of bounded maximum degree can be

[1]With probability that tends to one as $n \to \infty$; abbreviated w.h.p. henceforth.

constructed w.h.p. in $O(n)$ rounds in this model. Formally, define $\mathcal{P}_H$ as the property of containing an (unlabeled) copy of H, i.e., as the property that there exists an injection $\varphi\colon H \to G$ (where G is the graph constructed by Builder), so that $\{\varphi(i), \varphi(j)\} \in E(G)$ for every $\{i, j\} \in E(H)$.

Question 1.1 (N. Alon, Question 6.2 in [4])**.** *Consider the semi-random graph process over n vertices. Is it true that for every graph H on n vertices with bounded maximum degree, there exists a strategy which enables Builder to w.h.p. construct a copy of H in $O(n)$ rounds?*

The main result of this paper gives a positive answer to the above question (see Theorem 1.3).

The offline case Before addressing Question 1.1, let us consider the easier *offline* setting, where Builder is provided *in advance* with the full sequence of vertices offered to them in all rounds of the process. In other words, in this setting Builder does not need to make his decisions online, but can rather choose all edges at once after seeing the sequence of random vertices.

As an example, consider the case where Builder's goal is to construct a triangle-factor[2]. Observe that if at some point of the process, at least $\frac{2n}{3}$ vertices have been offered at least twice, then Builder can already construct a triangle-factor (in the offline setting). Indeed, Builder simply partitions the vertices into triples $\{u, v, w\}$, where u, v were offered at least twice, and chooses the edges $\{u, v\}$ and $\{u, w\}$ at rounds where u was offered, and the edge $\{v, w\}$ at a round where v was offered. It is easy to check that $O(n)$ rounds suffice to have at least $\frac{2n}{3}$ vertices offered at least twice w.h.p. Thus, in the offline setting Builder can construct a triangle-factor in $O(n)$ rounds. This $O(n)$ bound can in fact be generalized in a strong sense to *any* bounded-degree target graph H. Proposition 4.1 in [4] – which presents necessary and sufficient general winning conditions for Builder in the offline setting – implies that Builder wins the game as soon as the list of offered vertices allows the construction of a suitable *orientation* of H (see Lemma 2.4 for the definition). In Section 3.4 we show how this can be used to get the following general offline result.

Proposition 1.2. *Let Δ, n be positive integers, and let H be an n-vertex graph of maximum degree Δ. In the offline version of the semi-random process on n vertices, Builder has a strategy guaranteeing that w.h.p., after $(\Delta/2+o(\Delta))n$ rounds of the process, the constructed graph will contain a copy of H.*

The $o(\Delta)$ term here converges to zero as $\Delta \to \infty$. Proposition 1.2 substantially extends Theorem 1.9 in [4], which showed a similar result for the property of having minimum degree at least k (with an explicit dependence on k). Proposition 1.2 is clearly optimal up to the $o(\Delta)$ term, since Δ-regular graphs on n vertices have exactly $\Delta n/2$ edges (and hence trivially require at least this number of rounds). More interestingly, it turns out that the $o(\Delta)$-term is unavoidable; it follows from [4, Theorem 1.9] that at least $(\frac{1}{2} + \varepsilon_\Delta)\Delta n$ rounds are required for Builder to construct a graph of minimum degree at least Δ, where $\varepsilon_\Delta > 0$ (and $\varepsilon_\Delta \to 0$ with Δ).

Main result: online strategy for constructing bounded-degree spanning graphs We now return to the more interesting *online* setting, where Builder is offered vertices one-by-one and must (irrevocably) decide which edge to add immediately after being offered a vertex. Our main result in this paper, Theorem 1.3, asserts that Builder can construct any given bounded-degree spanning graph in $O(n)$ rounds w.h.p.

[2]For a graph F, an n-vertex F-factor is the graph which consists of $n/|V(F)|$ vertex-disjoint copies of F.

Theorem 1.3. *Let Δ, n be positive integers, and let H be an n-vertex graph of maximum degree Δ. In the online version of the semi-random graph process on n vertices,* Builder *has a strategy guaranteeing that w.h.p., after*

$$\begin{cases} (\Delta/2 + o(\Delta))n & \text{if } \Delta = \omega(\log n) \\ (3\Delta/2 + o(\Delta))n & \text{otherwise} \end{cases}$$

rounds of the process, the constructed graph will contain a copy of H.

As before, the $o(\Delta)$ term tends to zero as $\Delta \to \infty$. Note that Δ is allowed to depend on n arbitrarily. Theorem 1.3 answers Question 1.1 in a strong sense: not only can any bounded-degree graph be constructed w.h.p. in a linear number of rounds, but in fact, the dependence on the maximum degree is very modest. This result clearly illustrates the power of semi-random algorithms compared to their truly random counterparts, see the discussion below on the appearance of clique-factors in the random graph process.

The notion of *competitive ratio* [7] refers to the performance of an online algorithm compared to the best offline algorithm for the same problem. In view of the trivial $\Delta n/2$ lower bound and Proposition 1.2, our algorithm is 3-competitive for general H. As an open question, it will be very interesting to determine the optimal competitive ratio of an online algorithm for this problem.

Problem 1.4. *Is it true that for every n-vertex graph H of maximum degree Δ,* Builder *has a strategy that constructs a copy of H w.h.p. in $(\Delta/2 + o(\Delta))n$ rounds of the online version?*

Non-adaptive strategies It is fairly natural to inquire whether imposing the restriction of *non-adaptivity* handicaps Builder, and if so, to which extent exactly. Here, by non-adaptivity we mean that, in a sense, Builder's choices are decided upon beforehand, and do not depend on the situation at any given round of the process. The precise definition that we use is as follows: a *non-adaptive strategy* consists of a family $\mathcal{L}$ of adjacency lists $\mathcal{L} = \{L^w : w \in [n]\}$, where each $L^w = (L^w(i) : i = 1, \ldots, n-1)$ is a permutation of $[n] \setminus \{w\}$, which are specified *in advance* (i.e., before the sequence of random vertices starts being exposed). Playing according to such a strategy means that during the vertex exposure process $w_1, w_2, \ldots$, if vertex w appears for the ith time, $i \geq 1$, then Builder is obliged to connect w to the ith vertex $L^w(i)$ on its list. (To avoid ambiguities, let us assume that if w has already been connected to $L^w(i)$, then Builder simply skips his move, this assumption will not change much in our analysis.)

It turns out that the non-adaptivity assumption indeed hampers Builder — it takes him typically $\Omega(n\sqrt{\log n})$ rounds to get rid of isolated vertices, as stated in Theorem 1.5 below. This is in rather sharp contrast with the situation for general (i.e. adaptive) strategies. Indeed, it is easy to see that Builder can construct a connected graph in $n - 1$ rounds (with probability 1); constructing a graph with no isolated vertices can be done w.h.p. even faster, in $(\ln 2 + o(1))n$ rounds (see [4]); and finally, Theorem 1.3 shows that in fact every bounded-degree graph can be constructed in $O(n)$ rounds.

Theorem 1.5. *Consider the (online version of the) semi-random graph process on n vertices.*

1. *Any non-adaptive strategy requires w.h.p. $\Omega(n\sqrt{\log n})$ rounds to construct a graph in which none of the n vertices is isolated.*

2. *On the other hand, for every $r \geq 2$ there is $C = C(r)$ such that for every n which is divisible by r, there is a non-adaptive strategy for constructing a K_r-factor in $Cn\sqrt{\log n}$ rounds w.h.p.*

We remark that while the above definition of non-adaptivity is deterministic in nature (in the sense that the lists $\{L^w : w \in [n]\}$ are predetermined), the proof of Item 1 in Theorem 1.5 can be easily adapted (with the same asymptotic lower bound) to "random non-adaptive strategies", i.e. strategies in which for every $w \in [n]$ and $i \geq 1$, the vertex which Builder connects to w the ith time that w is sampled is drawn from some predetermined probability distribution on $[n] \setminus \{w\}$.

Constructing bounded-degree spanning trees Theorem 1.3 establishes that the (typical) number of rounds needed to construct a general spanning graph H of maximum degree Δ is $O(\Delta n)$. This is clearly tight for graphs whose average degree is $\Theta(\Delta)$. It is now natural to ask if we can break the $\Theta(\Delta n)$ barrier for graphs with a much smaller average degree, such as trees. The next result answers this question positively.

Theorem 1.6. *Let Δ, n be positive integers and let T be an n-vertex forest of maximum degree Δ. In the online version of the semi-random process on n vertices, Builder has a strategy guaranteeing that w.h.p., after $O(\log \Delta)n$ rounds of the process, the constructed graph will contain a copy of T.*

The next proposition shows that the dependence on n and Δ in Theorem 1.6 is tight even for the offline version of the semi-random process.

Proposition 1.7. *For every $\Delta \geq 2$ and for every $n \geq n_0(\Delta)$, there exists a forest T with n vertices and maximum degree Δ satisfying the following. In the offline version of the semi-random graph process, w.h.p. Builder needs $\Omega(\log \Delta)n$ rounds in order to construct a copy of T.*

The situation in the ("purely"-)random graph process A common theme in our results is that introducing "intelligent choices" into a random setting can allow for a dramatic improvement (perhaps the first appearance of this theme is in the aforementioned work of Azar et al. [2]). To illustrate this phenomenon in the setting of random graph processes, let us compare our Theorem 1.3 with the situation in which all edge-choices are made completely randomly. Recall that the *random graph process* $\tilde{G} = (G_m)_{m=1}^N$ is defined by choosing a random permutation $e_1, \dots, e_N$ of all $N = \binom{n}{2}$ edges of K_n, and letting G_m be the graph whose edge-set is $\{e_1, \dots, e_m\}$ (we refer the reader to [15] for an overview of this classical object). Note that for each $0 \leq m \leq \binom{n}{2}$, the graph G_m is distributed as the Erdős-Rényi graph $G(n, m)$, i.e. as a random graph chosen uniformly among all graphs with m edges and n (labeled) vertices. This model was introduced in the seminal papers [8, 13].

The appearance of bounded-degree spanning graphs in the random graph process (or equivalently, in $G(n, m)$) was thoroughly investigated[3]. For starters, a standard first moment argument (see e.g. [15]) shows that a copy of $K_{\Delta+1}$ only appears in the random graph process after roughly $n^{2-2/\Delta}$ rounds. Thus, to obtain even a single $K_{\Delta+1}$-copy (let alone a $K_{\Delta+1}$-factor), requires much more than a linear number of rounds. Determining the typical time of the appearance of a $K_{\Delta+1}$-factor turned

[3]We note that the results surveyed here were actually proved for the binomial random graph $G(n, p)$, which is the graph obtained by selecting each of the $\binom{n}{2}$ edges of K_n with probability p and independently. It is well known (see e.g. [15, Section 1.4] or [11, Section 1.1]) that $G(n, m)$ is closely related to $G(n, p)$ with $p = m/\binom{n}{2}$, which allows to transfer results between the two models.

out to be a difficult problem. Following a long line of reseach, this problem was settled by a (special case of a) celebrated result by Johansson, Kahn and Vu [16], which states that a $K_{\Delta+1}$-factor appears in the random graph process at around $m = n^{2-2/(\Delta+1)}(\log n)^{1/\binom{\Delta+1}{2}}$. A more general discussion on the appearance of bounded-degree graphs H other than clique-factors (and the closely-related notion of *universality*) can be found in the recent work of Ferber, Kronenberg, and Luh [9]; see in particular Conjecture 1.5 there.

Similar superlinear lower bounds (on the number of edges required in order to typically contain a $K_{\Delta+1}$-factor, or even a single $K_{\Delta+1}$) are known or can be shown for various other random graph models, like the random regular graph, or the model $G_{k\text{-}out}$ (where one connects each vertex to exactly k other randomly chosen vertices, discarding repetitions). Thus, while it was shown in [4] that the semi-random graph process can *simulate* the random models $G(n, p)$, $G(n, m)$, and $G_{k\text{-}out}$, the above discussion indicates that these cannot help in solving Question 1.1, and we must utilize the power of the intelligent player, Builder, in a more imaginative way.

A similar comparison can be made between Theorem 1.6 and the emergence of (bounded-degree) spanning trees in the random graph process. For the random graph process, it is well known that w.h.p. the last isolated vertex disappears only at around $m = \frac{1}{2}n \log n$. Thus, a superlinear number of rounds is required in order to contain a spanning tree w.h.p. Again, we observe here that intelligent choices speed up the time required to reach the goal: it takes $\Theta(n \log n)$ rounds for the random graph process to contain even a single spanning tree, whilst in our semi-random graph process, the number of rounds required to contain any prescribed bounded-degree spanning tree is only $O(n)$.

It is worth mentioning that a recent breakthrough of Montgomery [24], which confirms a conjecture of Kahn [17], shows that for a fixed Δ, w.h.p. all spanning trees with maximum degree Δ appear in G_m after $m = Cn \log n$ rounds (where $C = C(\Delta)$ is a large enough constant). We refer the reader to [24] for further references to many other related works on this subject.

1.1 Related Work

1.1.1 Semi-random processes

Perhaps the most famous semi-random graph process is now known as the *Achlioptas process*, and was proposed by Dimitris Achlioptas in 2000. This process runs in rounds, where in each round, two uniformly random edges are picked from the set of all $\binom{n}{2}$ possible edges, or alternatively (depending on the version of the process) from all edges untaken at this point. These two edges are offered to Builder, who then must choose exactly one of them and add it to the graph.

While in our random graph process Builder's goal is always to make his graph *satisfy* some given graph property, in the context of the Achlioptas process the goal is often to *avoid* satisfying a given property for as long as possible. In fact, Achlioptas's original question was whether Builder can delay the appearance of a giant component beyond its typical time of appearance in the ("purely"-)random graph process. This question was answered positively by Bohman and Frieze [5], see also [6, 18, 28, 1, 27]. Similar problems have also been studied for other properties; for example, the problem of avoiding a fixed subgraph [19], or the problem of speeding up the appearance of a Hamilton cycle [20]. Achlioptas-like processes involving two choices were also investigated in other contexts, see e.g. [25] for a geometric perspective.

1.1.2 The neuroscience connection: memory formation in the brain

An intriguing natural process that seems somewhat related to our semi-random graph model is that of *memory formation in the human brain*, whose investigation is one of the main frontiers of modern neuroscience. While this process is not yet completely understood, it is already well-known that it shares several characteristics with our semi-random process [3, 21, 22]: crucially, long-term memories are stored in the brain using physical structures that can essentially be viewed as weighted subgraphs, which makes the efficient construction (and destruction) of such graphs desirable for creation or modification of memories.

The memory formation process is typically governed by a centralized decision maker, usually the part of the brain called the *hippocampus*. In this context, *neurons* play the role of vertices, whereas the physical connections between them – known as *synapses* – play the role of edges. It is known that a combination of electric signals ("spikes") and specialized proteins (see [21]) sent from the hippocampus to neurons, control the long-term storage of memories, and that returning signals from the neurons are also involved in the process.

In any case, while the exact mechanisms of memory formation and destruction are not yet fully understood, they are probably far more complicated than the semi-random graph process we investigate here. However, there are various random models for communication between neurons, see e.g. the book of Gerstner and Kistler on spiking neuron models [12]. We believe that further study of suitable semi-random processes (with biologically-inspired models for randomness) and their connections to complex natural processes is an interesting venue for future research.

1.2 Paper Organization

In Section 2 we state several auxiliary results. Section 3 contains the proofs of Theorem 1.3 and Proposition 1.2, as well as the description and analysis of Strategy 3.6, which is the key tool used in our proofs. Theorem 1.6 and Proposition 1.7 are proved in Section 4. Finally, the proof of Theorem 1.5 appears in Section 5. Since the statements of Theorems 1.3 and 1.6 are asymptotic (in both n and Δ), we will always assume, where needed, that n and Δ are sufficiently large. All logarithms are base e. We will omit floor and ceiling signs whenever these are not crucial.

2 Preliminaries

We start by stating three known concentration inequalities that will be used in this paper. The first is a standard Chernoff-type bound (see, e.g., [15]), the second is a simplified version of Azuma's inequality (see, e.g., [15, Theorem 2.27]), and the third is a simplified version of Talagrand's inequality (see, e.g., [23]).

Lemma 2.1. *Let X be a binomial random variable. Then, for every $\lambda \geq 0$, it holds that*

$$\mathbb{P}[X \leq \mathbb{E}[X] - \lambda] \leq e^{-\frac{\lambda^2}{2\mathbb{E}[X]}}$$

and that

$$\mathbb{P}[X \geq \mathbb{E}[X] + \lambda] \leq e^{-\frac{\lambda^2}{2(\mathbb{E}[X]+\lambda/3)}} .$$

Lemma 2.2. *[15, Theorem 2.27] Let X be a non-negative random variable, not identically 0, which is determined by T independent trials $w_1, \dots, w_T$. Suppose that $c \in \mathbb{R}$ is such that changing the outcome of any one of the trials can change the value of X by at most c. Then, for every $\lambda \geq 0$, it holds that*

$$\mathbb{P}\left[X \leq \mathbb{E}[X] - \lambda\right] \leq e^{-\frac{\lambda^2}{2c^2T}}$$

and that

$$\mathbb{P}\left[X \geq \mathbb{E}[X] + \lambda\right] \leq e^{-\frac{\lambda^2}{2c^2T}}.$$

Lemma 2.3. *[23, Pages 80-81] Let X be a non-negative random variable, not identically 0, which is determined by T independent trials $w_1, \dots, w_T$. Suppose that $c, g > 0$ are such that*

1. *Changing the outcome of any one of the trials can change the value of X by at most c.*
2. *For every ℓ, if $X \geq \ell$ then there is a set of at most $g\ell$ trials whose outcomes certify[4] that $X \geq \ell$.*

Then, for every $0 \leq \lambda \leq \mathbb{E}[X]$, it holds that

$$\mathbb{P}\left[|X - \mathbb{E}[X]| > \lambda + 60c\sqrt{g\mathbb{E}[X]}\right] \leq 4e^{-\frac{\lambda^2}{8c^2g\mathbb{E}[X]}} \quad .$$

We will also need the following lemma regarding “balanced” orientations of graphs.

Lemma 2.4. *Let H be an n-vertex graph of maximum degree Δ. Then, there exists an orientation D of the edges of H which satisfies the following two conditions:*

(a) $d^+_D(u) \leq \lfloor \Delta/2 \rfloor + 1$ *for every* $u \in V(H)$,

(b) *there exists a set* $A \subseteq V(H)$ *of size* $|A| \geq \frac{n}{\Delta^2+1}$ *such that* $d^+_D(u) = 0$ *for every* $u \in A$.

Proof. Let H^2 denote the square of H, that is, the graph obtained from H by adding an edge between any two vertices at distance 2 in H. Let $A \subseteq V(H)$ be a maximum independent set in H^2; clearly $|A| \geq \frac{n}{\Delta^2+1}$. Let $H_0 = H \setminus A$. If $d_{H_0}(u)$ is even for every $u \in V(H_0)$, then let $H_1 = H_0$. Otherwise, let H_1 denote the graph obtained from H_0 by adding a new vertex x and connecting it by an edge to every vertex of odd degree in H_0. Orient the edges of each connected component of H_1 along some Eulerian cycle of that component. Orient every edge of $E_H(V(H) \setminus A, A)$ from $V(H) \setminus A$ to A. Delete x and denote the resulting oriented graph by D. Observe that A is independent in H and thus D is an orientation of all the edges of H. It is evident that D satisfies (b). Moreover, D satisfies (a) as A is independent in H^2 and thus $d_H(u, A) \leq 1$ for every $u \in V(H) \setminus A$. □

3 Constructing Spanning Graphs of Given Maximum Degree

In this section we prove the main part of Theorem 1.3 and Proposition 1.2. The tools we develop here in order to prove Theorem 1.3 will also be used in the proof of Theorem 1.6 in Section 4. We start by introducing some definitions and sketching a rough outline of the proof. From this point onward, we fix an n-vertex graph H with maximum degree Δ. We assume that the ground-set of vertices for

[4] To be precise, this means that if $(w_1, \dots, w_T)$ is such that $X \geq \ell$, then there is $k \leq g\ell$ and $1 \leq i_1 < \dots < i_k \leq T$ such that changing the outcome of any trials other than $w_{i_1}, \dots, w_{i_k}$ does not change the fact that $X \geq \ell$.

the semi-random process is $[n]$. For a graph G with vertex-set $[n]$ and a bijection $\varphi : V(H) \to [n]$, we say that a subset $A \subseteq V(H)$ is *(G, φ)-good* if φ maps all edges of H contained in A to edges of G; that is, A is (G, φ)-good if $\{\varphi(x), \varphi(y)\} \in E(G)$ for every $x, y \in A$ such that $\{x, y\} \in E(H)$. The graph G under consideration will always be Builder's graph at some given moment during the process. Often, it will be clear which moment we are considering, and so we will omit G from the notation, simply writing φ-good. Note that, if at some point during the process, there is a bijection $\varphi : V(H) \to [n]$ such that $V(H)$ is φ-good, then Builder has succeeded in constructing a copy of H.

Our strategy consists of two stages: in the first stage, Builder fixes an arbitrary bijection $\varphi : V(H) \to [n]$ and plays so as to construct a φ-good set which is as large as possible. We show (see Lemma 3.4 below) that Builder can w.h.p. guarantee the existence of a φ-good set which covers almost all the vertices of H, within $(\Delta/2 + o(\Delta))n$ rounds. This part of the argument is fairly straightforward. In the second stage, which is far more involved and constitutes the heart of the proof, Builder tries to iteratively extend this φ-good set by updating the embedding φ. We will show that by using a suitable "role-switching" strategy (i.e., Strategy 3.6), Builder can ensure that w.h.p. $V(H)$ will be φ-good after $(\Delta + o(\Delta))n$ additional rounds. This will be done in Lemma 3.11.

3.1 The Initial Embedding

In this subsection we describe and analyze the first stage of Builder's strategy. Along the way we also prove the easy part of Theorem 1.3, namely, the regime $\Delta = \omega(\log n)$. In both cases Builder uses the following simple (non-adaptive) strategy:

Strategy 3.1. *Let H be an n-vertex graph of maximum degree Δ. Fix an orientation D of the edges of H which satisfies property (a) of Lemma 2.4. Let $\varphi : V(H) \to [n]$ be an arbitrary bijection. At any given round of the process, having been offered a random vertex w, Builder chooses an arbitrary vertex $u \in \varphi\left(N_D^+(\varphi^{-1}(w))\right)$ which is not adjacent to w in his current graph, and claims the edge $\{u, w\}$; if no such vertex u exists, then Builder claims an arbitrary edge incident with w.*

Builder's goal when using Strategy 3.1 is to construct his graph in such a way that the predetermined bijection φ will be an embedding of H into his graph (or, when obtaining a full embedding of H is w.h.p. impossible, to construct as large a φ-good set as possible). In the analysis we will need the following simple lemma.

Lemma 3.2. *Let n, Δ be positive integers, let $\varepsilon \in (0, 1)$, and consider the execution of the semi-random process on K_n for $T = (1/2 + \varepsilon)\Delta n$ rounds. Then for every $i \in [n]$, the probability that i was offered at most $\Delta/2$ times in the course of these T rounds, is at most $e^{-\varepsilon^2\Delta/3}$.*

Proof. Let Z_i be the random variable counting the number of times i is offered during the T rounds. Then $Z_i \sim \text{Bin}(T, 1/n)$, implying that $\mathbb{E}[Z_i] = T/n = (1/2 + \varepsilon)\Delta$. Applying Lemma 2.1 with $\lambda = \mathbb{E}[Z_i] - \Delta/2$, we obtain

$$\mathbb{P}[Z_i \le \Delta/2] = \mathbb{P}[Z_i \le \mathbb{E}[Z_i] - (\mathbb{E}[Z_i] - \Delta/2)] \le \exp\left(-\frac{(\mathbb{E}[Z_i] - \Delta/2)^2}{2\mathbb{E}[Z_i]}\right)$$
$$\le \exp\left(-\frac{\varepsilon^2\Delta^2}{(1 + 2\varepsilon)\Delta}\right) < e^{-\varepsilon^2\Delta/3} ,$$

as required. □

We now prove the easy part of Theorem 1.3.

Proposition 3.3. *For every $\varepsilon > 0$ there exists an integer C for which the following holds. Let n and $\Delta = \Delta(n) \geq C \log n$ be positive integers and let H be an n-vertex graph of maximum degree Δ. Then, in the online version of the semi-random process on K_n, Builder has a strategy guaranteeing that w.h.p., after $(1/2 + \varepsilon)\Delta n$ rounds of the process, his graph will contain a copy of H.*

Proof. Builder executes Strategy 3.1 for $T := (1/2 + \varepsilon)\Delta n$ rounds. In the notation of Strategy 3.1, it is evident that if for every $x \in V(H)$, the vertex $\varphi(x)$ is offered at least $d_D(x)$ times, then Builder is successful in building a copy of H. Therefore, in order to complete the proof of the proposition, it suffices to show that w.h.p., for every $1 \leq i \leq n$, the vertex i will be offered at least $\lfloor \Delta/2 \rfloor + 1$ times in the course of these T rounds of the process. By Lemma 3.2, the probability that any given $1 \leq i \leq n$ is offered at most $\Delta/2$ times is at most

$$e^{-\varepsilon^2 \Delta/3} \leq n^{-C\varepsilon^2/3} \leq 1/n^2 ,$$

where the penultimate inequality holds since $\Delta \geq C \log n$ and the last inequality holds for $C \geq 6\varepsilon^{-2}$. A union bound then implies that with probability at least $1 - \frac{1}{n} = 1 - o(1)$, every $1 \leq i \leq n$ was offered at least $\lfloor \Delta/2 \rfloor + 1$ times. □

We now return to the main case of Theorem 1.3, assuming henceforth that $\Delta = O(\log n)$. In the following lemma, we analyze the aforementioned first stage of Builder's strategy.

Lemma 3.4. *Let n, Δ be positive integers, let $\alpha \in (0, 1)$, and suppose that $n \gg \alpha^{-2}\Delta^3$. Let H be an n-vertex graph of maximum degree Δ and let $\varphi : V(H) \to [n]$ be a bijection. Then, Builder has a strategy guaranteeing that after $T := \left(\Delta/2 + \sqrt{3\Delta \log(6\Delta/\alpha)}\right)n$ rounds of the process, w.h.p. there will be a φ-good set $A \subseteq V(H)$ of size at least $(1 - \alpha)n$.*

Proof. Builder executes Strategy 3.1 for T rounds. For every vertex $x \in V(H)$, let I_x be the indicator random variable for the event that $\varphi(x)$ was offered at least $\geq \lfloor \Delta/2 \rfloor + 1$ times. By Lemma 3.2 with $\varepsilon := \sqrt{\frac{3\log(6\Delta/\alpha)}{\Delta}}$, the probability that for a given $x \in V(H)$ we have $I_x = 0$, is at most $e^{-\varepsilon^2\Delta/3} = e^{-\log(6\Delta/\alpha)} = \frac{\alpha}{6\Delta}$. Let A' be the set of all vertices $x \in V(H)$ such that $I_x = 1$. Since $|A'| = \sum_{x \in V(H)} I_x$, it follows that $n \geq \mathbb{E}[|A'|] \geq (1 - \frac{\alpha}{6\Delta})n$. We will use Lemma 2.3 to prove that $|A'|$ is concentrated around its expected value. Consider the sequence of random vertices $(w_1, \ldots, w_T) \in [n]^T$, and observe that changing any single coordinate of this sequence can change the value of $|A'|$ by at most 1. Moreover, if $|A'| \geq \ell$, then there are at least ℓ vertices $x \in V(H)$ for each of which $\varphi(x)$ was offered at least $\lfloor \Delta/2 \rfloor + 1$ times, and so there is a set of at most $(\lfloor \Delta/2 \rfloor + 1)\,\ell$ entries in the sequence $(w_1, \ldots, w_T)$ which certify that $|A'| \geq \ell$. Hence, applying Lemma 2.3 with parameters $c = 1$, $g = \lfloor \Delta/2 \rfloor + 1$ and $\lambda = \frac{\alpha}{6\Delta} \cdot n$ yields

$$\mathbb{P}\left[|A'| < \mathbb{E}[|A'|] - \frac{\alpha}{6\Delta} \cdot n - 60\sqrt{(\lfloor \Delta/2 \rfloor + 1)\,\mathbb{E}[|A'|]}\right] \leq 4e^{-\frac{(\alpha n/(6\Delta))^2}{8(\lfloor \Delta/2 \rfloor + 1)\mathbb{E}[|A'|]}} \leq 4e^{-\frac{\alpha^2 n^2}{288\Delta^3 n}} = o(1),$$

where the last equality holds since $n \gg \alpha^{-2}\Delta^3$ by assumption. We conclude that w.h.p.,

$$|A'| \geq \mathbb{E}[|A'|] - \frac{\alpha}{6\Delta} \cdot n - 60\sqrt{(\lfloor \Delta/2 \rfloor + 1)\,\mathbb{E}[|A'|]} \geq \left(1 - \frac{2\alpha}{6\Delta}\right)n - 60\sqrt{\Delta n} \geq \left(1 - \frac{\alpha}{2\Delta}\right)n. \quad (1)$$

Now, let $B = \{u \in V(H) : N_D^-(u, V(H) \backslash A') \neq \emptyset\}$, where D is the orientation from Strategy 3.1. It readily follows from the description of Builder's strategy (namely, Strategy 3.1) that if $x \in A' \backslash B$, then after T rounds of the process, $\varphi(x)$ is adjacent in Builder's graph to every vertex of $\varphi(N_H(x))$. Thus, $A := A' \backslash B$ is φ-good. Since the maximum degree of H is Δ, it follows that $|B| \leq \Delta \cdot |V(H) \backslash A'| \leq \frac{\alpha n}{2}$, where the last inequality holds w.h.p. by (1). We conclude that w.h.p. $|A| \geq (1 - \alpha)n$. □

3.2 Improving the Embedding

In this subsection we introduce and analyze Builder's strategy for the second stage (see Strategy 3.6 below). Our starting point is the set A whose (likely) existence is guaranteed by Lemma 3.4 (the parameter α with which we apply Lemma 3.4 will be chosen later, and the bijection φ with which we apply this lemma is immaterial and can be chosen arbitrarily). Our goal is to iteratively update φ, so as to maintain a φ-good set which gradually increases in size until it equals $V(H)$.

Before delving into the details, let us illustrate the idea behind Strategy 3.6 by considering the following "toy" example: suppose that at some point during the process, Builder has already managed to obtain a bijection $\varphi : V(H) \to [n]$ such that there is a φ-good set A of size $n - 1$. Let b denote the unique element of $V(H) \setminus A$. The fact that A is φ-good means that $\varphi(A)$ spans a copy of $H[A]$ in Builder's graph. The problem is that in order for φ to be an embedding of H into Builder's graph, $\varphi(b)$ needs to be adjacent to all of the vertices in $\varphi(N_H(b))$, which might fail to hold. A naive way of trying to solve this problem would be for Builder to wait until $\varphi(b)$ will have been offered $d_H(b)$ times, and at each such time, to connect $\varphi(b)$ to a new vertex in $\varphi(N_H(b))$. This, however, will not work, since the probability that $\varphi(b)$ is offered (even once) in the course of $O(n)$ rounds does not tend to 1. So instead, Builder will try to find another vertex in $[n]$ to "play" the role $\varphi(b)$, and to have $\varphi(b)$ play the role which was previously played by that other vertex. To this end, Builder fixes (a large number of) vertices $a_1, \ldots, a_m \in A$ (we note that in order to make the strategy work, some assumptions on $a_1, \ldots, a_m$ are required (see Setting 3.5), but the reader may ignore this issue at the moment). Now Builder acts as follows: each time a vertex of $\{\varphi(a_1), \ldots, \varphi(a_m)\}$ is offered, Builder connects it to some new vertex of $\varphi(N_H(b))$; and each time a vertex of $\bigcup_{i=1}^m \varphi(N_H(a_i))$ is offered, Builder connects it to $\varphi(b)$. Now, if at some point there is an index $1 \leq i \leq m$ such that $\varphi(a_i)$ has already been offered at least $\Delta(H)$ times and every vertex in $\varphi(N_H(a_i))$ has already been offered at least once, then at this point $\varphi(a_i)$ is adjacent in Builder's graph to every vertex of $\varphi(N_H(b))$, and $\varphi(b)$ is adjacent in Builder's graph to every vertex of $\varphi(N_H(a_i))$. Hence, Builder can now safely "switch" the roles of $\varphi(a_i)$ and $\varphi(b)$. Formally, Builder defines a new bijection $\varphi' : V(H) \to [n]$ by setting $\varphi'(b) = \varphi(a_i)$, $\varphi'(a_i) = \varphi(b)$, and $\varphi'(x) = \varphi(x)$ for every $x \in V(H) \setminus \{b, a_i\}$. Then φ' is an embedding of H into Builder's graph. A key point of this method is that, since there are many "candidates" for the role of b (i.e. the vertices $a_1, \ldots, a_m$), it is very likely that one of them will indeed be chosen to be swapped with b.

We now give a precise definition of the setting in which we will apply our "role-switching" strategy, including some technical assumptions which are necessary for this strategy to work.

Setting 3.5. *We are given a graph*[5] *G on the vertex-set $[n]$, a bijection $\varphi : V(H) \to [n]$, and a φ-good set $A \subseteq V(H)$. We set $B = V(H) \setminus A$ and write $B = \{b_1, \ldots, b_r\}$. We are also given an*

[5] We think of G as Builder's graph immediately after employing the strategy whose existence is guaranteed in Lemma 3.4 for T rounds.

integer $m > 0$ and distinct vertices $a_{i,k} \in A$, where $1 \leq i \leq r$ and $1 \leq k \leq m$. We assume that the following two properties are satisfied.

1. *There is no edge of H between $a_{i,k}$ and b_j for any $1 \leq i, j \leq r$ and $1 \leq k \leq m$.*

2. *The sets $\{a_{i,k}\} \cup N_H(a_{i,k})$ are pairwise-disjoint, where (i,k) run over all pairs in $[r] \times [m]$.*

For every $(i,k) \in [r] \times [m]$, we let $L_{i,k} = N_H(a_{i,k})$, noting that $a_{i,k} \notin L_{i,k}$.

For the remainder of this subsection, we fix an arbitrary integer d satisfying $d_H(x) \leq d$ for every $x \in \bigcup_{i=1}^{r}\{b_i, a_{i,1}, \ldots, a_{i,m}\}$. The reason for allowing this flexibility in the choice of d (as opposed to simply letting d be the maximum degree of H), is that in one application (namely, Theorem 1.6), we will be able to make sure that the degrees of the vertices $\{b_i, a_{i,1}, \ldots, a_{i,m} : 1 \leq i \leq r\}$ are much smaller than $\Delta(H)$, and this will be crucial for obtaining the desired bound.

Throughout the second stage of his strategy, Builder maintains and updates sets $A_t \subseteq V(H)$ and bijections $\varphi_t : V(H) \to [n]$. Initially, $A_0 = A$ and $\varphi_0 = \varphi$. For every positive integer t, the pair (A_t, φ_t) will be defined immediately after round t of the second stage. Moreover, we put $B_t = V(H) \setminus A_t$ (so in particular, $B_0 = B$). Finally, we put $G_0 = G$, and for every integer $t \geq 1$, we let $G_t = G_{t-1} \cup \{e_t\}$, where e_t is the edge claimed by Builder in round t of the second stage. We are now ready to describe Builder's strategy for round t of the second stage (for any integer $t \geq 1$).

Strategy 3.6. *Let $w_t \in [n]$ be the random vertex Builder is offered at round t of the second stage.*

1. *If some pair $(i,k) \in [r] \times [m]$ is such that $w_t \in \varphi(\{a_{i,k}\} \cup L_{i,k})$ and $b_i \in B_{t-1}$, then do:*

   (a) *If $w_t \in \varphi(L_{i,k})$, then claim the edge $\{w_t, \varphi(b_i)\}$.*

   (b) *If $w_t = \varphi(a_{i,k})$, then choose an arbitrary vertex $u \in \varphi_{t-1}(N_H(b_i) \cap A_{t-1})$ which is not adjacent to $\varphi(a_{i,k})$ in G_{t-1}, and claim the edge $\{\varphi(a_{i,k}), u\}$.*

   (c) *Check whether $\varphi(b_i)$ is adjacent in G_t to every vertex of $\varphi(L_{i,k})$ and, moreover, $\varphi(a_{i,k})$ is adjacent in G_t to every vertex of $\varphi_{t-1}(N_H(b_i) \cap A_{t-1})$. If so, then set $A_t = A_{t-1} \cup \{b_i\}$ (and hence $B_t = V(H) \setminus A_t = B_{t-1} \setminus \{b_i\}$), and*

$$\varphi_t(x) = \begin{cases} \varphi_{t-1}(b_i) & x = a_{i,k}, \\ \varphi_{t-1}(a_{i,k}) & x = b_i, \\ \varphi_{t-1}(x) & x \in V(H) \setminus \{a_{i,k}, b_i\}. \end{cases}$$

   (d) *Otherwise (i.e., if the condition in Item 1(c) does not hold), set $A_t = A_{t-1}$ and $\varphi_t = \varphi_{t-1}$.*

2. *Else (i.e., if there is no pair $(i,k) \in [r] \times [m]$ which satisfies the condition in Item 1), claim an arbitrary edge which is incident with w_t; this edge will not be considered as part of Builder's graph in our analysis. Set $A_t = A_{t-1}$ and $\varphi_t = \varphi_{t-1}$.*

We refer to the operation of defining A_t and φ_t as in Item 1(c) as *switching $a_{i,k}$ and b_i*. This name stems from the fact that we swap the vertices which play the roles of b_i and $a_{i,k}$ in our current partial embedding φ_t of H into G. Note that, when switching $a_{i,k}$ and b_i, every other vertex, i.e., every vertex of $V(H) \setminus \{a_{i,k}, b_i\}$, retains its role. Switching $a_{i,k}$ and b_i is only done if, roughly speaking, $\varphi(b_i)$ can play the role of $a_{i,k}$ and $\varphi(a_{i,k})$ can play the role of b_i; the exact requirement is stated in

Item 1(c) above. Note that the only pairs of vertices which can be switched, are of the form $(a_{i,k}, b_i)$ for some $1 \leq i \leq r$ and $1 \leq k \leq m$.

In the following lemma we collect several simple facts regarding Strategy 3.6.

Lemma 3.7. *Consider the execution of Strategy 3.6 for ℓ consecutive rounds, where ℓ is an arbitrary positive integer. Then the following statements hold.*

1. *If $a_{i,k}$ and b_i were switched in round t, then none of the vertices $b_i, a_{i,1}, \ldots, a_{i,m}$ was switched at any other round.*

2. *If $x \in V(H)$ was not switched at any round, then $\varphi_s(x) = \varphi(x)$ for every $0 \leq s \leq \ell$. If x was switched in round t, then $\varphi_s(x) = \varphi(x)$ for every $0 \leq s \leq t-1$ and $\varphi_s(x) = \varphi_t(x)$ for every $t \leq s \leq \ell$.*

3. *For every $x \in V(H) \setminus (\bigcup_{i=1}^{r}\{b_i, a_{i,1}, \ldots, a_{i,m}\})$, we have $\varphi_s(x) = \varphi(x)$ for every $0 \leq s \leq \ell$.*

4. *Fix $1 \leq i \leq r$, and let $N_t := \varphi_{t-1}(N_H(b_i) \cap A_{t-1})$ for every $1 \leq t \leq \ell$. Then, $N_t \subseteq N_{t'}$ holds for every $1 \leq t < t' \leq \ell$.*

Proof. We start with Item 1. Switching $a_{i,k}$ and b_i in round t forces $b_i \in A_t$. It then follows by the description of Strategy 3.6 that $b_i \in A_q$ for every $t < q \leq \ell$, making the condition in Item 1 of Strategy 3.6 false for b_i in every subsequent round. Hence, if $a_{i,k}$ and b_i were switched in round t, then none of the vertices $b_i, a_{i,1}, \ldots, a_{i,m}$ could be switched in any subsequent round. Moreover, none of these vertices could have been switched in any round prior to round t as this would have made b_i ineligible for switching in round t. This proves Item 1. Item 2 can be easily proved by induction, using Item 1 and the definition of the functions $(\varphi_s : 0 \leq s \leq \ell)$ in Strategy 3.6. Item 3 follows from Item 2 and the fact that only vertices in $\bigcup_{i=1}^{r}\{b_i, a_{i,1}, \ldots, a_{i,m}\}$ can be switched.

Let us prove Item 4. Fix $1 \leq i \leq r$ and $t' > t \geq 1$, and let $v \in N_t$ be an arbitrary vertex; we will prove that $v \in N_{t'}$. Set $x = \varphi_{t-1}^{-1}(v)$, and note that $x \in N_H(b_i) \cap A_{t-1} \subseteq N_H(b_i) \cap A_{t'-1}$. We will show that $\varphi_{t'-1}(x) = \varphi_{t-1}(x) = v$, which would imply that $v \in \varphi_{t'-1}(N_H(b_i) \cap A_{t'-1}) = N_{t'}$, as required. Assume first that $x \in A$. It then follows by Item 1 of Setting 3.5 that $x \notin \bigcup_{j=1}^{r}\{b_j, a_{j,1}, \ldots, a_{j,m}\}$. By Item 3 of Lemma 3.7 we then have $\varphi_s(x) = \varphi(x)$ for every $0 \leq s \leq \ell$; in particular, $\varphi_{t'-1}(x) = \varphi_{t-1}(x) = v$, as claimed. Suppose now that $x \in B$, that is, $x = b_j$ for some $1 \leq j \neq i \leq r$. Since $b_j = x \in A_{t-1}$, it must have been switched with some $a_{j,k}$ prior to round t. Now, Item 2 of Lemma 3.7 implies that $\varphi_{t'-1}(x) = \varphi_{t-1}(x) = v$ in this case as well. □

The following lemma can be thought of as a proof of the "correctness" of Strategy 3.6.

Lemma 3.8. *For every non-negative integer t, the set A_t is (G_t, φ_t)-good.*

Proof. The proof is by induction on t. The base case $t = 0$ is immediate from our assumption that A is φ-good (see Setting 3.5), and the fact that $A_0 = A$, $\varphi_0 = \varphi$ and $G_0 = G$. For the induction step, fix some $t \geq 1$ and suppose that the assertion of the lemma holds for $t-1$. Consider the execution of Strategy 3.6 in round t. If either the condition in Item 1 or the condition in Item 1(c) does not hold, then there is nothing to prove, as then $A_t = A_{t-1}$ and $\varphi_t = \varphi_{t-1}$. Suppose then that both of these conditions hold, and let $(i, k) \in [r] \times [m]$ be the pair satisfying the condition in Item 1 of Strategy 3.6. In other words, we assume that $a_{i,k}$ and b_i were switched in round t.

We need to show that for every $x, y \in A_t$, if $\{x, y\} \in E(H)$, then $\{\varphi_t(x), \varphi_t(y)\} \in E(G_t)$. Hence, let $x, y \in A_t$ be such that $\{x, y\} \in E(H)$. Note that $\{x, y\} \neq \{a_{i,k}, b_i\}$, as $\{a_{i,k}, b_i\} \notin E(H)$ by Item 1 in Setting 3.5. If $x, y \notin \{a_{i,k}, b_i\}$ then we have $x, y \in A_{t-1}$, $\varphi_t(x) = \varphi_{t-1}(x)$ and $\varphi_t(y) = \varphi_{t-1}(y)$; so the assertion of the lemma for x, y follows from the induction hypothesis for $t-1$. Therefore, without loss of generality, we may assume that $x \in \{a_{i,k}, b_i\}$ and $y \notin \{a_{i,k}, b_i\}$. This assumption implies that $\varphi_t(y) = \varphi_{t-1}(y)$ and that $y \in A_{t-1}$.

Suppose first that $x = a_{i,k}$. Since $\{x, y\} \in E(H)$, it follows that $y \in N_H(x) = N_H(a_{i,k}) = L_{i,k}$. Therefore, $\varphi_t(y) = \varphi_{t-1}(y) = \varphi(y) \in \varphi(L_{i,k})$, where the second equality holds by Item 3 of Lemma 3.7 and Items 1 and 2 of Setting 3.5. It follows by the definition of φ_t in Item 1(c) of Strategy 3.6 that $\varphi_t(x) = \varphi_t(a_{i,k}) = \varphi_{t-1}(b_i) = \varphi(b_i)$, where the last equality holds by Item 2 of Lemma 3.7. Since $a_{i,k}$ and b_i were switched in round t, it follows by Item 1(c) of Strategy 3.6 that $\varphi_t(x) = \varphi(b_i)$ is adjacent in G_t to all vertices of $\varphi(L_{i,k})$. In particular, $\{\varphi_t(x), \varphi_t(y)\} \in E(G_t)$ as required.

Suppose now that $x = b_i$. Since $\{x, y\} \in E(H)$, it follows that $y \in N_H(b_i)$. Therefore, $\varphi_t(y) = \varphi_{t-1}(y) \in \varphi_{t-1}(N_H(b_i) \cap A_{t-1})$. Observe that $\varphi_t(x) = \varphi_t(b_i) = \varphi_{t-1}(a_{i,k}) = \varphi(a_{i,k})$, where the second equality holds since $a_{i,k}$ and b_i were switched in round t, and the third equality holds by Item 2 of Lemma 3.7. Since $a_{i,k}$ and b_i were switched in round t, it follows by Item 1(c) of Strategy 3.6 that $\varphi(a_{i,k})$ is adjacent in G_t to all vertices of $\varphi_{t-1}(N_H(b_i) \cap A_{t-1})$. In particular, $\{\varphi_t(x), \varphi_t(y)\} \in E(G_t)$. This concludes the proof of the lemma. □

In the following three lemmas, we consider the execution of Strategy 3.6 for ℓ rounds for some positive integer ℓ. For every $1 \leq i \leq r$ and $1 \leq k \leq m$, we denote by $\mathcal{A}_{i,k}$ the event: "$\varphi(a_{i,k})$ was offered at least d times **after** each of the vertices in $\varphi(L_{i,k})$ had already been offered". In other words, $\mathcal{A}_{i,k}$ is the event that there are indices $1 \leq t_1 < \ldots < t_q < s_1 < \ldots < s_d \leq \ell$, where $q = |L_{i,k}|$, such that each element of $\varphi(L_{i,k})$ was offered in one of the rounds $t_1, \ldots, t_q$, and $\varphi(a_{i,k})$ was offered in each of the rounds $s_1, \ldots, s_d$.

Lemma 3.9. *Let $1 \leq i \leq r$. If there exists some $1 \leq k \leq m$ for which $\mathcal{A}_{i,k}$ occurred, then $b_i \in A_\ell$.*

Proof. Suppose for a contradiction that $\mathcal{A}_{i,k}$ occurred for some $1 \leq i \leq r$ and $1 \leq k \leq m$, but $b_i \notin A_\ell$, i.e., $b_i \in B_\ell$. This means that b_i was not switched at any of the ℓ rounds for which we execute Strategy 3.6. Set $q = |L_{i,k}|$ and let $1 \leq t_1 < \ldots < t_q < s_1 < \ldots < s_d \leq \ell$ be the round numbers appearing in the definition of $\mathcal{A}_{i,k}$. Item 1(a) of Strategy 3.6 dictates that whenever a vertex from $\varphi(L_{i,k})$ is sampled, Builder connects it to $\varphi(b_i)$. This implies that, for every $t_q \leq t \leq \ell$, every vertex of $\varphi(L_{i,k})$ is adjacent in G_t to $\varphi(b_i)$.

As in Lemma 3.7, we let $N_t = \varphi_{t-1}(N_H(b_i) \cap A_{t-1})$ for each $1 \leq t \leq \ell$. Suppose first that there exists some $1 \leq j \leq d$ such that $\varphi(a_{i,k})$ is adjacent in G_{s_j} to every vertex of N_{s_j}. Then by Item 1(c) of Strategy 3.6, Builder would have switched $a_{i,k}$ and b_i in round s_j. This is a contradiction to our assumption that $b_i \in B_\ell$. Hence, suppose that for every $1 \leq j \leq d$, there exists a vertex of N_{s_j} which is not adjacent in G_{s_j} to $\varphi(a_{i,k})$. It follows by Item 1(b) of Strategy 3.6 that, in round s_j, Builder claims an edge $\{\varphi(a_{i,k}), u_j\}$ for some $u_j \in N_{s_j}$. Note that $u_1, \ldots, u_d$ are distinct. It follows by Item 4 of Lemma 3.7 that $u_1, \ldots, u_d \in N_{s_d}$. On the other hand, $|N_{s_d}| \leq |N_H(b_i)| \leq d$, implying that $N_{s_d} = \{u_1, \ldots, u_d\}$. But this means that in the graph G_{s_d}, the vertex $\varphi(a_{i,k})$ is adjacent to every vertex of N_{s_d}, contrary to the above. □

The following technical lemma provides lower bounds on the probability of the events $\mathcal{A}_{i,k}$.

Lemma 3.10. *Fix any $1 \le i \le r$ and $1 \le k \le m$.*

(a) *If $\ell \ge 2d$, then*

$$\mathbb{P}[\mathcal{A}_{i,k}] \ge \left(\frac{\ell^2}{12dn^2}\right)^d \cdot e^{-\frac{(d+1)(\ell-2d)}{n-d-1}} .$$

(b) *If $\ell \ge (\log(2d) + d + 3\sqrt{d})n$, then $\mathbb{P}[\mathcal{A}_{i,k}] \ge \frac{1}{4}$.*

Proof. We start with Item (a). Since $|L_{i,k}| = d_H(a_{i,k}) \le d$, it follows that

$$\begin{aligned}\mathbb{P}[\mathcal{A}_{i,k}] &\ge \binom{\ell}{2d} \cdot d! \cdot \left(\frac{1}{n}\right)^{2d} \cdot \left(1 - \frac{d+1}{n}\right)^{\ell-2d} \\ &\ge \left(\frac{\ell}{2dn}\right)^{2d} \cdot \left(\frac{d}{e}\right)^d \cdot e^{-\frac{(d+1)(\ell-2d)}{n-d-1}} \\ &\ge \left(\frac{\ell^2}{12dn^2}\right)^d \cdot e^{-\frac{(d+1)(\ell-2d)}{n-d-1}} ,\end{aligned}$$

where in the second inequality we used the estimates $1-x \ge e^{-\frac{x}{1-x}}$ (which holds for every $0 < x < 1$) and $d! \ge \left(\frac{d}{e}\right)^d$ (which holds for every $d \ge 1$).

Next, we prove (b). Let $\mathcal{E}_k$ be the event that every vertex of $\varphi(L_{i,k})$ was offered in the course of the first $\log(2d)n$ rounds, and let $\mathcal{F}_k$ be the event that $\varphi(a_{i,k})$ was offered at least d times in the course of the last $(d + 3\sqrt{d})n$ rounds. Since $\ell \ge (\log(2d) + d + 3\sqrt{d})n$ by assumption, the events $\mathcal{E}_k$ and $\mathcal{F}_k$ are independent. Note that $\mathcal{E}_k \cap \mathcal{F}_k \subseteq \mathcal{A}_{i,k}$; that is, if both $\mathcal{E}_k$ and $\mathcal{F}_k$ occur, then so does $\mathcal{A}_{i,k}$. Therefore, $\mathbb{P}[\mathcal{A}_{i,k}] \ge \mathbb{P}[\mathcal{E}_k \cap \mathcal{F}_k] = \mathbb{P}[\mathcal{E}_k] \cdot \mathbb{P}[\mathcal{F}_k]$. The probability that $\mathcal{E}_k$ did not occur is at most

$$d \cdot \left(1 - \frac{1}{n}\right)^{\log(2d)n} \le d \cdot e^{-\log(2d)} = \frac{1}{2},$$

and the probability that $\mathcal{F}_k$ did not occur equals the probability that $\mathrm{Bin}\left(\left(d + 3\sqrt{d}\right)n, 1/n\right)$ is smaller than d, which is at most

$$\mathbb{P}\left[\mathrm{Bin}\left(\left(d + 3\sqrt{d}\right)n, 1/n\right) < d\right] \le e^{-\frac{9d}{2(d+3\sqrt{d})}} \le e^{-\frac{9d}{8d}} \le \frac{1}{2} ,$$

Here in the first inequality we used Lemma 2.1 with $\lambda = 3\sqrt{d}$. We thus conclude that $\mathbb{P}[\mathcal{A}_{i,k}] \ge 1/2 \cdot 1/2 = 1/4$ as claimed. $\square$

The following lemma forms our main result in this subsection, and plays the key role in the proofs of Theorems 1.3 and 1.6. Roughly speaking, this lemma states that if $\varphi : V(H) \to [n]$ is a bijection admitting a φ-good set that misses only a small fraction of $V(H)$, then by following Strategy 3.6 for a suitable (and not too large) number of rounds, Builder can obtain a bijection $\varphi' : V(H) \to [n]$ which admits a φ'-good set that misses significantly fewer vertices. The proof of Lemma 3.11 utilizes Lemmas 3.8, 3.9 and 3.10, as well as some of the concentration inequalities from Section 2.

Lemma 3.11. *Let H and G be graphs on the vertex-set $[n]$, let $\varphi : V(H) \to V(G)$ be a bijection and let $V(H) = A \cup B$ be a partition such that A is (G, φ)-good. Write $B = \{b_1, \ldots, b_r\}$. Let m be a positive integer, and let $\{a_{i,k} \in A : 1 \le i \le r$ and $1 \le k \le m\}$ be vertices which satisfy Items*

1 and 2 of Setting 3.5. Let d be such that $d_H(x) \leq d$ for every $x \in \bigcup_{i=1}^{r}\{b_i, a_{i,1}, \ldots, a_{i,m}\}$. Let $\ell_1 = (\log(2d) + d + 3\sqrt{d})n$, $\ell_2 = \lceil n \cdot m^{-1/4d} \rceil$, $q_1 = \frac{1}{4}$, and $q_2 = d^{-2d}m^{-1/2}$. Fix any $j \in \{1, 2\}$, and assume that $mq_j \geq 10^6 d$. Suppose that Builder executes Strategy 3.6 for ℓ_j consecutive rounds. Let

$$p = \begin{cases} o(1), & \frac{mq_j}{64d} \geq \log n, \\ e^{-\frac{\sqrt{n}}{4d}}, & \frac{mq_j}{64d} < \log n. \end{cases}$$

Then, with probability at least $1 - p$, we have

$$|B_{\ell_j}| \leq \begin{cases} 0, & \frac{mq_j}{64d} \geq \log n, \\ 5n \cdot e^{-\frac{mq_j}{256d}}, & \frac{mq_j}{64d} < \log n. \end{cases} \tag{2}$$

In particular, under the assumptions of this lemma, executing Strategy 3.6 guarantees that with probability at least $1 - p$, after ℓ_j rounds of the process Builder's graph G' will satisfy the following property: there will be a bijection $\varphi' : V(H) \to V(G)$ and a partition $V(H) = A' \cup B'$ such that A' is (G', φ')-good and such that the bounds in (2) hold for B'.

Proof. The "In particular" part of the lemma follows from the first part by setting $\varphi' = \varphi_{\ell_j}$, $A' = A_{\ell_j}$, and $B' = B_{\ell_j}$, and applying Lemma 3.8. It thus remains to prove the first part of the lemma.

Fix arbitrary indices $1 \leq i \leq r$ and $1 \leq k \leq m$. For $j \in \{1, 2\}$, let us denote by $\mathbb{P}_j(\mathcal{A}_{i,k})$ the probability that $\mathcal{A}_{i,k}$ occurred in the course of the first ℓ_j rounds of the random graph process. We claim that $\mathbb{P}_j[\mathcal{A}_{i,k}] \geq q_j$. Starting with $j = 1$, recall that $\ell_1 = (\log(2d) + d + 3\sqrt{d})n$, and so by Item (b) of Lemma 3.10 we have $\mathbb{P}_1[\mathcal{A}_{i,k}] \geq \frac{1}{4} = q_1$, as required. As for $j = 2$, recall that $\ell_2 = \lceil n \cdot m^{-1/4d} \rceil$, implying that $\ell_2 \geq 2d$ holds for n which is sufficiently large with respect to d (since, trivially, $m \leq n$, something like $n \geq d^2$ would suffice). Therefore, Item (a) of Lemma 3.10 yields

$$\mathbb{P}_2[\mathcal{A}_{i,k}] \geq \left(\frac{\ell_2^2}{12dn^2}\right)^d \cdot e^{-\frac{(d+1)(\ell_2 - 2d)}{n-d-1}} \geq \left(\frac{1}{12d}\right)^d \cdot m^{-1/2} \cdot e^{-d-1} \geq d^{-2d}m^{-1/2} = q_2 \ ,$$

where the last inequality holds for sufficiently large d.

We have thus proved our assertion that $\mathbb{P}_j[\mathcal{A}_{i,k}] \geq q_j$ holds for every $j \in \{1, 2\}$. From now on, let us fix an arbitrary $j \in \{1, 2\}$ and assume that $mq_j \geq 10^6 d$. For convenience, we put $\ell := \ell_j$ and $q := q_j$. For every $1 \leq i \leq r$, let X_i be the random variable counting the number of indices $1 \leq k \leq m$ for which $\mathcal{A}_{i,k}$ occurred in the course of the first ℓ rounds. It follows by linearity of expectation that $\mathbb{E}[X_i] \geq mq$.

Now, consider the sequence $(w_1, \ldots, w_\ell)$ of random vertices offered to Builder, and observe that changing any one coordinate in this sequence can change the value of X_i by at most 1. Furthermore, for every s, if $X_i \geq s$, then there is a set of at most $2ds$ coordinates in the sequence $(w_1, \ldots, w_\ell)$ which certify that $X_i \geq s$ (indeed, each event $\mathcal{A}_{i,k}$ that occurred is certified by a set of at most $2d$ coordinates). It thus follows by Lemma 2.3 with $c = 1$, $g = 2d$, and $\lambda = \frac{\mathbb{E}[X_i]}{2}$, that

$$\mathbb{P}\left[X_i < \frac{\mathbb{E}[X_i]}{2} - 60\sqrt{2d\mathbb{E}[X_i]}\right] \leq 4e^{-\frac{(\mathbb{E}[X_i]/2)^2}{16d\mathbb{E}[X_i]}} = 4e^{-\frac{\mathbb{E}[X_i]}{64d}} \leq 4e^{-\frac{mq}{64d}} \ .$$

Therefore, with probability at least $1 - 4e^{-\frac{mq}{64d}}$, it holds that

$$X_i \geq \frac{\mathbb{E}[X_i]}{2} - 60\sqrt{2d\mathbb{E}[X_i]}$$

$$
\begin{aligned}
&= \sqrt{\mathbb{E}[X_i]} \cdot \left(\frac{\sqrt{\mathbb{E}[X_i]}}{2} - 60\sqrt{2d} \right) \\
&\geq \sqrt{\mathbb{E}[X_i]} \cdot \left(\frac{\sqrt{mq}}{2} - 60\sqrt{2d} \right) \\
&> 0,
\end{aligned}
$$

where the last inequality follows from our assumption that $mq \geq 10^6 d$.

Now let $\mathcal{I}$ be the set of all $1 \leq i \leq r$ such that $X_i = 0$. It follows by Lemma 3.9 that if some $1 \leq i \leq r$ satisfies $X_i > 0$, then $b_i \notin B_\ell$. Hence, we have $B_\ell \subseteq \{b_i : i \in \mathcal{I}\}$. So to complete the proof it is enough to show that the bounds in (2) hold for the set $\mathcal{I}$. We have seen that $\mathbb{P}[i \in \mathcal{I}] \leq 4e^{-\frac{mq}{64d}}$ holds for every $1 \leq i \leq r$. Therefore,

$$
\mathbb{E}[|\mathcal{I}|] \leq r \cdot 4e^{-\frac{mq}{64d}} = |B| \cdot 4e^{-\frac{mq}{64d}} .
$$

Suppose first that $\frac{mq}{64d} \geq \log n$. Note that we have $|B| \leq \frac{n}{m} \leq \frac{n}{\log n}$. It follows that

$$
\mathbb{E}[|\mathcal{I}|] \leq |B| \cdot 4e^{-\frac{mq}{64d}} \leq \frac{n}{\log n} \cdot \frac{4}{n} = o(1).
$$

So by Markov's inequality, we have $|\mathcal{I}| = 0$ w.h.p., as required.

Suppose then that $\frac{mq}{64d} < \log n$. Observe that changing any one coordinate in the sequence $(w_1, \ldots, w_\ell)$ of random vertices, can change the value of $|\mathcal{I}|$ by at most 1. Hence, it follows by Lemma 2.2 with $c = 1$ and $\lambda = n \cdot e^{-\frac{mq}{256d}} \geq n^{3/4}$, that

$$
\mathbb{P}\left[|\mathcal{I}| \geq \mathbb{E}[|\mathcal{I}|] + \lambda\right] \leq e^{-\frac{\lambda^2}{2\ell}} \leq e^{-\frac{n^{3/2}}{2\ell}} \leq e^{-\frac{\sqrt{n}}{4d}} ,
$$

where the last inequality holds since $\ell_1, \ell_2 \leq 2dn$. We conclude that with probability at least $1 - e^{-\frac{\sqrt{n}}{4d}}$, we have $|\mathcal{I}| \leq \mathbb{E}[|\mathcal{I}|] + \lambda \leq (4|B| + n) \cdot e^{-\frac{mq}{256d}} \leq 5n \cdot e^{-\frac{mq}{256d}}$, as required. □

3.3 Putting it All Together

In this subsection we prove Theorem 1.3. The proof follows by combining Lemma 3.4 with (multiple applications of) Lemma 3.11. We will need the following simple claim, which states that we can satisfy the conditions listed in Setting 3.5 with a relatively large choice of m.

Claim 3.12. *Let H be an n-vertex graph with maximum degree Δ. Let A' and B' be disjoint subsets of $V(H)$ such that $|A'| \geq (\Delta+1)|B'|$, and let $m = \lfloor \frac{|A'| - (\Delta+1)|B'|}{(\Delta^2+1)|B'|} \rfloor$. Then, there are vertices $\{a_{i,k} \in A' : 1 \leq i \leq r := |B'| \text{ and } 1 \leq k \leq m\}$ which satisfy Conditions 1 and 2 of Setting 3.5.*

Proof. Let $C = \{a \in A' : N_H(a) \cap B' \neq \emptyset\}$. Note that $\Delta(H) = \Delta$ implies that $|C| \leq \Delta \cdot |B'|$. Using again the fact that $\Delta(H) = \Delta$, we infer that there exists an integer

$$
M \geq \frac{|A'| - |C|}{\Delta^2 + 1} \geq \frac{|A'| - (\Delta+1)|B'|}{\Delta^2 + 1}
$$

and vertices $a_1, \ldots, a_M \in A' \setminus C$ such that $\mathrm{dist}_H(a_i, a_j) \geq 3$ for every $1 \leq i < j \leq M$. Now index (a subset of) the vertices $a_1, \ldots, a_M$ by pairs $(i,k) \in [r] \times [m]$. For every $(i,k) \in [r] \times [m]$, let $a_{i,k}$ be the vertex of $\{a_1, \ldots, a_M\}$ which is indexed by the pair (i,k). Then Condition 1 of Setting 3.5 holds since $a_{i,k} \notin C$ for every $(i,k) \in [r] \times [m]$, and Condition 2 of Setting 3.5 holds since $\mathrm{dist}_H(a_{i,k}, a_{i',k'}) \geq 3$ for every choice of distinct pairs $(i,k), (i',k') \in [r] \times [m]$. □

Proof of Theorem 1.3. Let n, Δ, and H be as in the statement of the theorem. Throughout the proof we will apply Lemma 3.11 with $d = \Delta$ (the distinction between d and Δ is immaterial in this proof). We will now describe Builder's strategy, and then prove that w.h.p. Builder can follow all parts of this strategy, and that by doing so, w.h.p. he builds a copy of H within $(3\Delta/2 + o(\Delta))n$ rounds. Builder employs the following strategy.

Stage 1: Fix an arbitrary bijection $\varphi : V(H) \to [n]$ and let $\alpha = 10^{-7}\Delta^{-5}$. For the first $(\Delta/2 + \sqrt{3\Delta \log(6\Delta/\alpha)})n = (\Delta/2 + o(\Delta))n$ rounds, Builder invokes the strategy whose existence is guaranteed by Lemma 3.4.

Stage 2: Let $A \subseteq V(H)$ be a φ-good set such that $B := V(H) \setminus A$ satisfies $|B| \leq \alpha n = 10^{-7}\Delta^{-5}n$. Let $m = \frac{n}{2\Delta^2|B|} \leq \lfloor \frac{|A|-(\Delta+1)|B|}{(\Delta^2+1)|B|} \rfloor$. Find vertices $\{a_{i,k} \in A : 1 \leq i \leq |B| \text{ and } 1 \leq k \leq m\}$ for which Conditions 1 and 2 of Setting 3.5 are satisfied. Apply the strategy whose existence is guaranteed by Lemma 3.11 with $j = 1$ and with input φ and $A \cup B$. Lemma 3.11 (with $j = 1$) ensures that after $\ell_1 = (\log(2\Delta) + \Delta + 3\sqrt{\Delta})n = (\Delta + o(\Delta))n$ rounds, there will be a bijection $\varphi_0 : V(H) \to [n]$ and a partition $V(H) = A_0 \cup B_0$ such that A_0 is φ_0-good with respect to Builder's graph, and such that

$$|B_0| \leq \begin{cases} 0, & \frac{m}{256\Delta} \geq \log n, \\ 5n \cdot e^{-\frac{m}{1024\Delta}}, & \frac{m}{256\Delta} < \log n. \end{cases}$$

If $B_0 = \emptyset$ then $A_0 = V(H)$ is φ'-good, implying that Builder has successfully embedded H into his graph, and so Builder is done. Otherwise, proceed to Stage 3.

Stage 3: Define a sequence of bijections $\varphi_1, \varphi_2, \ldots$ from $V(H)$ to $[n]$, and a sequence of partitions $A_1 \cup B_1, A_2 \cup B_2, \ldots$ of $V(H)$, by performing the following steps for every positive integer t for which $B_{t-1} \neq \emptyset$.

(a) Find vertices $\{a_{i,k} \in A_{t-1} : 1 \leq i \leq |B_{t-1}| \text{ and } 1 \leq k \leq m_{t-1} := \frac{n}{2\Delta^2|B_{t-1}|}\}$ for which Conditions 1 and 2 of Setting 3.5 are satisfied.

(b) Invoke the strategy whose existence is guaranteed by Lemma 3.11 with $j = 2$ and with input φ_{t-1} and $A_{t-1} \cup B_{t-1}$, to obtain a bijection $\varphi_t : V(H) \to [n]$ and a partition $A_t \cup B_t$ of $V(H)$ such that A_t is φ_t-good with respect to Builder's graph, and such that

$$|B_t| \leq \begin{cases} 0, & m'_{t-1} \geq \log n, \\ 5n \cdot e^{-m'_{t-1}/4}, & m'_{t-1} < \log n, \end{cases} \tag{3}$$

where

$$m'_{t-1} := \frac{m_{t-1} \cdot q_2}{64\Delta} = \frac{\sqrt{m_{t-1}}}{64\Delta^{2\Delta+1}},$$

and q_2 is as in Lemma 3.11 (here and later on we abuse notation a bit by omitting from the notation the fact that $q_2 = \Delta^{-2\Delta}/\sqrt{m_{t-1}}$ depends on t).

Having described Builder's strategy, we now turn to prove that w.h.p. Builder can follow it. Note first that Builder can follow his strategy for Stage 1 and that doing so, he can guarantee that w.h.p. there will be sets A and B as in the beginning of Stage 2 of his strategy (here we use Lemma 3.4).

Now, by Claim 3.12 with $A' = A$ and $B' = B$, there are vertices $\{a_{i,k} \in A : 1 \le i \le |B| \text{ and } 1 \le k \le m\}$ satisfying Conditions 1 and 2 of Setting 3.5. Moreover, the conditions required for the application of Lemma 3.11 with $j = 1$ are satisfied as $mq_1 = \frac{m}{4} \ge \frac{n}{8\Delta^2|B|} > 10^6\Delta$, where the last inequality holds since $|B| \le 10^{-7}\Delta^{-5}n$. This shows that w.h.p. Builder can follow Stage 2 of his strategy as well. Finally, we need to show that w.h.p. Builder can follow Stage 3 of his strategy. Similarly to Stage 2, the existence of the vertices $\{a_{i,k} \in A : 1 \le i \le |B_{t-1}| \text{ and } 1 \le k \le m_{t-1}\}$ for every positive integer t, follows from Claim 3.12 with input $A' = A_{t-1}$ and $B' = B_{t-1}$, and the fact that $\lfloor \frac{|A_{t-1}|-(\Delta+1)|B_{t-1}|}{(\Delta^2+1)|B_{t-1}|} \rfloor \ge m_{t-1}$. It remains to prove that the conditions of Lemma 3.11 are met whenever Builder wishes to apply it (with $j = 2$). The fact that A_{t-1} is φ_{t-1}-good for every positive integer t is guaranteed by the previous applications of Lemma 3.11. In order to show that $\sqrt{m_{t-1}} \cdot \Delta^{-2\Delta} = m_{t-1} \cdot q_2 \ge 10^6\Delta$ holds for every positive integer t for which $B_{t-1} \neq \emptyset$, we will first prove the following claim.

Claim 3.13. *Let $\zeta = e^{-\Delta^2}$ and, for every non-negative integer t, let $\beta_t = |B_t|/n$. If $t \ge 0$ is such that $m'_{t-1} < \log n$, then $\beta_t \le \zeta^{t+1}$.*

Proof. Our proof proceeds by induction on t. The base case $t = 0$ holds, because the description of Stage 2 of Builder's strategy implies that

$$|B_0| = |B'| \le 5n \cdot e^{-\frac{m}{1024\Delta}} \le 5n \cdot e^{-\frac{n}{2048\Delta^3|B|}} \le n \cdot e^{-\Delta^2} \le \zeta n\ .$$

Let then $t \ge 1$ and suppose that $m'_{t-1} < \log n$. Note that the sequence m'_s is monotone non-decreasing in s (this follows from the fact that $B_0 \supseteq B_1 \supseteq \ldots$). So $m'_{t-2} < \log n$ as well. By the induction hypothesis for $t-1$, we have $\beta_{t-1} \le \zeta^t$. Now,

$$m'_{t-1} = \frac{\sqrt{m_{t-1}}}{64\Delta^{2\Delta+1}} \ge \frac{\sqrt{n}}{128\Delta^{2\Delta+2}\sqrt{|B_{t-1}|}}\ , \tag{4}$$

and therefore

$$\begin{aligned}\beta_t &\le 5 \cdot e^{-m'_{t-1}/4} \le 5 \cdot \exp\left(-\frac{1}{512\Delta^{2\Delta+2}\sqrt{\beta_{t-1}}}\right) \le 5 \cdot \exp\left(-\frac{1}{512\Delta^{2\Delta+2}\zeta^{t/2}}\right)\\ &= 5 \cdot \exp\left(-\frac{e^{\Delta^2 t/2}}{512\Delta^{2\Delta+2}}\right) \le e^{-\Delta^2(t+1)} = \zeta^{t+1}\ ,\end{aligned}$$

where the first inequality holds by (3), the second inequality holds by (4), the third inequality holds by the induction hypothesis for $t-1$, and the last inequality holds for every $t \ge 1$, provided that Δ is larger than some suitable absolute constant. This proves the claim. □

Returning to the proof of the theorem, observe that

$$\begin{aligned} m_{t-1} \cdot q_2 = \sqrt{m_{t-1}} \cdot \Delta^{-2\Delta} &\ge \frac{\sqrt{n}}{2\Delta^{2\Delta+1}\sqrt{|B_{t-1}|}} = \frac{1}{2\Delta^{2\Delta+1}\sqrt{\beta_{t-1}}}\\ &\ge \frac{1}{2\Delta^{2\Delta+1}\zeta^{t/2}} = \frac{e^{\Delta^2 t/2}}{2\Delta^{2\Delta+1}} \ge 10^6\Delta, \end{aligned} \tag{5}$$

where the second inequality holds by Claim 3.13 and the last holds for sufficiently large Δ. This shows that we can indeed apply Lemma 3.11 with $j = 2$ and with input $A_{t-1} \cup B_{t-1}$ for every positive integer t for which $B_{t-1} \neq \emptyset$. We conclude that Builder can follow Stage 3 of his strategy.

Next, we prove the correctness of Builder's strategy. For the time being, we will assume that all of the applications of Lemmas 3.4 and 3.11 throughout Builder's strategy are successful; later we will show that w.h.p. this is indeed the case. It follows from an intermediate calculation appearing in (5) that $m'_{t-1} \geq \log n$ must hold for some $t \leq \log\log n$ (and in fact much faster). But if $m'_{t-1} \geq \log n$ then by (3) we have $B_t = \emptyset$, which in turn implies that Builder has successfully embedded H into the graph he is constructing.

Next, we estimate the probability that Builder's strategy fails. Recall that Lemma 3.4 is only applied once (in Stage 1), and Lemma 3.11 is only applied once with $j = 1$ (in Stage 2). Both of these applications are w.h.p. successful. Let us now consider the applications of Lemma 3.11 with $j = 2$ (in Stage 3). As previously noted, there is at most one such application with $m'_{t-1} \geq \log n$, and at most $\log\log n$ such applications with $m'_{t-1} < \log n$. The failure probability of the former application is $o(1)$, and the failure probability of each of the latter applications is at most $e^{-\frac{\sqrt{n}}{4\Delta}}$. We thus conclude that w.h.p. all of the above applications of Lemmas 3.4 and 3.11 are successful, as required. This concludes the proof of correctness of Builder's strategy.

It remains to estimate the overall number of rounds required for implementing Builder's strategy. Recall that the application of Lemma 3.4 requires $(\Delta/2 + o(\Delta))n$ rounds, and the sole application of Lemma 3.11 with $j = 1$ requires $(\Delta + o(\Delta))n$ rounds. Hence, Stages 1 and 2 of Builder's strategy together require at most $(3\Delta/2 + o(\Delta))n$ rounds. It thus remains to bound from above the number of rounds required for Stage 3 of Builder's strategy. To this end, let t^* denote the smallest integer t satisfying $m'_{t-1} \geq \log n$, and note that $t^* \leq \log\log n$. Then in Stage 3 we must have applied Lemma 3.11 at most t^* times. Moreover, for each $1 \leq t \leq t^*$, applying Lemma 3.11 with input $A_{t-1} \cup B_{t-1}$ (and with $j = 2$) requires at most

$$\lceil n \cdot m_{t-1}^{-1/4\Delta} \rceil \leq n \cdot m_{t-1}^{-1/4\Delta} + 1 \leq n \cdot e^{-\Delta t/16} + 1$$

rounds, where in the last inequality we used an intermediate calculation appearing in (5). Therefore, the overall number of rounds required for the (at most) t^* applications of Lemma 3.11 in Stage 3 is no more than

$$\sum_{t=1}^{t^*} \left(n \cdot e^{-\Delta t/16} + 1 \right) \leq n \sum_{t=1}^{\infty} e^{-\Delta t/16} + \log\log n = o(\Delta)n.$$

We conclude that the overall number of rounds required for implementing Builder's strategy is at most $(3\Delta/2 + o(\Delta))n$, thus completing the proof of Theorem 1.3. □

3.4 Offline Construction of Spanning Graphs

Proof of Proposition 1.2. Let D be an orientation of the edges of H as in Lemma 2.4. Let $w_1, w_2, \ldots$ denote the sequence of random vertices Builder is offered. Let $m(D)$ denote the smallest integer m for which there exists a bijection $\varphi : V(H) \to [n]$ such that $\varphi(u)$ appears in $(w_1, w_2, \ldots, w_m)$ at least $d_D^+(u)$ times for every $u \in V(H)$. It follows by Proposition 4.1 in [4] that the number of rounds needed for Builder to construct a copy of H in the offline version of the semi-random process is at most $m(D)$. Hence, in order to complete the proof of this proposition, it suffices to prove that w.h.p., in the course of $m := (\Delta/2 + \sqrt{60\Delta \log \Delta})n$ rounds, at least $n - n/(\Delta^2 + 1)$ vertices are offered at least $\lfloor \Delta/2 \rfloor + 1$ times each.

For every $1 \leq i \leq n$, let I_i be the indicator random variable for the event that vetrex i was offered at least $\Delta/2\rfloor + 1$. By Lemma 3.2 with parameter $\varepsilon = \sqrt{\frac{60\log(\Delta)}{\Delta}}$, for any given $i \in [n]$ we have

$$\mathbb{P}[I_i = 0] \leq e^{-\varepsilon^2 \Delta/3} = e^{-20 \log \Delta} = \Delta^{-20}. \tag{6}$$

Let A denote the set of vertices $1 \leq i \leq n$ which were offered at least $\lfloor \Delta/2 \rfloor + 1$ times in $(w_1, w_2, \ldots, w_m)$; clearly $|A| = \sum_{i=1}^{n} I_i$. If $\Delta \geq n^{1/10}$ then it follows by (6) and a union bound that

$$\mathbb{P}[|A| < n] = \mathbb{P}[\exists 1 \leq i \leq n \text{ such that } I_i = 0] \leq n \cdot \Delta^{-20} \leq 1/n.$$

Assume then that $2 \leq \Delta \leq n^{1/10}$ (note that we are allowed to assume that Δ is not too small; moreover, the case $\Delta = 1$ was handled in [4]). It follows by (6) and the linearity of expectation that

$$n \geq \mathbb{E}[|A|] \geq n - n \cdot \Delta^{-20}.$$

We will use Lemma 2.3 to prove that $|A|$ is concentrated around its expected value. Consider the sequence of random vertices $(w_1, w_2, \ldots, w_m)$, and observe that changing any single coordinate of this sequence can change the value of $|A|$ by at most 1. Moreover, if $|A| \geq \ell$, then there are at least ℓ indices $1 \leq i \leq n$ which appear at least $\lfloor \Delta/2 \rfloor + 1$ times in $(w_1, w_2, \ldots, w_m)$, and so there is a set of $(\lfloor \Delta/2 \rfloor + 1)\,\ell$ entries in the sequence $(w_1, w_2, \ldots, w_m)$ which certify that $|A| \geq \ell$. Therefore, applying Lemma 2.3 with parameters $c = 1$, $g = \lfloor \Delta/2 \rfloor + 1$ and $\lambda = n/\Delta^4$ yields

$$\mathbb{P}\left[|A| < \mathbb{E}[|A|] - n/\Delta^4 - 60\sqrt{(\lfloor \Delta/2 \rfloor + 1)\,\mathbb{E}[|A|]}\right] \leq 4e^{-\frac{n^2/\Delta^8}{8(\lfloor \Delta/2 \rfloor + 1)\mathbb{E}[|A|]}} \leq 4e^{-\frac{n^2}{8\Delta^9 n}} = o(1),$$

where the last equality holds since $\Delta \leq n^{1/10}$. We conclude that w.h.p. we have

$$|A| \geq \mathbb{E}[|A|] - n/\Delta^4 - 60\sqrt{(\lfloor \Delta/2 \rfloor + 1)\,\mathbb{E}[|A|]} \geq n - 2n/\Delta^4 - 60\sqrt{\Delta n} \geq n - n/(\Delta^2 + 1),$$

where the last inequality holds since $\Delta \geq 2$ and n is sufficiently large. □

4 Building Spanning Trees and Forests

In this section we prove Theorem 1.6 and Proposition 1.7. We start with the following simple lemma, whose proof demonstrates a strategy for embedding an almost-spanning forest.

Lemma 4.1. *Let n be a positive integer and let $\alpha \in (0, 1)$ such that $n \gg \alpha^{-2} \log(1/\alpha)$. Let T be a forest on $(1 - \alpha)n$ vertices and let $\ell = \log(2/\alpha) \cdot n$. Then, in the semi-random process on K_n, Builder has a strategy which w.h.p. allows him to construct a copy of T within ℓ rounds.*

Proof. Assume without loss of generality that T is a tree (otherwise simply replace T with a tree containing it). Let $t = (1 - \alpha)n$ and let $v_1, \ldots, v_t$ be an ordering of the vertices of T such that $T[\{v_1, \ldots, v_i\}]$ is a tree for every $1 \leq i \leq t$. Throughout the process, Builder maintains a partial function φ which is initially empty. For every positive integer i, let w_i denote the vertex Builder is offered in the ith round. In the first round, Builder connects w_1 to an arbitrary vertex u; he then sets $\varphi(v_1) = w_1$ and $\varphi(v_2) = u$. For every $i \geq 2$, Builder plays the ith round as follows. Let r denote the largest integer for which $\varphi(v_r)$ was previously defined. If $w_i \notin \{\varphi(v_1), \ldots, \varphi(v_r)\}$, then Builder connects w_i to $\varphi(v_j)$, where $j \leq r$ is the unique integer for which $\{v_{r+1}, v_j\} \in E(T)$; he then sets $\varphi(v_{r+1}) = w_i$. Otherwise, Builder claims an arbitrary edge which is incident with w_i, but does not consider this edge to be part of the tree he is building.

It is evident that, by following the proposed strategy, Builder's graph contains a copy of T as soon as t different vertices are offered. Hence, it suffices to prove that w.h.p. at least t different vertices are offered during the first ℓ rounds. For every $1 \leq j \leq n$, let I_j be the indicator random variable for the event: "vertex j was not offered during the first ℓ rounds of the process". Let $X = \sum_{j=1}^{n} I_j$; then

$$\mathbb{E}(X) = \sum_{j=1}^{n} \mathbb{E}(I_j) = n(1 - 1/n)^{\ell} \leq n \cdot e^{-\ell/n} = \alpha n/2.$$

Observe that changing any one coordinate in the sequence $(w_1, \ldots, w_\ell)$ of random vertices, can change the value of X by at most 1. Hence, it follows by Lemma 2.2 with $c = 1$ and $\lambda = \alpha n/2$, that

$$\mathbb{P}[X \geq \alpha n] \leq \mathbb{P}\left[X \geq \mathbb{E}(X) + \alpha n/2\right] \leq e^{-\frac{(\alpha n/2)^2}{2\ell}} \leq e^{-\frac{\alpha^2 n^2}{8 \log(2/\alpha) n}} = o(1) ,$$

where the last equality holds by our assumption that $n \gg \alpha^{-2} \log(1/\alpha)$. □

We are now in a position to prove Theorem 1.6. The proof has the same scheme as the proof of Theorem 1.3, but with two crucial differences: firstly, instead of using Lemma 3.4 to create an initial embedding, we achieve this task by using Lemma 4.1 (with an appropriate choice of a subforest T); and secondly, we apply Lemma 3.11 with $d = 2$ instead of $d = \Delta$.

Proof of Theorem 1.6. Let n, Δ, and T be as in the statement of the theorem. Assume first that $\Delta \geq n^{1/3}$. In this case Builder employs the strategy presented in the proof of Lemma 4.1. It is easy to see that as soon as each of the n vertices has been offered, Builder's graph contains a copy of T. It is well-known (and easy to prove) that this will happen w.h.p. in $(1 + o(1))n \log n = \Theta(n \log \Delta)$ rounds. For the remainder of the proof we thus assume that $\Delta < n^{1/3}$.

Define $D_{\leq 2} = \{v \in V(T) : d_T(v) \leq 2\}$. As T is a forest, we have $2n - 2 \geq \sum_{v \in V(T)} d_T(v) \geq |D_{\leq 2}| + 3(n - |D_{\leq 2}|)$. It follows that $|D_{\leq 2}| > n/2$. Let $B \subseteq D_{\leq 2}$ be a set of size $2^{-52} n/\Delta^3$, and let $T' = T \setminus B$. We can now describe Builder's strategy.

Stage 1: Builder invokes the strategy which is described in the proof of Lemma 4.1 (with $\alpha = 2^{-52}/\Delta^3$) to construct a copy of T' in $\log(2/\alpha)n = O(\log \Delta)n$ rounds.

Stage 2: Define a sequence of bijections $\varphi_0, \varphi_1, \ldots$ from $V(T)$ to $[n]$, and a sequence of partitions $A_0 \cup B_0, A_1 \cup B_1, \ldots$ of $V(T)$ with the property that $A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots$, as follows. Let $A_0 = V(T')$, $B_0 = B$, and let $\varphi_0 : V(T) \to [n]$ be any bijection whose restriction to $A_0 = V(T')$ is an embedding of T' into Builder's graph. Now, perform the following steps for every positive integer t for which $B_{t-1} \neq \emptyset$.

(a) Find vertices $\{a_{i,k} \in A_{t-1} : 1 \leq i \leq |B_{t-1}| \text{ and } 1 \leq k \leq m_{t-1} := \frac{n}{3\Delta^2 |B_{t-1}|}\}$ such that Conditions 1 and 2 of Setting 3.5 are satisfied, and, moreover, $d_T(a_{i,k}) \leq 2$ for every $1 \leq i \leq |B_{t-1}|$ and $1 \leq k \leq m_{t-1}$.

(b) Invoke the strategy whose existence is guaranteed by Lemma 3.11 with $j = 2$, with $d = 2$, and with input φ_{t-1} and $A_{t-1} \cup B_{t-1}$, to obtain a bijection $\varphi_t : V(T) \to [n]$ and a partition $A_t \cup B_t$ of $V(T)$ such that A_t is φ_t-good with respect to Builder's graph, and such that

$$|B_t| \leq \begin{cases} 0, & 2^{-11} \cdot \sqrt{m_{t-1}} \geq \log n, \\ 5n \cdot e^{-2^{-13} \cdot \sqrt{m_{t-1}}}, & 2^{-11} \cdot \sqrt{m_{t-1}} < \log n. \end{cases} \tag{7}$$

Having described Builder's strategy, we now turn to prove that Builder can w.h.p. follow its instructions and that, by doing so, w.h.p. he builds a copy of T in $O(\log \Delta)n$ rounds. First, note that Builder can follow Stage 1 of his strategy and that, by Lemma 4.1, by doing so, w.h.p. he embeds T' into his graph in $O(\log \Delta)n$ rounds. In particular, at the end of Stage 1 there exists w.h.p. a bijection φ_0 as in the beginning of Stage 2. Next, we show that Builder can follow Stage 2(a) of his strategy for every positive integer t for which $B_{t-1} \neq \emptyset$. First, note that $|A_{t-1} \cap D_{\leq 2}| \geq 0.49n$, because $|A_{t-1}| \geq |A_0| \geq n - 2^{-52}n/\Delta^3$ and $|D_{\leq 2}| > n/2$. Now apply Claim 3.12 with $A' = A_{t-1} \cap D_{\leq 2}$ and $B' = B_{t-1}$ to get the required vertices $\{a_{i,k} \in A_{t-1} : 1 \leq i \leq |B_{t-1}| \text{ and } 1 \leq k \leq m_{t-1}\}$. Here we use the fact that $\lfloor \frac{|A_{t-1} \cap D_{\leq 2}| - (\Delta+1)|B_{t-1}|}{(\Delta^2+1)|B_{t-1}|} \rfloor \geq \frac{n}{3\Delta^2 |B_{t-1}|} = m_{t-1}$.

It remains to show that w.h.p. Builder can follow Stage 2(b) of his strategy for every positive integer t for which $B_{t-1} \neq \emptyset$. In order to do so, we first need to prove that the conditions of Lemma 3.11 are met. The fact that A_{t-1} is φ_{t-1}-good for every positive integer t is guaranteed by the previous applications of Lemma 3.11. Finally, observe that for $d = 2$, we have

$$\frac{m_{t-1} \cdot q_2}{64d} = \frac{\sqrt{m_{t-1}}}{64 d^{2d+1}} = 2^{-11} \cdot \sqrt{m_{t-1}} \geq 2^{-11} \cdot \sqrt{\frac{n}{3\Delta^2 |B_{t-1}|}} \geq 2^{14}, \tag{8}$$

where q_2 is as in Lemma 3.11, and the last inequality holds since $|B_{t-1}| \leq |B_0| \leq 2^{-52}n/\Delta^2$. Now (8) justifies (7) and shows that $m_{t-1} \cdot q_2 = \frac{1}{16} \cdot \sqrt{m_{t-1}} \geq 2^{21} >= 2 \cdot 10^6 = 10^6 d$, as required by Lemma 3.11.

Next, we prove the correctness of Builder's strategy. For the time being, we will assume that all of the applications of Lemma 3.11 throughout Builder's strategy are successful; later we will show that w.h.p. this is indeed the case. We first prove the following simple claim.

Claim 4.2. *For every non-negative integer t, let $\beta_t = |B_t|/n$. If $t \geq 0$ is such that $2^{-11} \cdot \sqrt{m_{t-1}} < \log n$, then $\beta_t \leq \Delta^{-2t-3}$.*

Proof. Our proof proceeds by induction on t. The base case $t = 0$ holds since $\beta_0 = |B_0|/n \leq \Delta^{-3}$. For the induction step, fix some $t \geq 1$, and suppose that $2^{-11} \cdot \sqrt{m_{t-1}} < \log n$. Since the sequence m_s is monotone non-decreasing in s, we have $2^{-11} \cdot \sqrt{m_{t-2}} < \log n$ as well. It then follows by the induction hypothesis that $\beta_{t-1} \leq \Delta^{-2t-1}$. Then

$$\beta_{t+1} \leq 5 \cdot e^{-2^{-13} \cdot \sqrt{m_t}} \leq 5 \cdot \exp\left(-\frac{1}{2^{14}\Delta\sqrt{\beta_t}}\right) \leq 5 \cdot \exp\left(-\frac{\Delta^{t+1/2}}{2^{14}\Delta}\right) = 5 \cdot e^{-2^{-14}\Delta^{t-1/2}} \leq \Delta^{-2t-3},$$

where the first inequality holds by (7), the second from our choice of m_{t-1}, and the third from the induction hypothesis for $t - 1$. The last inequality holds for every $t \geq 1$ provided that Δ is larger than some suitable absolute constant. □

Combining Claim 4.2 and our choice of m_t implies that $\sqrt{m_t} \geq 1/2 \cdot \sqrt{\frac{1}{\Delta^2 \beta_t}} \geq 1/2 \cdot \Delta^t$ as long as $2^{-11} \cdot \sqrt{m_{t-1}} < \log n$. Therefore, there must exist some $t \leq O(\log \log n)$ such that $2^{-11} \cdot \sqrt{m_t} \geq \log n$.

Let t^* be the minimum such t. It then follows by (7) that $B_{t^*+1} = \emptyset$, implying that Builder has successfully embedded T into the graph he is constructing.

Next, we estimate the probability that Builder's strategy fails. The execution of Stage 1 of Builder's strategy is successful w.h.p. by Lemma 4.1. The only thing which might fail in the execution of Stage 2 of Builder's strategy, are the applications of Lemma 3.11. However, as previously noted, there are at most $O(\log \log n)$ applications of this lemma, and each of them fails with probability at most $e^{-\sqrt{n}/8}$, except maybe the last which fails with probability $o(1)$. This concludes the proof of correctness of Builder's strategy.

It remains to estimate the overall number of rounds required for implementing Builder's strategy. By Lemma 4.1, Stage 1 of the strategy requires w.h.p. $\log(2/\alpha)n = \log(2^{61}\Delta^3)n = O(\log \Delta)n$ rounds. As guaranteed by Lemma 3.11, Stage 2 of Builder's strategy requires w.h.p.

$$\sum_{t=0}^{t^*} \lceil n \cdot m_t^{-1/8} \rceil \leq n \sum_{t=0}^{t^*} m_t^{-1/8} + t^* + 1 \leq n \sum_{t=0}^{t^*} (3\Delta^2 \beta_t)^{1/8} + o(n) \leq n \sum_{t=0}^{\infty} 2 \cdot \Delta^{-t/4} + o(n) = O(n)$$

rounds, where the second inequality holds by an intermediate calculation in (8) and the third inequality holds by Claim 4.2. We conclude that w.h.p., Builder can construct a copy of T in $O(\log \Delta)n$ rounds. This completes the proof of the theorem. □

Proof of Proposition 1.7. Let Δ and $n \geq n_0(\Delta)$ be as in the statement of the proposition. Since Builder clearly needs at least $n-1$ rounds in order to build a tree on n vertices, we can assume that Δ is a sufficiently large constant. We prove the proposition for the n-vertex forest T consisting of $\lfloor \frac{n}{\Delta+1} \rfloor$ pairwise-disjoint $(\Delta+1)$-vertex stars, and some additional isolated vertices (if needed). Let us denote the center of the ith star by u_i, and its leaves by $x_{i,1}, \ldots, x_{i,\Delta}$ $(1 \leq i \leq \lfloor \frac{n}{\Delta+1} \rfloor)$.

Let $w_1, w_2, \ldots$ denote the sequence of random vertices offered to Builder, and let $m = 0.1n \log \Delta$. Suppose that Builder did manage to build a copy of T within m rounds, and let $\varphi : V \to [n]$ be a bijection such that $\{\varphi(u), \varphi(v)\}$ is an edge in Builder's graph for every $\{u, v\} \in E(T)$. It is then evident that, for every $1 \leq i \leq \lfloor \frac{n}{\Delta+1} \rfloor$, either $\varphi(u_i)$ appears at least $\sqrt{\Delta}$ times in $(w_1, w_2, \ldots, w_m)$, or at least $\Delta - \sqrt{\Delta}$ of the elements of $\{\varphi(x_{i,j}) : 1 \leq j \leq \Delta\}$ appear at least once in $(w_1, w_2, \ldots, w_m)$. A straightforward calculation then shows that, either at least $\frac{1}{\sqrt{\Delta}} \cdot \lfloor \frac{n}{\Delta+1} \rfloor \geq \frac{n}{2\Delta^{3/2}}$ of the vertices in $\{\varphi(u_i) : 1 \leq i \leq \lfloor \frac{n}{\Delta+1} \rfloor\}$ were offered at least $\sqrt{\Delta}$ times each; or all but at most

$$\Delta + (\Delta+1) \cdot \frac{1}{\sqrt{\Delta}} \cdot \left\lfloor \frac{n}{\Delta+1} \right\rfloor \cdot \Delta + \left\lfloor \frac{n}{\Delta+1} \right\rfloor \cdot (\sqrt{\Delta}+1) \leq \frac{2n}{\Delta}$$

of the n vertices were offered at least once. So in order to prove that w.h.p. Builder needs more than m rounds to build T, it suffices to show that w.h.p. there are more than $2n/\sqrt{\Delta}$ vertices $1 \leq i \leq n$ that do not appear in $(w_1, w_2, \ldots, w_m)$, and less than $\frac{n}{2\Delta^{3/2}}$ vertices $1 \leq i \leq n$ which appear in $(w_1, w_2, \ldots, w_m)$ at least $\sqrt{\Delta}$ times.

Let X be the random variable which counts the number of vertices $1 \leq i \leq n$ that do not appear in $(w_1, w_2, \ldots, w_m)$. Our goal is to show that w.h.p. $X > 2n/\sqrt{\Delta}$. We have

$$\mathbb{E}(X) = m(1-1/n)^m \geq 0.01n \log \Delta \cdot e^{-0.1 \log \Delta} > n/\Delta^{1/4},$$

where the last inequality holds for sufficiently large Δ. We will use Lemma 2.2 to prove that w.h.p. X is not much smaller than its expected value. Observe that changing any single coordinate in the sequence of random vertices $(w_1, w_2, \ldots, w_m)$ can change the value of X by at most 1. Therefore, applying Lemma 2.2 with parameters $c = 1$ and $\lambda = \mathbb{E}(X) - n/\sqrt{\Delta}$ yields

$$\mathbb{P}\left[X \leq 2n/\sqrt{\Delta}\right] = \mathbb{P}\left[X \leq \mathbb{E}(X) - (\mathbb{E}(X) - 2n/\sqrt{\Delta})\right] \leq e^{-\frac{(\mathbb{E}(X)-2n/\sqrt{\Delta})^2}{2m}} \leq \exp\left(-\frac{\left(\frac{n}{2\Delta^{1/4}}\right)^2}{n \log \Delta}\right)$$
$$= e^{-\frac{n}{4\sqrt{\Delta}\log\Delta}} = o(1),$$

where the last equality holds since n is sufficiently large with respect to Δ.

Let Z be the random variable which counts the number of vertices $1 \leq i \leq n$ that appear at least $\sqrt{\Delta}$ times in $(w_1, w_2, \ldots, w_m)$. Our goal is to prove that w.h.p. $Z < \frac{n}{2\Delta^{3/2}}$. For every $1 \leq i \leq n$, let Z_i be the random variable counting the number of times i appears in $(w_1, w_2, \ldots, w_m)$. Then $Z_i \sim \text{Bin}(m, 1/n)$, implying that $\mathbb{E}[Z_i] = m/n = 0.1 \log \Delta$. Applying Lemma 2.1 with parameter $\lambda = \sqrt{\Delta} - 0.1 \log \Delta$, we obtain

$$\mathbb{P}[Z_i \geq \sqrt{\Delta}] = \mathbb{P}[Z_i \geq \mathbb{E}[Z_i] + (\sqrt{\Delta} - 0.1 \log \Delta)] \leq \exp\left(-\frac{(\sqrt{\Delta} - 0.1\log\Delta)^2}{2(0.1\log\Delta + \sqrt{\Delta}/3)}\right) \leq \frac{1}{4\Delta^{3/2}}\,, \quad (9)$$

where the last inequality holds for sufficiently large Δ. For every $1 \leq i \leq n$, let I_i be the indicator random variable for the event $Z_i \geq \sqrt{\Delta}$; note that $Z = \sum_{i=1}^{n} I_i$. It follows by (9) and by the linearity of expectation that $\mathbb{E}(Z) \leq \frac{n}{4\Delta^{3/2}}$. Since changing any single coordinate in the sequence of random vertices $(w_1, w_2, \ldots, w_m)$ can change the value of Z by at most 1, applying Lemma 2.2 with parameters $c = 1$ and $\lambda = \frac{n}{4\Delta^{3/2}}$ yields

$$\mathbb{P}\left[Z \geq \frac{n}{2\Delta^{3/2}}\right] \leq \mathbb{P}\left[Z \geq \mathbb{E}(Z) + \frac{n}{4\Delta^{3/2}}\right] \leq e^{-\frac{n^2}{32\Delta^3 m}} \leq e^{-\frac{n^2}{32\Delta^3 n\log\Delta}} = o(1),$$

where the equality holds since n is sufficiently large with respect to Δ. □

5 Non-Adaptive Strategies

In this section we prove Theorem 1.5. For the sake of readability, we prove each of its two parts separately.

Proof of Theorem 1.5, Part 1. Let $\mathcal{L} = \{L^w : w \in [n]\}$ be a family of lists as in the definition of a non-adaptive strategy. Recall that for each $w \in [n]$, the list L^w is a permutation of $[n] \setminus \{w\}$. Our goal is to show that the strategy corresponding to $\mathcal{L}$ requires w.h.p. at least $\Omega(n\sqrt{\log n})$ rounds to make all n vertices non-isolated.

Set $t = n\sqrt{\log n}/4$, and let $w_1, \ldots, w_t$ be the first t random vertices Builder is offered. For every $v \in [n]$, let t_v denote the number of appearances of v in the sequence $(w_1, \ldots, w_t)$. Let $U = \{v \in [n] : t_v > \sqrt{\log n}/2\}$ and let $W = \bigcup_{v \in U}\{L^v(i) : 1 \leq i \leq t_v\}$. Our main observation (which follows immediately from the definitions of U and W) is that a vertex $u \in [n]$ will be left isolated after t rounds, if all of the following conditions hold:

(1) u does not appear in $(w_1, \ldots, w_t)$;

(2) none of the vertices v, for which u is included among the first $\sqrt{\log n}/2$ elements of L^v, appear in $(w_1, \dots, w_t)$;

(3) $u \notin W$.

So in order to complete the proof of the theorem, it remains to prove that w.h.p. there exists a vertex $u \in [n]$ which satisfies Conditions (1), (2), and (3) as above. To this end, we will use a two-round exposure argument. Let Z denote the set of vertices which do not appear in $(w_1, \dots, w_t)$; clearly $Z \cap U = \emptyset$. In the following claim we collect some simple facts regarding the sets U, Z and the integers $(t_v : v \in [n])$.

Claim 5.1. *The following hold w.h.p.*

(a) $|U| \le e^{-\Omega(\sqrt{\log n})} n$.

(b) $t_v < \log n$ *for every* $v \in [n]$.

(c) $|Z| \ge (1 - o(1)) e^{-\sqrt{\log n}/4} n$.

Proof. We start with Item (a). Recall that for a given vertex $v \in [n]$, we have $t_v \sim \mathrm{Bin}(t, \frac{1}{n})$; hence, $\mathbb{E}[t_v] = \sqrt{\log n}/4$. Now, by Lemma 2.1 with $\lambda = \sqrt{\log n}/4$, we have

$$\mathbb{P}[v \in U] = \mathbb{P}[t_v > \sqrt{\log n}/2] \le \mathbb{P}[t_v > \mathbb{E}[t_v] + \sqrt{\log n}/4] \le e^{-\frac{\log n}{O(\sqrt{\log n})}} = e^{-\Omega(\sqrt{\log n})} .$$

It thus follows by Markov's inequality that w.h.p. $|U| \le e^{-\Omega(\sqrt{\log n})} n$.

We now prove Item (b). Observe that for every $v \in [n]$ we have

$$\mathbb{P}[t_v \ge \log n] \le \binom{t}{\log n} \left(\frac{1}{n}\right)^{\log n} \le \left(\frac{et}{n \log n}\right)^{\log n} \le \left(\frac{1}{\sqrt{\log n}}\right)^{\log n} = o(1/n).$$

A union bound over $[n]$ then shows that w.h.p. $t_v < \log n$ for every $v \in [n]$.

Finally, we prove Item (c). For each $v \in [n]$, the probablity that $v \in Z$ is $(1 - 1/n)^t = (1 \pm o(1)) e^{-\sqrt{\log n}/4}$. Therefore, $\mathbb{E}[|Z|] = (1 \pm o(1)) e^{-\sqrt{\log n}/4} n$. To show that $|Z|$ is concentrated around its expected value, observe that changing any single coordinate in the sequence $(w_1, \dots, w_t)$ of random vertices, can change the value of $|Z|$ by at most 1. Hence, by Lemma 2.2 with $c = 1$ and (say) $\lambda = n^{2/3}$, we have

$$\mathbb{P}\left[|Z| \le \mathbb{E}[|Z|] - n^{2/3}\right] \le e^{-\frac{n^{4/3}}{2t}} = e^{-\frac{n^{4/3}}{O(n\sqrt{\log n})}} = o(1).$$

We conclude that w.h.p. $|Z| \ge (1 - o(1)) e^{-\sqrt{\log n}/4} n$. $\square$

From now on we condition on the events stated in Items (a)-(c) of Claim 5.1 (which hold w.h.p. by that claim). Items (a) and (b) imply that $|W| \le |U| \log n \le \log n \cdot e^{-\Omega(\sqrt{\log n})} n = o(n)$. Observe that conditioning on their sizes, U, Z are uniformly distributed among all pairs of disjoint subsets of $[n]$ of the corresponding sizes. From this point on we condition on U, which in turn determines W.

For each $u \in [n]$, let A_u be the set of all vertices $v \in [n] \setminus \{u\}$ such that u is included among the first $\sqrt{\log n}/2$ elements of L^v. Let V_0 be the set of all $u \in [n]$ satisfying $|A_u| \le \sqrt{\log n}$. Since the union (as a multiset) of the first $\sqrt{\log n}/2$ elements in all lists has size $n\sqrt{\log n}/2$ altogether, we deduce that $|V_0| \geq n/2$; hence $|V_0 \setminus W| \geq (1/2 - o(1))n$.

Now expose Z, conditioning on its size. Observe that if $u \in [n] \setminus W$ is such that $A_u \cup \{u\} \subseteq Z$, then u satisfies Conditions (1), (2) and (3). So from now on our goal is to show that w.h.p. there exists a vertex $u \in [n] \setminus W$ for which $A_u \cup \{u\} \subseteq Z$. Since each $u \in [n]$ belongs to at most $\sqrt{\log n}/2+1$ of the sets $\{A_v \cup \{v\} : v \in [n]\}$, and since $|A_u \cup \{u\}| \leq \sqrt{\log n} + 1$ for each $u \in V_0$, one can find a collection $B_1, \dots, B_{s'}$ of

$$s' \geq \left\lfloor \frac{|V_0 \setminus W|}{(\sqrt{\log n}/2+1)(\sqrt{\log n}+1)+1} \right\rfloor = \Omega\left(\frac{n}{\log n}\right)$$

pairwise-disjoint sets among the sets $\{A_u \cup \{u\} : u \in V_0 \setminus W\}$. Since $B_1, \dots, B_{s'}$ are pairwise-disjoint, at least $s := s' - |U| \geq \Omega(n/\log n) - e^{-\Omega(\sqrt{\log n})}n = \Omega(n/\log n)$ of these sets are contained in $[n] \setminus U$. So suppose, without loss of generality, that $B_1, \dots, B_s \subseteq [n] \setminus U$, and let us show that w.h.p., there is $1 \leq i \leq s$ such that $B_i \subseteq Z$. To this end, we will couple Z with a binomial random set of slightly smaller size. Recall that we are conditioning on the size of Z and on the event $|Z| \geq (1 - o(1))e^{-\sqrt{\log n}/4}n$, which occurs w.h.p. by Claim 5.1. We denote $z = |Z|$, recalling that (under this conditioning), Z is distributed uniformly among all subsets of $[n] \setminus U$ of size z. We generate Z by performing the following experiment: set $p = \frac{z}{2n}$, and let R be a random subset of $[n] \setminus U$, obtained by independently including each element of $[n] \setminus U$ with probability p. If $|R| \leq z$, then we uniformly choose a set $Z' \subseteq [n] \setminus U$ of size z which contains R. It is easy to see that, conditioned on $|R| \leq z$, the set Z' is distributed uniformly among all subsets of $[n] \setminus U$ of size z. Hence (conditioned on $|R| \leq z$), Z' has the same distribution as Z (conditioned on $|Z| = z$). Note that $|R|$ is stochastically dominated by $\mathrm{Bin}(n, \frac{z}{2n})$, so by Lemma 2.1 with $\lambda = \frac{z}{2}$ we have

$$\mathbb{P}[|R| > z] \leq \mathbb{P}\left[|R| \geq \mathbb{E}[|R|] + \frac{z}{2}\right] \leq e^{-\frac{(z/2)^2}{2(\mathbb{E}[|R|]+z/6)}} \leq e^{-\Omega(z)} = o(1).$$

Since the sets $B_1, \dots, B_s$ are pairwise-disjoint and of size at most $\sqrt{\log n} + 1$ each, the probability that R contains none of these sets is at most

$$\begin{aligned}\left(1 - p^{\sqrt{\log n}+1}\right)^s &\leq \exp\left(-p^{\sqrt{\log n}+1} \cdot s\right) = \exp\left(-\left(\frac{z}{2n}\right)^{\sqrt{\log n}+1} \cdot s\right) \\ &\leq \exp\left(-e^{-\log n/2} \cdot s\right) = e^{-s/\sqrt{n}} = e^{-\Omega(\sqrt{n}/\log n)} = o(1),\end{aligned}$$

where in the second inequality we used the assumption that $z \geq (1 - o(1))e^{-\sqrt{\log n}/4}n$. We conclude that w.h.p. there will be some $1 \leq i \leq s$ such that $B_i \subseteq Z$, as required. □

Proof of Theorem 1.5, Part 2. Partition $[n]$ into $k = n/\sqrt{\log n}$ parts $V_1, \dots, V_k$ whose sizes are all divisible by r and are as close to each other as possible. Let $s_{\min} = \min\{|V_i| : 1 \leq i \leq k\}$ and let $s_{\max} = \max\{|V_i| : 1 \leq i \leq k\}$; observe that $s_{\min}, s_{\max} = (1 \pm o(1))\sqrt{\log n}$. Then, for every $1 \leq i \leq k$ and every $v \in V_i$, Builder sets the adjacency list L^v so that first appear all the vertices of $V_i \setminus \{v\}$ (in an arbitrary order), and then all other vertices (in an arbitrary order).

Now, set $t = Cn\sqrt{\log n}$ (where $C = C(r)$ will be chosen later), and let $(w_1, \dots, w_t)$ be the random vertices Builder is offered. Let W be the set of vertices appearing at most $s_{\max} - 2$ times in $(w_1, \dots, w_t)$. Observe that if $|W \cap V_i| < s_{\min}/r$, then the resulting induced subgraph of Builder $G[V_i]$

has minimum degree at least $(1 - 1/r)|V_i|$ and thus admits a K_r-factor by the Hajnal-Szemerédi Theorem [14]. If this happens for every $1 \le i \le k$, then the union over $1 \le i \le k$ of these K_r-factors obviously forms a K_r-factor of G. It remains to prove that w.h.p. $|W \cap V_i| < s_{\min}/r$ holds for every $1 \le i \le k$. Fix some $1 \le i \le k$. Then

$$\begin{aligned}
\mathbb{P}[|W \cap V_i| \ge s_{\min}/r] &\le \binom{s_{\max}}{s_{\min}/r} \cdot \mathbb{P}\left[Bin(t, s_{\min}/(rn)) \le (s_{\max} - 2) \cdot s_{\min}/r\right] \\
&\le (3r)^{s_{\min}/r} \cdot \exp\left\{-\frac{rn}{2ts_{\min}} \cdot \frac{C^2 \cdot s_{\min}^4}{4r^2}\right\} \\
&\le (3r)^{s_{\min}/r} \cdot \exp\left\{-\frac{C \cdot s_{\min}^3}{8r\sqrt{\log n}}\right\} \\
&\le (3r)^{\sqrt{\log n}} \cdot \exp\left\{-\frac{C \cdot (1 - o(1)) \log n}{8r}\right\} \\
&\le \exp\left\{-\frac{C \log n}{9r}\right\} = o(1/k),
\end{aligned}$$

where the second inequality holds by Lemma 2.1 with $\lambda = \frac{C}{2} \cdot \frac{s_{\min}^2}{r}$, and the equality holds if, say, $C = 9r$. We also assumed throughout that n is large enough with respect to r. A union bound over all $1 \le i \le k$ then shows that w.h.p. $|W \cap V_i| < s_{\min}/r$ holds for every $1 \le i \le k$, as required. $\square$

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