Optimal Universal Graphs with Deterministic Embedding

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Abstract

Let $\mathcal{H}$ be a finite family of graphs. A graph G is $\mathcal{H}$-universal if it contains a copy of each $H \in \mathcal{H}$ as a subgraph. Let $\mathcal{H}(k,n)$ denote the family of graphs on n vertices with maximum degree at most k. For all admissible k and n, we construct an $\mathcal{H}(k,n)$-universal graph G with at most $c_k n^{2-\frac{2}{k}}$ edges, where c_k is a constant depending only on k. This is optimal, up to the constant factor c_k, as it is known that $c'_k n^{2-2/k}$ is a lower bound for the number of edges in any such graph. The construction of G is explicit, and there is an efficient deterministic algorithm for finding a copy of any given $H \in \mathcal{H}(k,n)$ in G.

1 Introduction

For a family $\mathcal{H}$ of graphs, a graph G is $\mathcal{H}$*-universal* if, for each $H \in \mathcal{H}$, the graph G contains a subgraph isomorphic to H. The construction of sparse universal graphs for various families arises in the study of VLSI circuit design. See, for example, [9] and [16], for applications motivating the study of universal graphs with a small number of edges for various families of graphs. There is an extensive literature on universal graphs. In particular, universal graphs for forests have been studied in [8], [12], [13], [14], [15], [17], and universal graphs for planar graphs and other related families have been investigated in [1], [7], [8], [10], [11], [20].

Universal graphs for general bounded-degree graphs have also been considered in various papers. For positive integers k and n, let $\mathcal{H}(k,n)$ denote the family of all graphs on n vertices with maximum degree at most k. The authors of [3] constructed $\mathcal{H}(k,n)$-universal graphs with at most $O(n^{2-1/k}\log^{1/k} n)$ edges, as well as $\mathcal{H}(k,n)$-universal graphs on n vertices with $O(n^{2-c/(k\log k)})$ edges. In addition, it is shown in [3], by a simple counting argument, that any $\mathcal{H}(k,n)$-universal graph must have at least $\Omega(n^{2-2/k})$ edges. A better construction is given in [4], where the authors present such graphs with at most $O(n^{2-2/k}\log^{1+8/k} n)$ edges.

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In [2] we gave an explicit construction of $\mathcal{H}(k,n)$-universal graphs with at most $O(n^{2-2/k}\log^{4/k} n)$ edges, thus nearly closing the gap between the upper and lower bounds for the minimum possible number of edges. In the present paper we close this gap, proving the following.

Theorem 1.1 *For every $k \geq 3$ there exist positive constants $c_1 = c_1(k)$ and $c_2 = c_2(k)$ so that for every n there is an (explicitly constructable) $\mathcal{H}(k,n)$-universal graph G with at most $c_1 n$ vertices and at most $c_2 n^{2-2/k}$ edges.*

Besides getting rid of the logarithmic factor, the result improves the one in [2] (and in the previous papers) in the sense that it provides an efficient, deterministic algorithm for finding a copy of any given $H \in \mathcal{H}(k,n)$ in the constructed graph G. Curiously, the previous proofs gave only randomized algorithms for finding such an embedding.

Our construction is similar to the one in [2], but the proof requires some new techniques based on several intriguing properties of high girth expanders.

The rest of the paper is organized as follows. In Section 2 we describe the construction, and in Section 3 we describe the basic ideas in the proof that it is universal, leaving a crucial argument to the next section. The detailed proof of this argument, which is based on properties of high girth expanders, is given in Section 4. The final Section 5 contains some concluding remarks and open problems. Throughout the paper we make no attempts to optimize the constants and omit all floor and ceiling signs whenever these are not crucial. We further assume that n is sufficiently large whenever this is needed. All graphs considered here have no self loops and no parallel edges, and all logarithms are in base 2, unless otherwise specified.

2 The construction

Note, first, that (as mentioned in [2]) the case $k = 2$ is trivial; the square of a cycle of length n is $\mathcal{H}(2,n)$-universal, and has a linear number of edges. We thus assume $k > 2$.

Let $k > 2$ be an integer and put $m = 20n^{1/k}$. Let F be a constant degree high girth expander on m vertices. (The construction works equally well if the number of vertices of F is any integer between m and, say, $10m$, but to simplify the presentation we assume the number is exactly m.) Specifically, we assume that F is an (m,d,λ)-graph, where d is an absolute constant to be chosen later. This means that F is d-regular and all its eigenvalues but the largest have absolute value at most λ. It is convenient to assume that F is Ramanujan, that is, $\lambda \leq 2\sqrt{d-1}$. We also assume that the girth of F is at least $\frac{2}{3}\log m/\log(d-1)$. Explicit constructions of such high girth expanders, for every $d = p+1$, where p is a prime congruent to 1 modulo 4, have been given in [18], [19]. Let $G = G_{k,n}$ be the graph whose vertex set is $V(G) = (V(F))^k$, where two vertices $(x_1, x_2, \ldots, x_k)$ and $(y_1, y_2, \ldots, y_k)$ are adjacent iff there exist at least two indices i such that x_i and y_i are within distance 4 in F. Note that G has $m^k = O(n)$ vertices and $O(nm^{k-2}) = O(n^{2-2/k})$ edges.

Our main result is that the graph $G_{k,n}$ is $\mathcal{H}(k,n)$-universal. This is proved in the following two sections.

3 Embedding in the universal graph

A *homomorphism* from a graph Z to a graph T is a mapping of the vertices of Z to those of T such that adjacent vertices in Z are mapped to adjacent ones in T. Note that there is an injective homomorphism from Z to T iff Z is a subgraph of T.

The *k-th power* T^k of a graph $T = (V(T), E(T))$ is the graph whose vertices are the vertices of T, and two are adjacent iff the distance between them in T is at most k. Let $P = P_n$ denote the path on n vertices, that is, the graph whose set of vertices is $[n] = \{1, 2, \ldots, n\}$, where i, j are connected iff $|i - j| = 1$.

An *augmentation* of a graph $T = (V, E)$ is any graph obtained from T by choosing an arbitrary (possibly empty) subset $U \subset V$, adding a new set U' of $|U|$ vertices, and adding a matching between U and U'. Thus, an augmentation of T is obtained from it by connecting new vertices of degree 1 to some of its vertices.

Call a graph *thin* if its maximum degree is at most 3 and each connected component of it is either an augmentation of a path or of a cycle, or a graph with at most two vertices of degree 3. It is easy to check that every thin graph H on n vertices is a (spanning) subgraph of the forth power of the path P_n, that is, there is a bijective homomorphism from each such H to P_n^4.

We need the following result, proved in [2].

Theorem 3.1 ([2]) *Let $k \geq 2$ be an integer, and let H be an arbitrary graph of maximum degree at most k. Then there are k spanning subgraphs $H_1, H_2, \ldots, H_k$ of H such that each H_i is thin, and every edge of H lies in precisely two graphs H_i.* ∎

The assertion of the theorem for even values of k is an immediate consequence of Petersen's Theorem. The proof for odd values of k, given in [2], requires some work based on techniques from Matching Theory. The proof is algorithmic and supplies an efficient algorithm to find the graphs H_i for a given input graph H.

To prove Theorem 1.1 we have to show that every graph $H \in \mathcal{H}(k, n)$ is a subgraph of $G = G_{k,n}$. Given such an $H = (V, E)$, let $H_1, H_2, \ldots, H_k$ be as in Theorem 3.1, and note that as all of them are spanning subgraphs of H, the set of vertices of each of them is V. As each H_i is thin, there are injective homomorphisms $g_i : V \mapsto [n]$ from H_i to P_n^4. The main part of the proof is to show that there are homomorphisms $f_i : [n] \mapsto V(F)$ from the path P_n to the expander F, such that the mapping $f : V(H) \mapsto V(G)$ given by $f(v) = (f_1(g_1(v)), f_2(g_2(v), \ldots, f_k(g_k(v))$ forms an injective homomorphism from H to G, thus implying that H is a subgraph of G. To do so, we define each f_i as a homomorphism from the path P_n to F, given by a non-backtracking walk. Since the girth of F exceeds 4, this ensures that each composition $f_i(g_i(\cdot))$ is a homomorphism from H_i to the forth power F^4 of F. By the definition of G, this implies that f is indeed a homomorphism from H to G. Indeed, for any pair u, v of adjacent vertices of H there are two indices i such that u, v are adjacent in H_i, as each edge of H is covered by two of the graphs H_i. For each such index i, $g_i(u)$ and $g_i(v)$ are distinct and within distance 4 in P, implying that $f_i(g_i(u))$ and $f_i(g_i(v))$ are distinct and within distance 4 in F, that is, they are adjacent in F^4. Hence $f(u)$ and $f(v)$ are adjacent in G, and f is a homomorphism, as needed.

The only remaining piece, which is the crucial part of the proof, is to show that the homomorphisms f_i can be defined so that f is injective. To do so, we define the non-backtracking mappings f_i one by one in order, ensuring, while defining f_i, that the following condition holds.

For every i (not necessarily distinct) vertices $v_1, v_2, \ldots, v_i \in V(F)$,

$$|\{v \in V(H) : \; f_1(g_1(v)) = v_1, f_2(g_2(v)) = v_2, \ldots, f_i(g_i(v)) = v_i\}| \leq n^{(k-i)/k} \quad (1)$$

This holds trivially for $i = 0$, and in the next section we show how to define the mappings f_i so that it holds for all i. The case $i = k$ asserts that f is injective, as needed. Moreover, the proof provides an efficient way to find the mappings with the required properties. To complete the proof it thus remains to prove that the mappings f_i can defined so that they satisfy (1). This is done in the next section.

4 The crucial argument

The proof is based on several properties of high girth Ramanujan graphs. These are stated and proved in the following subsection.

4.1 Properties of high girth expanders

Throughout this subsection, $F = (V, E)$ is an (m, d, λ)-graph, that is, a d-regular graph on m vertices in which all nontrivial eigenvalues have absolute value at most λ. We start with the following well known fact.

Lemma 4.1 ([5], Lemma 9.2.4) *For every set B of $|B| = bm$ vertices of F,*

$$\sum_{v \in V} (|N_B(v)| - bd)^2 \leq \lambda^2 b(1-b)m,$$

where $|N_B(v)|$ is the number of neighbors of v in B. ■

This easily implies the following (see also [6] for a similar claim).

Lemma 4.2 *Let B, C be two sets of vertices of F, and suppose that each vertex of C has at least $3d/4$ neighbors in B. Then*

$$|B| \geq min\{m/2, |C|\frac{d^2}{16\lambda^2}\}.$$

In particular, if $\lambda \leq 2\sqrt{d-1}$ then

$$|B| \geq min\{m/2, \frac{d}{64}|C|\}.$$

Proof: Put $b = |B|/m$. If $b \geq 1/2$ there is nothing to prove, hence assume $b < 1/2$. By Lemma 4.1,

$$|C|\frac{d^2}{16} \leq \sum_{v \in C} (|N_B(v)| - bd)^2 \leq \sum_{v \in V} (|N_B(v)| - bd)^2 \leq \lambda^2 b(1-b)m \leq \lambda^2 |B|,$$

implying the desired result. ■

Another simple corollary of Lemma 4.1 is the following.

Lemma 4.3 *If B is a set of at least $9m/10$ vertices of F, then the number of vertices that have less than $8d/10$ neighbors in B is at most $9\lambda^2 m/d^2$. In particular, if $\lambda \leq 2\sqrt{d-1}$ and $d > 720$, then this number is smaller than $m/20$.*

Proof: Put $C = \{v \in V : |N_B(v)| < 8d/10\}$. By Lemma 4.1 we get, as before,

$$|C|\frac{d^2}{100} \leq \sum_{v \in V}(|N_B(v)| - bd)^2 \leq \lambda^2 b(1-b)m \leq \lambda^2 \frac{9m}{100},$$

as needed. ■

Let q be a positive integer. A (simple) *walk* of length q in F is a sequence $W = (w_0, w_1, \ldots, w_q)$ of **distinct** vertices of F, where $w_i w_{i+1}$ is an edge for all $i < q$. Let $S_1, S_2, \ldots, S_q$ be subsets of the set of vertices V of F. We say that a walk W as above *slips by* the sets S_i if for all $1 \leq i \leq q$, $w_i \notin S_i$. Call a vertex $w \in V$ *nice* with respect to the sets S_i if there are at least $m/2$ vertices z so that there is a walk $w = w_0, w_1, \ldots, w_q = z$ of length q that starts at w, slips by the sets S_i and ends at z. Call a vertex $w \in V$ *very nice* with respect to the sets S_i if for every set Q of vertices that contains at most $d/20 - \log d$ neighbors of each vertex, the vertex w is nice with respect to the sets $S_1 \cup Q, S_2 \cup Q, \ldots, S_q \cup Q$.

Lemma 4.4 *Let F be as before, assume $\lambda \leq 2\sqrt{d-1}$, and suppose that the girth g of F exceeds $\frac{1}{2}\log m/\log d$ and that $d > 720$. Put $q = \lceil \log m/\log 10 \rceil$. Then, for every collection of sets $S_1, S_2, \ldots, S_q$ of vertices of F satisfying $|S_i| \leq m/20$ for all i, the number of vertices w that are very nice with respect to the sets S_i is at least $9m/10$.*

Proof: Define sets of vertices $T_q, T_{q-1}, \ldots, T_1, T_0$ of V as follows. Put $T_q = V - S_q$. Let Z_{q-1} denote the set of all vertices that have less than $0.8d$ neighbors in T_q and define $T_{q-1} = V - (Z_{q-1} \cup S_{q-1})$. Assuming we have already defined $T_q, T_{q-1}, \ldots, T_{q-i+1}$, let Z_{q-i} denote the set of all vertices that have less than $0.8d$ neighbors in T_{q-i+1} and define $T_{q-i} = V - (Z_{q-i} \cup S_{q-i})$. By Lemma 4.3 and induction, $|Z_{q-i}| \leq m/20$ for all i, and hence $|T_{q-i}| \geq m - m/20 - m/20 = 0.9m$ for all i. To complete the proof we show that every $w \in T_0$ is very nice with respect to the sets S_i. To do so, fix $w \in T_0$ and let Q be a set of vertices of F that contains at most $d/20 - \log d$ neighbors of each vertex of F. Define sets of vertices $Y_0, Y_1, Y_2, \ldots, Y_q$ as follows. Put $Y_0 = \{w\}$ and let Y_1 be the set of all neighbors of w in $T_1 - Q$. Assuming $Y_1, Y_2, \ldots, Y_j$ have already been defined, we define Y_{j+1} as follows. For each $y \in Y_j$ fix a walk W of length j from w to y that slips by the sets $S_1 \cup Q, S_2 \cup Q, \ldots, S_j \cup Q$. Now consider all neighbors of y that lie in $T_{j+1} - (Q \cup W)$. Y_{j+1} consists of the union of all these sets of neighbors. Note that by construction, for every $y_{j+1} \in Y_{j+1}$ there is a walk of length $j+1$ that slips by the sets $S_1 \cup Q, S_2 \cup Q, \ldots, S_{j+1} \cup Q$, starts at w and ends at y_{j+1}. (We have omitted the vertices of W from the set of neighbors of y to ensure this walk consists of distinct vertices.) By construction, each vertex of Y_i has at least $0.8d$ neighbors in T_{i+1}. In addition, the assumption on the girth implies that the walk W cannot contain more than $\frac{q}{g-2} + 1 < \log d$ neighbors of any single vertex, and hence, by the assumption on Q, each member of Y_i has at least $0.8d - (d/20 - \log d) - \log d = 3d/4$ neighbors in Y_{i+1}. Therefore, by Lemma 4.2, $|Y_{i+1}| \geq min\{m/2, |Y_i|\frac{d}{64}\}$. By the definition of q (and as $d > 720$)

this implies that $|Y_q| \geq m/2$ (with room to spare), and as any vertex of Y_q can be reached from w by a walk of length q that slips by the sets $S_i \cup Q$, it follows that w is indeed very nice, as needed. ■

We need another lemma, which is a simple consequence of the definitions of the notions nice and very nice.

Lemma 4.5 *Let F be as in Lemma 4.4, let $q = \lceil \log m / \log 10 \rceil$ and let $S_1, S_2, \ldots, S_q$ be sets of vertices of F. If a vertex w is very nice with respect to the sets S_i, and Q is the set of vertices of a walk of length $q+1$ in F, then w is nice with respect to the sets $S_i \cup Q$.*

Proof: By the assumption on the girth g, Q contains at most $(q+1)/(g-2)+1$ neighbors of any given vertex, and the assumptions on the girth g and the degree d ensure that this is smaller than $d/20 - \log d$. The result thus follows from the definitions. ■

4.2 Completing the proof

Proof of Theorem 1.1: By the discussion in Section 3, it suffices to prove the existence of mappings f_i satisfying (1). This is done by induction on i. The assertion trivially holds for $i = 0$. Assuming it holds for $i-1$, we prove it for i, where $i \geq 1$. To this end, we define $f_i : [n] \mapsto V(F)$ by defining the values of $f_i(1), f_i(2), \ldots$ in order, making sure that throughout the process, the condition (1) will be kept. Starting with $t = 0$, assume we have already defined $f_i(1), f_i(2), \ldots, f_i(t)$. It is convenient to think about t as denoting time, hence we say that at time t we have already defined the values of f_i up to $f_i(t)$. For every $j > t$, define a set $S_{j,t}$, consisting, intuitively, of all vertices $v_i \in V(F)$ so that we cannot define $f_i(j) = v_i$ based on the information we have at time t. Formally, $S_{j,t}$ is defined as follows. Put $v = g_i^{-1}(j)$, that is, v is the unique vertex in $V(H) = V(H_i)$ mapped by g_i to j. A vertex $v_i \in V(F)$ is *dangerous for place j at time t* if there are at least $n^{(k-i)/k}$ vertices $u \in V(H)$ such that $g_i(u) = \ell \leq t$, and

$$f_1(g_1(u)) = f_1(g_1(v)), f_2(g_2(u)) = f_2(g_2(v)), \ldots, f_{i-1}(g_{i-1}(u)) = f_{i-1}(g_{i-1}(v)) \text{ and } f_i(g_i(u)) = v_i.$$

Thus, if v_i is dangerous for place j at time t, we cannot define $f_i(j) = v_i$, as this will violate the condition (1). The set $S_{j,t}$ consists of all vertices v_i which are dangerous for place j in time t. Note that, crucially, $|S_{j,t}| \leq m/20$ for all j and t. Indeed, by the induction hypothesis, the number of vertices u satisfying

$$f_1(g_1(u)) = f_1(g_1(v)), f_2(g_2(u)) = f_2(g_2(v)), \ldots, f_{i-1}(g_{i-1}(u)) = f_{i-1}(g_{i-1}(v))$$

is at most $n^{(k-i+1)/k}$, and therefore there cannot be more than $n^{1/k} = m/20$ vertices v_i so that at least $n^{(k-i)/k}$ of these vertices u satisfy $f_i(g_i(u)) = v_i$ as well.

It is obvious that our objective is to define the mapping f_i so that $f_i(j) \notin S_{j,t}$ for any t smaller than j, that is, the walk given by the values of $f_i(j), f_i(j+1), \ldots$ has to slip by the sets $S_{j,t}$. The difficulty in deducing the existence of such a walk from Lemma 4.4 is that the sets $S_{j,t}$ keep changing as the time t increases. The notions "nice" and "very nice" have been introduced in order to overcome this difficulty, enabling us to define the values of $f_i(t)$ in steps, where in each step we pick the values for $q = \lceil \log m / \log 10 \rceil$ additional consecutive values of t. Here are the details.

Starting with $t=0$, consider the sets $S_{1,0}, S_{2,0}, \ldots, S_{q,0}$. (Note that in fact these sets are all empty, as the time is $t=0$, but we do not use this fact here as it will not hold in the general case). By Lemma 4.4 there is a set M_0 of at least $0.9m$ vertices that are very nice with respect to these sets. Similarly, by Lemma 4.4 there is a set M_q of at least $0.9m$ vertices that are very nice with respect to the sets $S_{q+1,0}, S_{q+2,0}, \ldots, S_{2q,0}$. By definition, for each $w_0 \in M_0$ there are at least $m/2$ vertices w_q so that for each of them there is a walk of length q that slips by the sets $S_{1,0}, S_{2,0}, \ldots, S_{q,0}$, starts at w_0 and ends at w_q. Therefore, there is at least one (in fact, at least $0.4m$) such possible w_q that lies in M_q as well. We can now define $f_i(1), \ldots, f_i(q)$ according to this walk, ending at $f_i(q) = w_q$. The crucial point is that as the vertex w_q has been very nice with respect to the sets $S_{q+1,0}, S_{q+2,0}, \ldots, S_{2q,0}$, it is nice with respect to these sets even after adding to them all the vertices of the walk $f_i(1), \ldots, f_i(q)$. This enables us to continue by induction, in a similar way.

Suppose we have already defined the values of $f_i(1), f_i(2), \ldots, f_i(t)$ for some multiple t of q, and assume, by induction, that $f_i(t) = w_t$ is nice with respect to the sets

$$S_{t+1,t} \cup \{f_i(t-1)\}, S_{t+2,t} \cup \{f_i(t-1)\}, \ldots, S_{t+q,t} \cup \{f_i(t-1)\}.$$

(The addition of the point $f_i(t-1)$ is a technical point, required in order to make sure the walk defined by f_i is non backtracking even near the multiples of q. Note that the proof of the induction step maintains this property, as we may add all vertices of the walk from $f_i(t-q+1)$ to $f_i(t)$ to the sets $S_{j,t-q}$ while replacing the notion of very nice by nice.)

By Lemma 4.4 there are at least $0.9m$ vertices w_{t+q} which are very nice with respect to the sets $S_{t+q+1,t}, S_{t+q+2,t}, \ldots, S_{t+2q,t}$. As before, since $w_t = f_i(t)$ is nice with respect to the sets

$$S_{t+1,t} \cup \{f_i(t-1)\}, S_{t+2,t} \cup \{f_i(t-1)\}, \ldots, S_{t+q,t} \cup \{f_i(t-1)\}$$

there is a walk from w_t to some w_{t+q}, which slips by these sets, where w_{t+q} is very nice with respect to the sets $S_{t+q+1,t}, S_{t+q+2,t}, \ldots, S_{t+2q,t}$. Defining the values of $f_i(t+1), \ldots, f_i(t+q)$ according to this walk, and adding all vertices of this walk to the sets $S_{t+q+1,t}, \ldots, S_{t+2q,t}$ we conclude, by Lemma 4.5, that $w_{t+q} = f_i(t+q)$ is nice with respect to the sets

$$S_{t+q+1,t+q} \cup \{f_i(t+q-1)\}, S_{t+q+2,t+q} \cup \{f_i(t+q-1)\}, \ldots, S_{t+2q,t+q} \cup \{f_i(t+q-1)\}.$$

The process clearly ensures that the condition (1) will not be violated, completing the proof of the induction step. The assertion of Theorem 1.1 follows. ■

5 Concluding remarks and open problems

- Theorem 1.1 provides an explicit construction of an $\mathcal{H}(k,n)$-universal graph with at most $c_k n^{2-2/k}$ edges. All steps in the proof of universality are algorithmic, and it thus provides a polynomial time deterministic algorithm to embed any given $H \in \mathcal{H}(k,n)$ in G.

- As mentioned in the introduction, we have made no effort to optimize the constants in our construction, and indeed these constants are large, and grow exponentially with k. This can be improved by taking smaller expanders in the first $k-1$ coordinates of G, leaving only the final one as large as it is here. The embedding can be done by defining $f_1, f_2, \ldots, f_{k-1}$ by random walks, as done in [2], leaving only the last step (that of defining f_k) deterministic using the methods of the present paper. In addition, one can avoid taking the forth powers of the expander F, replacing it by a final blow-up of the whole construction. In this case, however, the application of the random walks will not provide a deterministic embedding algorithm. The (somewhat tediuos) details are omitted.

- The $\mathcal{H}(k,n)$-universal graph $G_{k,n}$ constructed here has an optimal number of edges up to a constant factor, but its number of vertices is (much) bigger than n. By combining it with an appropriate expander, as done in [4], we can reduce the number of vertices to $(1+\epsilon)n$, for any fixed $\epsilon > 0$, increasing the number of edges only by a constant factor (depending on ϵ). It remains open to decide if there are $\mathcal{H}(k,n)$-universal graphs with n vertices and $O_k(n^{2-2/k})$ edges. Note that the construction in [2] provides $\mathcal{H}(k,n)$-universal graphs with n vertices, but their number of edges exceeds that of the graphs constructed here by a logarithmic factor.

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