Sparse Universal Graphs for Bounded-Degree Graphs

Noga Alon * Michael Capalbo†

March 12, 2006

Abstract

Let $\mathcal{H}$ be a family of graphs. A graph T is $\mathcal{H}$-universal if it contains a copy of each $H \in \mathcal{H}$ as a subgraph. Let $\mathcal{H}(k,n)$ denote the family of graphs on n vertices with maximum degree at most k. For all positive integers k and n, we construct an $\mathcal{H}(k,n)$-universal graph T with $O_k(n^{2-\frac{2}{k}} \log^{\frac{4}{k}} n)$ edges and exactly n vertices. The number of edges is almost as small as possible, as $\Omega(n^{2-2/k})$ is a lower bound for the number of edges in any such graph. The construction of T is explicit, whereas the proof of universality is probabilistic, and is based on a novel graph decomposition result and on the properties of random walks on expanders.

1 Introduction

For a family $\mathcal{H}$ of graphs, a graph T is *$\mathcal{H}$-universal* if, for each $H \in \mathcal{H}$, the graph T contains a subgraph isomorphic to H. Thus, for example, the complete graph K_n is $\mathcal{H}_n$-universal, where $\mathcal{H}_n$ is the family of all graphs on at most n vertices. The construction of sparse universal graphs for various families arises in the study of VLSI circuit design, and has received a considerable amount of attention.

Universal graphs are of interest to chip manufacturers, as explained, for example, in [7], page 308. It is very expensive to design computer chips, but relatively inexpensive to make many copies of a computer chip with the same design. This encourages manufacturers to make their chip designs configurable, in the sense that the entire chip is prefabricated except for the last layer, and a final layer of metal is then added corresponding to the circuitry of a customer's particular specification. Hence, most of the design costs can be spread over many customers. One may view the circuitry of a computer chip as a graph, and model the problem of designing chips with few wires that are configurable for a particular family of applications as designing smaller universal graphs for a particular family of graphs. Similarly, as discussed in [14], the problem of designing an efficient single circuit that can be specialized for a variety of other circuits can be viewed as constructing a small universal graph.

*Schools of Mathematics and Computer Science, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel and IAS, Princeton, NJ 08540, USA. Email: nogaa@tau.ac.il. Research supported in part by the Israel Science Foundation, by a USA-Israeli BSF grant, by NSF grant CCR-0324906, by a Wolfensohn fund and by the State of New Jersey.

†DIMACS, Rutgers University, Piscataway, NJ. Email: mcapalbo@dimacs.rutgers.edu. Supported by NSF grant CCR98210-58 and ARO grant DAAH04-96-1-0013.

These applications motivate the study of universal graphs with a small number of vertices and edges for various families of graphs, and indeed there is an extensive literature on this subject. In particular, universal graphs for forests have been studied in [6], [10], [11], [12], [13], [16], and universal graphs for planar and other sparse graphs have been investigated in [2], [5], [6], [8], [9], [21].

Universal graphs for general bounded-degree graphs have also been considered in various papers. For positive integers k and n, let $\mathcal{H}(k,n)$ denote the family of all graphs on n vertices with maximum degree at most k. The authors of [3] constructed $\mathcal{H}(k,n)$-universal graphs with at most $O(n^{2-1/k}\log^{1/k} n)$ edges, as well as $\mathcal{H}(k,n)$-universal graphs on n vertices with $O(n^{2-c/(k\log k)})$ edges. In addition, it is shown in [3], by a simple counting argument, that any $\mathcal{H}(k,n)$-universal graph must have at least $\Omega(n^{2-2/k})$ edges. A better construction (for universal graphs with more than n vertices) is given in [4], where the authors present such graphs with at most $O(n^{2-2/k}\log^{1+8/k} n)$ edges. Universal graphs for $\mathcal{H}(3,n)$ that have only n vertices and less than $O(n^{1.87})$ edges are constructed in [2].

Our main result here is an explicit construction of $\mathcal{H}(k,n)$-universal graphs with n vertices and at most $O(n^{2-2/k}\log^{4/k} n)$ edges, as stated in the following theorem.

Theorem 1.1 *For every $k \geq 3$ there exists an (explicitly constructable) $\mathcal{H}(k,n)$-universal graph T with n vertices and at most $c(k)n^{2-2/k}\log^{4/k} n$ edges, for some constant $c(k)$.*

This improves the above mentioned results as the number of vertices in our graphs is n, the same as that of the graphs to be embedded, and yet the number of edges is nearly the minimum possible (and is in fact better by a logarithmic factor than the number of edges in the construction of [4], which uses more vertices). Our basic approach borrows some ideas from the one in [4], but contains several new techniques. In particular, we prove a new decomposition result for graphs with maximum degree k, and apply the properties of random walks on expanders in the analysis of the construction.

In the course of some of the proofs it is helpful to consider graphs with loops. However, all graphs considered here are simple, unless otherwise specified explicitly.

The rest of the paper is organized as follows. In Section 2 we describe the basic construction and outline the main ideas in the proof that it is universal. Since it is easier to handle the case of even k, we restrict our attention here to this case, and further construct in this section only universal graphs with $2n$ vertices. In order to deal with odd values of k we need a decomposition result, which is stated and proved in Section 3. In Section 4 we combine it with the approach described in the outline, which is based on the basic properties of random walks on expanders, to construct an auxiliary graph which plays a crucial role in constructing our final universal graph. This is done in Section 5, where we also show how to reduce the number of vertices to its optimal value n. The final, short Section 6 contains a few concluding remarks.

2 The basic construction and an outline of the proof

Note, first, that the case $k = 2$ is trivial; the square of a cycle of length n is $\mathcal{H}(2,n)$-universal, and has a linear number of edges. We thus assume $k > 2$.

Let $k > 2$ be an *even* integer. Put $m = (\frac{n}{c\log^2 n})^{2/k}$, where c is an absolute constant, and assume, for simplicity, that m is an integer. Let Z be a bounded-degree, C-regular expander on a set $V(Z)$ of m vertices, with a loop at each vertex, in which the absolute value of every eigenvalue of the adjacency matrix, besides the first, is at most $C/2$. Let G be the graph whose vertex set is $V(G) = (V(Z))^{k/2}$, where two vertices $(x_1, x_2, \ldots, x_{k/2})$ and $(y_1, y_2, \ldots, y_{k/2})$ are adjacent iff there exists an i such that $x_i y_i \in E(Z)$. Thus, in particular, G has a loop at each vertex. Let T be the graph obtained from G by replacing each vertex v of G by a clique V_v of size $2c\log^2 n$, where for each edge uv of G, T contains a complete bipartite graph with vertex classes V_u and V_v. Note that T has $2n$ vertices and $O(nm^{(k/2)-1}\log^2 n) = O(n^{2-2/k}\log^{4/k} n)$ edges.

It remains to prove that T is $\mathcal{H}(k,n)$-universal. To this end we need the notion of graph homomorphism. An (H,P)*-homomorphism* from a graph H to a graph P is a mapping of the vertices of H to those of P such that adjacent vertices in H are mapped to adjacent ones in P. Let $H \in \mathcal{H}(k,n)$ be a graph on n vertices with maximum degree at most k. Note that in order to prove that H is a subgraph of T, it suffices to show that there is a homomorphism from H to G which maps at most $2c\log^2 n$ vertices on each vertex v of G.

This is done as follows. By Petersen's Theorem (c.f., e.g., [22]), the edges of H can be decomposed into $k/2$ subgraphs $H_1, H_2, \ldots, H_{k/2}$, so that each edge of H lies in exactly one H_i, and each H_i has maximum degree at most 2. Thus, there are homomorphisms f_i, where f_i maps H_i into a path P_n of length n with a loop at every vertex, so that the inverse image $f_i^{-1}(v)$ of each vertex of P_n is of size at most 2.

For each i, $1 \le i \le k/2$, let g_i be a random walk on the expander Z, that is, a mapping from $V(P_n)$ to $V(Z)$ obtained by choosing the image of the first vertex of P_n uniformly at random in $V(Z)$, and by choosing the image of each vertex, in order along P_n, to be a random neighbor (in Z) of the image of its predecessor in P_n. Finally define a mapping f from $V(H)$ to $V(G)$ by putting $f(v) = (g_1(f_1(v)), g_2(f_2(v), \ldots, g_{k/2}(f_{k/2}(v)))$.

It is easy to see that f is an (H,G)-homomorphism. Moreover, the expected number of vertices of H mapped to any fixed vertex of G is precisely $|V(H)|/|V(G)| = c\log^2 n$. The crucial fact, proved in details in Section 4, is that since the random walk on Z is rapidly mixing, with high probability, for every vertex v of G, the size of $f^{-1}(v)$ is close to $c\log^2 n$, and in particular, it is smaller than $2c\log^2 n$. This provides an embedding of H in T, and shows that T is $\mathcal{H}(k,n)$-universal.

Besides the proof of this crucial fact we describe, in Section 5, how to reduce the number of vertices of T to the minimum possible number n, by combining T with another bounded-degree expander in a suitable manner. A more difficult task is to extend the construction for odd values of k, where there is no analog to Petersen's Theorem. We overcome this difficulty by proving, in Section 3, a decomposition result for graphs of maximum degree k. We show that each such graph can be covered by k subgraphs, so that every edge is covered twice, and each subgraph can be mapped homomorphically to a path, mapping at most 4 vertices to each vertex of the path. As the decomposition here covers every edge twice, we have to modify the definition of G a bit, making its vertex set the k-th power of the vertex set of an appropriate expander, where two vertices are adjacent iff they are adjacent in at least two coordinates of the expander. The details are given in Sections 4.

3 A graph-decomposition result

An *augmentation* of a graph $H = (V, E)$ is any graph obtained from H by choosing an arbitrary (possibly empty) subset $U \subset V$, adding a new set U' of $|U|$ vertices, and adding a matching between U and U'. Thus, an augmentation of H is obtained from it by connecting new vertices of degree 1 to some of its vertices.

Call a graph *thin* if its maximum degree is at most 3 and each connected component of it is either an augmentation of a path or a cycle, or a graph with at most two vertices of degree 3.

It is easy to see that any thin graph H on n vertices has an (H, P)-homomorphism to a path P on n vertices with a loop at each vertex, such that the inverse image of any vertex of P consists of at most 4 vertices.

The main result in this section is the following theorem.

Theorem 3.1 *Let $k \geq 2$ be an integer, and let H be an arbitrary graph of maximum degree at most k. Then there are k spanning subgraphs $H_1, H_2, \ldots, H_k$ of H such that each H_i is thin, and every edge of H lies in precisely two graphs H_i.*

The assertion of the theorem for even values of k is very simple. To prove it for odd k as well, we need several lemmas.

Lemma 3.2 *Let $k \geq 3$ be an odd integer, and let H be an arbitrary k-regular graph. Then H contains a spanning subgraph W in which every vertex has degree 2 or 3, and there is a maximum matching M in W saturating only vertices whose degree in W is 3. In fact, every maximum matching M in H is contained in some W with the above properties.*

Proof: Let M be a maximum matching in H. By Hall's Theorem there is a matching M_2 in $H - M$ that saturates all vertices of H which are not covered by M. The graph $H - (M \cup M_2)$ has maximum degree $k - 1$, and hence, by Petersen's Theorem (c.f., e.g., [22]), its edges can be decomposed into $(k-1)/2$ pairwise edge-disjoint subgraphs $F_1, F_2, \ldots, F_{(k-1)/2}$, each having maximum degree 2. Consider the graph $T = F_1 \cup M$. Every vertex of H has degree at least 2 in T (as all other edges of H incident with this vertex are covered by M_2 and the other graphs F_i). In addition, every vertex of T has degree at most 3, and if it is not covered by M, then its degree is at most (and hence exactly) 2. Consider, now, the set of all vertices v of T which are covered by M and have degree 2 in T (if there are no such vertices, we are done). For each such v, as all other $k - 2$ edges incident with it lie in the other graphs F_i and in M_2, v must be saturated by M_2. Let uv be the unique edge of M_2 saturating v. By construction, u is not covered by M and hence its degree in T is 2. We can therefore add the edge uv to T, increasing the degrees of u and v to 3. Adding these edges for all the vertices v as above, we get a subgraph W of G which has the required properties. Indeed, all degrees of vertices of W are 2 or 3, the matching M in it saturates only vertices of degree 3, and it is a maximum matching in W, as it is a maximum matching even in H, that contains W. ■

Lemma 3.3 *Let $k \geq 3$ be an odd integer, and let H be an arbitrary k-regular graph. Then H contains a spanning subgraph H' in which every vertex has degree 2 or 3, and every connected component has at most 2 vertices of degree 3.*

Proof: Let W be a subgraph of H, and M a maximum matching in W, so that the conclusion of Lemma 3.2 holds. We next show that W contains a subgraph of the desired type. Clearly, we may assume that W is connected.

By the Gallai-Edmonds Theorem, (c.f., e.g., [18] pp. 94-99), there is a set A of vertices of W such that the set of connected components of $W - A$ is $\mathcal{S} \cup \mathcal{T}$, where $\mathcal{S}$ is the set of all odd connected components of $W - A$ and $\mathcal{T}$ is the set of all even connected components, and the following holds. The matching M contains a perfect matching in each component in $\mathcal{T}$, a near-perfect matching (that is, a matching covering all vertices but one) in each component in $\mathcal{S}$, and a matching saturating all vertices of A, connecting them to distinct components in $\mathcal{S}$. Moreover, each component C in $\mathcal{S}$ is factor-critical, namely, $C - v$ contains a perfect matching for each vertex v of C.

Note that as M covers only vertices whose degree in W is 3, it follows that all vertices of A and all vertices in the components $\mathcal{T}$ are of degree 3, and each component in $\mathcal{S}$ contains at most one vertex of degree 2. Let $\mathcal{S}_1$ denote the set of components in $\mathcal{S}$ in which all vertices have degree 3, and let $\mathcal{S}_2 = \mathcal{S} - \mathcal{S}_1$ denote the set of the components that contain a vertex of degree 2. Thus M contains no edge connecting a member of A to a vertex in a component in $\mathcal{S}_2$ (as M covers no vertex of degree 2).

Let I_1 be the set of all edges of M incident with vertices of A. By the above paragraph, these edges cover all vertices of A and connect them to vertices in distinct components in $\mathcal{S}_1$.

Let $\mathcal{S}_1'$ denote the set of all components C in $\mathcal{S}_1$, so that there is more than one edge (and hence at least 3 edges) connecting C to A. By Hall's Theorem, there is a matching I_2 connecting one vertex from each component in $\mathcal{S}_1'$ to a member of A. It is not difficult to see that the union of I_1 and I_2 contains a matching I that saturates all vertices of A, and covers a single vertex from each component of $\mathcal{S}_1'$. Indeed, each connected component of $I_1 \cup I_2$ is either an alternating cycle (including the case of a single edge) or an alternating path. If it is an alternating cycle, we let I include all the edges of, say, I_1 in it. If it is an alternating path of odd length, then it starts and ends with edges of the same I_j for some $j \in \{1, 2\}$, and we let I include all edges of that I_j in the component. If it is a path of even length, then it cannot start and end at a vertex of A (as I_1 saturates A), hence it starts and ends in vertices that lie in $\mathcal{S}_1$, and in this case we let I include all edges of I_2 of the component. This gives the required matching I. Adding to it a perfect matching in each member of $\mathcal{T}$, a near perfect matching in every component C in $\mathcal{S}_1$ covering all vertices of the component besides the one vertex covered by I (in case there is such a vertex), and a near perfect matching covering all vertices besides the one of degree 2 in each member of $\mathcal{S}_2$, we get a matching M' in W so that the only vertices not covered by M' are the vertices whose degree in W is 2, and at most one vertex in each component C in $\mathcal{S}_1$, but only when there is a unique edge from C to A.

The desired subgraph is $H' = W - M'$. Indeed, each vertex in this subgraph is of degree 2 or 3, and the only vertices of degree 3 lie in components in $\mathcal{S}_1$ with at most one edge from this component to A, where each such component can contain at most one such vertex. It is easy to see that at most 2 such vertices can lie in a component of $W - M'$. Indeed, consider a component that contains at least one vertex of degree 3. Omit all its edges that lie inside members of $\mathcal{S}_1$. This leaves a path, which can contain only two vertices in members of $\mathcal{S}_1$, as each of them must be an end-vertex. This completes the proof of the lemma. ■

Lemma 3.4 *Let $H = (V, E)$ be a 3-regular graph and let M be a (not necessarily perfect) matching in H. Then $H - M$ is the edge disjoint union of two subgraphs F_1 and F_2 so that each connected component of $F_1 \cup M$ and each connected component of $F_2 \cup M$ is an augmentation of a path or of a cycle.*

Proof: The graph obtained from H by contracting all edges of M has maximum degree 4 and thus, by Petersen's Theorem, it is the edge disjoint union of two subgraphs F_1' and F_2', each having maximum degree 2. Let F_1 consist of all edges of H that belong, after the contraction, to F_1', and let F_2 be defined analogously. Consider an edge uv of the matching M. Let u_1u, u_2u and v_1v, v_2v be the other edges incident with u and v (the four vertices u_i, v_i are not necessarily distinct). After the contraction, the new contracted vertex, denoted by x, is adjacent by edges to u_1, u_2, v_1, v_2. Two of these edges belong to F_1' and two to F_2'. If, for example, xu_1, xv_1 lie in F_1', then they lie in either a path or a cycle of this graph, and after the addition of the edge uv to F_1 this path or cycle simply becomes longer. If xu_1, xu_2 lie in F_1', then after adding the edge uv to F_1 the vertex v becomes a degree 1-vertex adjacent to u. All other cases are symmetric, giving the desired result. ■

Proof of Theorem 3.1: Since every graph with maximum degree at most k is a subgraph of a k-regular graph (possibly with more vertices), we may assume that H is k-regular. For even k, the assertion is a trivial consequence of Petersen's Theorem; in this case H is the edge-disjoint union of $k/2$ two-regular subgraphs, and we simply take each of them twice. For odd k, we apply induction. Starting with $k = 3$, consider a 3-regular graph H. By Lemma 3.3 it contains a spanning thin subgraph H' in which all degrees are either 2 or 3. Let M be the set of all edges of H besides those of H'. Then M is a matching and hence, by Lemma 3.4, $H - M$ is the edge disjoint union of two subgraphs F_1, F_2 so that $F_1 \cup M$, $F_2 \cup M$ are thin. We can therefore take $H_1 = F_1 \cup M, H_2 = F_2 \cup M$ and $H_3 = H' = F_1 \cup F_2$, obtaining the desired decomposition.

Assuming the result holds for $k - 2$, we prove it for k where $k \geq 5$ is odd. Given a k-regular graph H, we apply Lemma 3.3 to conclude it contains a spanning thin subgraph H' in which all degrees are either 2 or 3. Taking H' twice and applying the induction hypothesis to $H - H'$ (which has maximum degree $k - 2$ and is thus a subgraph of a $(k - 2)$-regular graph) the desired result follows. ■

4 An auxiliary construction

From now on we assume, whenever this is needed, that n is sufficiently large. We further omit all floor and ceiling signs whenever these are not crucial. Let $k \geq 3$ be an integer, let n be a large integer, and define $m = (\frac{n}{c \log^2 n})^{1/k}$, where $c = c(k)$ is a constant to be chosen later. Let C be an absolute constant, such that there is an $(m, C, C/2)$-graph Z, that is, a C-regular graph on m vertices, with a loop at each vertex, so that the absolute value of each of its eigenvalues but the first is at most $C/2$. Without trying to optimize the value of C, here is one way to construct such a Z explicitly. Start with any expander, for example, a 6-regular Ramanujan graph on $(1 + o(1))m$ vertices using the construction of [19] or [20]. Omit from it a set of $o(m)$-vertices in such a way that the resulting graph is still an expander on precisely m vertices (to do so, it suffices to omit vertices so that the distance between any two of them is at least, say, 20). Add to the resulting graph edges and a loop at every vertex to make it 7-regular. By the main result of [1], the absolute value of each of the eigenvalues of the resulting graph besides the first is bounded away from 7, and we can now simply take a power of it (that is, raise its adjacency matrix to some fixed power) to get an $(m, C, C/2)$-graph Z, as needed.

Using the graph Z, construct a graph $G = (V(G), E(G))$ on m^k vertices as follows. The set of vertices of G is $V(G) = \{1, 2, \ldots, m\}^k$. Two vertices $(x_1, x_2, \ldots, x_k)$ and $(y_1, y_2, \ldots, y_k)$ of G are adjacent if and only if there are at least two indices $1 \leq i < j \leq k$, such that $x_i y_i$ and $x_j y_j$ are edges of Z. Note that G has $m^k = \frac{n}{c \log^2 n}$ vertices and $O(m^{2k-2}) = O(\frac{n^{2-2/k}}{\log^{4-4/k} n})$ edges. The following result asserts that any member of $\mathcal{H}(k, n)$ can be mapped homomorphically into G, in a rather uniform way.

Theorem 4.1 *For every $k \geq 3$, there exists a constant $c_0 = c_0(k)$ such that if the constant c in the construction of G exceeds c_0, then the following holds. Let H be an arbitrary graph with n vertices and maximum degree at most k. Then there is an (H, G)-homomorphism $f : V(H) \mapsto V(G)$ so that for every vertex $v \in V(G)$,*

$$0.9c \log^2 n \leq |f^{-1}(v)| \leq 1.1c \log^2 n. \tag{1}$$

Proof: As outlined in Section 2, the proof is probabilistic, and is based on the basic properties of random walks in expanders. Given $H = (V(H), E(H))$ as above, apply Theorem 3.1 to conclude that there are k thin spanning subgraphs $H_1, H_2, \ldots, H_k$ of H so that every edge of H appears in precisely two of them. By the paragraph preceding the statement of Theorem 3.1, for each i, $1 \leq i \leq k$, there is an (H_i, P_n)-homomorphism $f_i : V(H_i) = V(H) \mapsto V(P_n)$, where P_n is a path on n vertices with a loop at each vertex, so that at most 4 vertices of H are mapped to any vertex of P_n. For each i, let $g_i : V(P_n) \mapsto V(Z)$ be a random walk in the expander Z, that is, a random mapping of the vertices $\{1, 2, \ldots, n\}$ of P_n to $V(Z)$ obtained by choosing $g_i(1) \in V(Z)$ randomly and uniformly, and then by choosing, for each $j > 1$ in order, $g_i(j)$ to be a random, uniformly chosen neighbor of $g_i(j-1)$ in Z. The desired (H, G)-homomorphism f is defined as follows. For each vertex $v \in V(H)$,

$$f(v) = (g_1(f_1(v)), g_2(f_2(v)), \ldots, g_k(f_k(v))).$$

Observe, first, that this is an (H, G)-homomorphism as claimed. Indeed, if u, v are two adjacent vertices of H, then the edge uv lies in at least two of the subgraphs H_t, say, in H_i and in H_j. Therefore, f_i maps u and v to adjacent vertices of P_n, and g_i maps these to adjacent vertices of Z. The same holds for f_j and g_j, implying that the two images $f(u)$ and $f(v)$ have two indices i and j with corresponding coordinates that are adjacent in Z. Thus, by the definition of G, $f(u)$ and $f(v)$ are adjacent in G, showing that f is indeed an (H, G)-homomorphism.

It remains to show that with positive probability (1) holds for every $v \in V(G)$. Fix a vertex $v = (x_1, x_2, \ldots, x_k) \in V(G)$. Consider a set of vertices $U \subset V(H)$ so that

$$\text{for every two distinct } u, u' \in U \text{ and for every } 1 \leq i \leq k,$$

$$\text{the distance between } f_i(u), f_i(u') \text{ in } P_n \text{ is at least } 4 \log n. \tag{2}$$

Claim: The number of vertices of U mapped by f to v is stochastically dominated by a binomial random variable $B(|U|,(1/m^k+1/n^3))$, and stochastically dominates a binomial random variable $B(|U|,(1/m^k-1/n^3))$.

Proof of claim: For each $j \in \{0,1,\dots,k\}$, let U_j denote the set of all vertices $u \in U$ such that the first j coordinates of $f(u)$ agree with those of v. Thus $U_0 = U$, and our objective is to bound $|U_k|$ as in the claim. We prove, by induction on j, that $|U_j|$ is stochastically dominated by a binomial random variable $B(|U|,p_+^j)$, where $p_+ = 1/m + 1/n^4$, and stochastically dominates $B(|U|,p_-^j)$, where $p_- = 1/m - 1/n^4$. The case $j=0$ is obvious, as $|U_0| = |U|$. Assuming the result holds for all $j < i$, we prove it for i. Clearly, $U_i \subset U_{i-1}$. Put $|U_{i-1}| = r$ and let $U_{i-1} = \{u_1,u_2,\dots,u_r\}$ be an enumeration of the vertices of U_{i-1} according to the order of the vertices $f_i(u_p)$ on the path P_n. Then, since Z is an $(m,C,C/2)$-graph, for any given values of $g_i(f_i(u_p))$, for all $p < s$, and for any vertex z of Z (and in particular for $z = x_i$), the conditional probability that $g_i(f_i(u_s)) = z$ deviates from $1/m$ by at most $1/n^4$. Indeed, the eigenvalues condition implies that the random walk in Z is rapidly mixing, and $4\log n$ steps starting from $g_i(f_i(u_{s-1}))$ suffice to ensure it reaches a nearly uniform distribution with the error term above. This implies that $|U_i|$ is stochastically dominated by $B(|U_{i-1}|,p_+)$ and stochastically dominates $B(|U_{i-1}|,p_-)$, completing the proof of the induction step. The assertion of the claim follows from the case $j=k$, as $p_+^k < 1/m^k + 1/n^3$ and $p_- > 1/m^k - 1/n^3$ (with room to spare). ■

Returning to the proof of the theorem, note that by the claim and by the standard known estimates for binomial distributions it follows that if $|U|$ is at least $c'm^k \log n$ for a sufficiently large universal constant c', then with probability at least $1-1/n^2$, the number of vertices of U mapped by f to every single vertex among the m^k vertices of G will not differ from $|U|/m^k$ by more than $0.1|U|/m^k$. Thus, to finish the proof of the theorem, it suffices to show that we can partition $V(H)$ into sets U, each satisfying the assumption (2) and each having at least $c'm^k \log n$ vertices. This is done in the next paragraph.

By a theorem of Hajnal and Szemerédi [17] (whose application here is not essential, but makes the computation a bit simpler), one can partition $V(H)$ into $32k\log n$ sets U of equal size, each satisfying the assumption (2). Indeed, the Hajnal-Szemerédi Theorem asserts that the vertices of any graph with maximum degree smaller than Δ can be partitioned into Δ independent sets of equal size. We apply it to the graph whose set of vertices is $V(H)$, in which two vertices u, u' are adjacent iff there exists an i such that the distance between $f_i(u)$ and $f_i(u')$ in P_n is smaller than $4\log n$. The maximum degree in this graph is smaller than $32k\log n$, as for each vertex v on a path there are less than $2\cdot 4\log n$ vertices of the path of distance smaller than $4\log n$ from v, and for each i, f_i maps at most 4 vertices of H to each vertex of the path. Thus we obtain a partition of $V(H)$ into sets U that satisfy (2) and each of them is of size $\frac{n}{32k\log n} = \frac{cm^k\log n}{32k}$ (which is at least $c'm^k\log n$ for a sufficiently large $c = c(k)$). This is the required partition of $V(H)$, completing the proof of Theorem 4.1. ■

5 A sparse universal graph

In this section we prove the main result of the paper.

Proof of Theorem 1.1: Let $G = (V(G), E(G))$ be the graph constructed in the previous section, and let $F = (V(F), E(F))$ be a bounded-degree expander on the set of vertices $V(F) = V(G)$, with a loop at each vertex, such that any set X of at most half the vertices of F has at least $|X|/9$ neighbors outside X. Let T be the graph constructed from F and G as follows. Its set of vertices, $V(T)$, is the disjoint union of m^k sets V_v, $v \in V(G) = V(F)$, each of size $c \log^2 n$. Its set of edges is defined as follows. If $x \in V_v$ and $y \in V_{v'}$, then x and y are neighbors in T iff there are two vertices z, z' in $V(G)$ that are adjacent in G, where z is adjacent to v in F, and z' adjacent to v' in F. (Note that as F has a loop at each vertex, this includes the cases $z = v$ and/or $z' = v'$.)

Clearly, T has $m^k c \log^2 n = n$ vertices, and $|E(G)| \cdot O(\log^4 n) = O(n^{2-2/k} \log^{4/k} n)$ edges.

Claim 5.1 *The graph T is $\mathcal{H}(k, n)$-universal.*

Proof: Let H be a graph with n vertices and maximum degree at most k. By Theorem 4.1 there is an (H, G)-homomorphism f so that $0.9c \log^2 n \leq |f^{-1}(v)| \leq 1.1c \log^2 n$ for every $v \in V(G)$. We now show how to modify f in order to obtain an (H, T)-homomorphism f' which is a bijection, thus proving Claim 5.1. Let D be the following bipartite graph. One side of D is $V(H)$, and the other is $V(T)$. A vertex $x \in V(H)$ is adjacent to a vertex $y \in V(T)$ iff x and y satisfy the following: Let v be the vertex in $V(G) = V(F)$ such that $v = f(x)$, and let v' be the vertex in $V(G)$ such that $y \in V_{v'}$ in the construction of T. Then x and y are adjacent in D iff v and v' are adjacent in the expander F (including, again, the case $v = v'$ as well). We claim that D satisfies the Hall condition, and thus contains a perfect matching M. To prove this claim, note, first that if x and x' are two vertices in $V(H)$ such that $f(x) = f(x')$, then $N_D(\{x\}) = N_D(\{x'\})$. Therefore, it suffices to check that subsets of $V(H)$ consisting of unions of complete sets $f^{-1}(v)$ satisfy the Hall condition. Let $S = \cup_{v \in X} f^{-1}(v)$ be such a subset, where $X \subset V(G)$. If X has no more than $|V(G)|/2$ vertices, then the number of neighbors $|N_D(S)|$ of S in D satisfies

$$|N_D(S)| = |N_F(X)| c \log^2 n \geq \frac{10}{9} |X| c \log^2 n > 1.1 |X| c \log^2 n \geq |S|,$$

the last inequality following from the upper bound of $1.1c \log^2 n$ on each $|f^{-1}(v)|$. If, on the other hand, X has more than $|V(F)|/2 = |V(G)|/2$ vertices, define $Y = V(F) - N_F(X)$ and observe that $|Y| < |V(F)|/2$. Therefore, $|N_F(Y)| \geq \frac{10}{9} |Y|$ and it follows that

$$| \cup_{v \in N_F(Y)} f^{-1}(v)| \geq \frac{10}{9} |Y| 0.9c \log^2 n = |Y| c \log^2 n.$$

Since all members of $\cup_{v \in N_F(Y)} f^{-1}(v)$ do not lie in S, it follows that

$$|S| \leq n - |Y| c \log^2 n = (|V(F)| - |Y|) c \log^2 n = |N_F(X)| c \log^2 n = |N_D(S)|.$$

This shows that D satisfies Hall's condition, and hence there is indeed a perfect matching M in D, as claimed.

Finally, for each vertex $u \in V(H)$, let $f'(u)$ be the vertex in T matched to u by M. This is clearly a bijection from $V(H)$ to $V(T)$, and by the construction of T, it is also an (H, T)-homomorphism because f is an (H, G)-homomorphism. This completes the proof of Claim 5.1. ■

Theorem 1.1 follows from Claim 5.1, and from the calculation of the number of vertices and edges of T. ■

6 Concluding remarks and open problems

- Theorem 1.1 provides an explicit construction of an $\mathcal{H}(k, n)$-universal graph T on n vertices, with a nearly optimal number of edges. Interestingly, the construction is explicit, whereas the proof of universality is probabilistic. Given a graph $H \in \mathcal{H}(k, n)$, the proof provides an efficient randomized algorithm to embed it in T.
- It will be interesting to decide if it is possible to omit the $\log^{4/k} n$-term in our construction and obtain an $\mathcal{H}(k, n)$-universal graph with $O(n^{2-2/k})$ edges, which will be optimal, up to a constant factor.

Acknowledgment We would like to thank Eli Berger and Muli Safra for helpful discussions, and an anonymous referee for lots of insightful comments concerning the presentation.

References

[1] N. Alon, Eigenvalues and expanders, Combinatorica 6(1986), 83-96.

[2] N. Alon and V. Asodi, Sparse universal graphs, Journal of Computational and Applied Mathematics 142 (2002), 1-11.

[3] N. Alon, M. Capalbo, Y. Kohayakawa, V. Rödl, A. Ruciński, and E. Szemerédi, Universality and tolerance, Proceedings of the 41st IEEE Annual Symposium on FOCS (2000), 14–21.

[4] N. Alon, M. Capalbo, Y. Kohayakawa, V. Rödl, A. Ruciński, and E. Szemerédi, Near-optimum universal graphs for graphs with bounded degrees, *RANDOM-APPROX 2001*, 170–180.

[5] L. Babai, F. R. K. Chung, P. Erdős, R. L. Graham, J. Spencer, On graphs which contain all sparse graphs, Ann. Discrete Math. 12 (1982), 21–26.

[6] S. N. Bhatt, F. Chung, F. T. Leighton and A. Rosenberg, Universal graphs for bounded-degree trees and planar graphs, SIAM J. Disc. Math. 2 (1989), 145–155.

[7] S. N. Bhatt and C. E. Leiserson, How to assemble tree machines, in: *Advances in Computing Research*, F. Preparata, ed., 1984.

[8] M. Capalbo, A small universal graph for bounded-degree planar graphs, *SODA* 1999, 150–154.

[9] M. Capalbo and S. R. Kosaraju, Small universal graphs, *STOC* 1999, 741–749.

[10] F. R. K. Chung and R. L. Graham, On graphs which contain all small trees, J. Combin. Theory Ser. B 24 (1978), 14–23.

[11] F. R. K. Chung and R. L. Graham, On universal graphs, Ann. New York Acad. Sci. 319 (1979), 136–140.

[12] F. R. K. Chung and R. L. Graham, On universal graphs for spanning trees, Proc. London Math. Soc. 27 (1983), 203–211.

[13] F. R. K. Chung, R. L. Graham, and N. Pippenger, On graphs which contain all small trees II, Proc. 1976 Hungarian Colloquium on Combinatorics (1978), 213–223.

[14] F. R. K. Chung, A. L. Rosenberg, and L. Snyder, Perfect storage representations for families of data structures, SIAM J. Alg. Disc. Methods 4 (1983), 548–565.

[15] J. Friedman, A Proof of Alon's Second Eigenvalue Conjecture, Memoirs of the AMS, to appear.

[16] J. Friedman and N. Pippenger, Expanding graphs contain all small trees, Combinatorica 7 (1987), 71–76.

[17] A. Hajnal and E. Szemerédi, Proof of a conjecture of Erdős, in: *Combinatorial Theory and its Applications*, Vol. II (P. Erdős, A. Rényi, and V. T. Sós, eds.), Colloq. Math Soc. J. Bolyai 4, North Holland, Amsterdam 1970, 601–623.

[18] L. Lovász and M. D. Plummer, *Matching Theory*, North Holland, Amsterdam (1986).

[19] A. Lubotzky, R. Phillips and P. Sarnak, Ramanujan Graphs, Combinatorica 8 (1988), 261-277.

[20] G. A. Margulis, Explicit group-theoretical constructions of combinatorial schemes and their application to the design of expanders and superconcentrators, Problems of Information Transmission 24(1988), 39-46.

[21] V. Rödl, A note on universal graphs, Ars Combin. 11 (1981), 225–229.

[22] D. P. West, *Introduction to Graph Theory*, Prentice-Hall, 1996.