Breaking the rhythm on graphs

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ABSTRACT. We study graph colorings avoiding periodic sequences with large number of blocks on paths. The main problem is to decide, for a given class of graphs $\mathcal{F}$, if there are absolute constants t, k such that any graph from the class has a t-coloring with no k identical blocks in a row appearing on a path. The minimum t for which there is some k with this property is called the *rhythm threshold* of $\mathcal{F}$, denoted by $t(\mathcal{F})$. For instance, we show that the rhythm threshold of graphs of maximum degree at most d is between $(d+1)/2$ and $d+1$. We give several general conditions for finiteness of $t(\mathcal{F})$, as well as some connections to existing chromatic parameters. The question whether the rhythm threshold is finite for planar graphs remains open.

1. Introduction

Let $k \geq 2$ be a fixed integer. A vertex coloring f of a graph G is k-*repetitive* if there is a positive integer n and a path on kn vertices $v_1, v_2, ..., v_{kn}$ such that $f(v_i) = f(v_{i+n}) = \ldots = f(v_{i+(k-1)n})$ for all $1 \leq i \leq n$. That is, if there is at least one path in G that looks like a periodic sequence with k blocks. Otherwise f is called k-*nonrepetitive*. In this case there are no k identical blocks in a row on any path of G. This type of coloring is a graph theoretic version of Thue sequences (see [**1**], [**13**], [**14**]). The minimum number of colors needed for a k-nonrepetitive coloring of G is denoted by $\pi_k(G)$. Unlike for most graph coloring invariants, determining the exact value of $\pi_k(G)$ is not trivial even for paths or cycles. By the results of Thue [**16**], [**17**] (see also [**6**], [**7**], [**8**]) we have $\pi_2(P_n) = 3$ and $\pi_3(P_n) = 2$ for all $n \geq 4$. Recently it was proved by Currie [**9**] that $\pi_2(C_n) = 3$ for all $n \geq 3$, except $n = 5, 7, 9, 10, 14, 17$, and by Currie and Fitzpatrick [**10**] that $\pi_3(C_n) = 2$ for all $n \geq 3$. Thus the picture is complete for graphs of maximum degree $d = 2$.

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Let $\pi_k(d)$ denote the supremum of $\pi_k(G)$, where G ranges over all graphs of maximum degree d. Extending the results of [**2**] we prove that there exist absolute positive constants c_1, c_2 such that for all $k \geq 2$

$$\frac{c_1}{k}\frac{d^{k/(k-1)}}{(\log d)^{1/(k-1)}} \leq \pi_k(d) \leq c_2 d^{k/(k-1)}.$$

We also study the threshold value $t = t(d)$, defined as the minimum number of colors guaranteeing that for some (possibly huge) k, each graph of maximum degree d has a k-nonrepetitive coloring using at most t colors. By the above mentioned results for paths and cycles it follows that $t(2) = 2$. Curiously, this is the only known exact value of this function for $d > 1$. We prove here that for every d

$$\frac{1}{2}(d+1) \leq t(d) \leq d+1.$$

This concept may be studied for other classes of graphs as well. Let $\mathcal{F}$ be a class of graphs. Define the *rhythm threshold* of $\mathcal{F}$ as the least number $t = t(\mathcal{F})$ for which there exists a finite number k such that each graph from $\mathcal{F}$ has a k-nonrepetitive vertex coloring using at most t colors. Thus, for every k there is a graph G_k in $\mathcal{F}$ such that any vertex coloring of G_k using less than t colors is k-repetitive. The main problem is to decide whether $t(\mathcal{F})$ is finite for a given class $\mathcal{F}$. The situation is especially interesting for planar graphs. We discuss it briefly at the end of the paper.

2. Probabilistic bounds for $\pi_k(d)$

In the proof of the upper bounds on $\pi_k(d)$ we use the following version of the Local Lemma (see, e.g., [**4**]).

LEMMA 1. (The Local Lemma; Multiple Version) *Let $A_1, A_2, ..., A_n$ be events in any probability space with dependency graph $D = (V, E)$. Let $V = V_1 \cup V_2 \cup ... \cup V_k$ be a partition such that all members of each part V_r have the same probability p_r. Suppose that the maximum number of vertices from V_s adjacent to a vertex from V_r is at most Δ_{rs}. If there are real numbers $0 \leq x_1, x_2, ..., x_k < 1$ such that $p_r \leq x_r \prod_{s=1}^{k}(1-x_s)^{\Delta_{rs}}$ then* $\Pr(\bigcap_{i=1}^{n} \overline{A_i}) > 0$.

We also need the following simple fact, obtained by substituting $x = 1/\theta$ in the identity $\sum_{s=1}^{\infty} sx^s = \frac{x}{(1-x)^2}$ which follows by differentiating $1 + x + x^2 + \ldots = \frac{1}{1-x}$, multiplying the resulting identity by x.

Fact: For every $\theta > 1$ the series $\sum_{s=1}^{\infty} \frac{s}{\theta^s}$ converges to $\theta/(\theta-1)^2$.

THEOREM 1. *For every $k, d \geq 2$ we have $\pi_k(d) \leq \lceil (6d)^{k/(k-1)} \rceil$.*

PROOF. Let G be a graph of maximum degree d. Consider a random coloring of the vertices of G with $N = \lceil (6d)^{k/(k-1)} \rceil$ colors. For each path

P in G let A_P be the event that the sequence of colors along P is periodic and consists of k identical blocks. Let V_r be the set of all events A_P with P having kr vertices. Clearly we have $p_r = N^{-r(k-1)}$.

Now define a dependency graph so that A_P is adjacent to A_Q iff the paths P and Q have a common vertex. Since a fixed path with kr vertices intersects at most k^2rsd^{ks} paths with ks vertices in G, we may take $\Delta_{rs} = k^2rsd^{ks}$. Next set $x_s = (5d)^{-ks}$. Since $(1 - x_s) \geq e^{-kx_s}$ we get

$$x_r \prod_s (1-x_s)^{\Delta_{rs}} \geq (5d)^{-kr} \prod_s e^{-kx_s\Delta_{rs}}.$$

Substituting for x_s and Δ_{rs} in the last expression gives

$$(5d)^{-kr} \prod_s e^{-k(5d)^{-ks}k^2rsd^{ks}} > (5d)^{-kr} \exp\left(-(kr)\sum_{s=1}^{\infty} \frac{k^2 s}{5^{ks}}\right).$$

By the above fact, the series $\sum_{s=1}^{\infty} \frac{k^2s}{5^{ks}}$ converges to $(k^2 5^k)/(5^k-1)^2$. Therefore we obtain

$$x_r \prod_s (1-x_s)^{\Delta_{rs}} > (5e^{(k^2 5^k)/(5^k-1)^2} d)^{-kr} > (6d)^{-kr} \geq p_r.$$

and by Lemma 1 the proof is complete. ∎

Theorem 2. *There is an absolute constant $c > 0$ such that for every $k, d \geq 2$ we have $\pi_k(d) \geq \frac{c}{k} \frac{d^{k/(k-1)}}{(\log d)^{1/(k-1)}}$.*

Proof. As this is not crucial for the proof, we omit all floor and ceiling signs. Clearly it suffices to prove the assertion for large values of d. Let $k \geq 2$ be a fixed integer. Put $p = p(n,k) = (36k^2)^{1/k}\left(\frac{\log n}{n}\right)^{1/k}$, assume n is large, and let $G = G(n,p)$ be the random graph on the set of n labelled vertices $\{1, 2, \ldots, n\}$ obtained by picking each pair of distinct vertices, randomly and independently, to be an edge with probability p. We claim that almost surely (that is, with probability that tends to 1 as n tends to infinity) G satisfies the following properties.

(1) The maximum degree $\Delta = \Delta(G)$ of G, is at most $20n^{(k-1)/k}(\log n)^{1/k}$.

(2) Let $m = \frac{n}{2k}$ and let U be any subset of $n/2$ vertices of G arranged in a $k \times m$ matrix $U = (u_{ij})$, $1 \leq i \leq k$, $1 \leq j \leq m$. Then there is a set $S \subset \{1, ..., m\}$, $|S| \geq m/3$, such that:
(a) The graph on the set S in which st is an edge iff $u_{is}u_{it}$ is an edge of G for all $i = 1, ..., k$, is connected.
(b)There is a pair of indices $s, t \in S$ such that G contains the following matching

$$u_{1t}u_{2s}, u_{2t}u_{3s}, \ldots, u_{(k-1)t}u_{ks}, u_{kt}u_{1s}. \tag{2.1}$$

Claim 1 is clear. To prove Claim 2 fix a set U and its order in the matrix, and consider the graph H on the set $\{1, ..., m\}$ in which st is an edge iff $u_{is}u_{it}$ is an edge of G for all $i = 1, ..., k$. This is a random graph with edge probability exactly $p^k = 36k^2 \frac{\log n}{n}$. Assume that there is no set S as required in 2a. This means that the set of vertices of H can be partitioned into two disjoint sets, each of size at least $m/3$ and at most $2m/3$, with no edges between them. As there are less than 2^m possibilities for the choice of these disjoint sets, the probability of this event is less than

$$2^m \left(1 - \frac{36k^2 \log n}{n}\right)^{\frac{2m^2}{9}} < n^{-n}.$$

Since the number of ordered sets U is less than $n^{n/2}$ it follows that the probability that there is no S satisfying the assertion of 2a is at most $n^{-n/2}$.

To prove that S satisfies Claim 2b as well, with high probability, it suffices to show that almost surely for **every** set S of $m/3$ ordered k-tuples of vertices u_{ij}, $1 \le i \le k, j \in S$ there are s and t satisfying (2.1). Indeed, for a fixed S, the probability that this does not hold is

$$(1 - p^k)^{\binom{m/3}{2}} = (1 - \frac{36k^2 \log n}{n})^{(\frac{1}{72} - o(1))n^2/k^2} < n^{-n/3}.$$

Since the number of choices of such an S is less than $n^{n/6}$, the desired result follows. This completes the proof of the claim.

Returning to the proof of the theorem, let G satisfy all three properties in the claim, and consider any vertex coloring of G by at most $\frac{n}{2k}$ colors. By omitting if necessary at most $k-1$ vertices from each color class, we are left with a set of more than $n/2$ vertices in which the size of each color class is divisible by k. Let U be a subset of cardinality $n/2$ of this set, and arrange its vertices in a matrix (u_{ij}), $1 \le i \le k$, $1 \le j \le m = n/(2k)$ so that each column of U consists of vertices of the same color. Consider a set $S \subset \{1, ...m\}$ satisfying the assertion in Claim 2a and 2b. Let st be the pair satisfying 2b and let $s = s_1, s_2, ..., s_l = t$ be a path in H, the existence of which is guaranteed by 2a. Then the path

$$u_{1s}, u_{1s_2}, ..., u_{1t}, u_{2s}, ..., u_{2t}, u_{3s}, ..., u_{(k-1)t}, u_{ks}, ..., u_{kt}$$

is colored repetitively, showing that $\pi_k(G) > \frac{n}{2k}$. By the first assertion of the claim this implies that there is an absolute constant $c > 0$ such that $\pi_k(G) > \frac{c}{k}\left(\frac{\Delta^k}{\log \Delta}\right)^{1/k-1}$. This, and the fact that we can take any large n in the proof imply the assertion of the theorem. ∎

3. The threshold function $t(d)$

For the upper bound of $t(d)$ we apply again the local lemma.

Theorem 3. *For every $d \ge 1$ we have $t(d) \le d + 1$.*

Proof. Let G be any graph of maximum degree $d \ge 2$. We will show that $d+1$ colors suffice to avoid all sufficiently long periodic sequences with

j blocks, for some integer j. Consider a random coloring of the vertices of G with $d+1$ colors. Choose positive integers j, s_0 and a real $\theta > 1$ so that

$$(*) \qquad 1 + 1/d \geq \theta(d+1)^{1/j} \exp\left(2 \sum_{s \geq s_0} s\theta^{-s}\right).$$

This can always be done since the series $\sum_{s=1}^{\infty} s\theta^{-s}$ converges for each $\theta > 1$. For a path $P \subseteq G$ let $A(P)$ denote the bad event that the sequence of colors along P consists of j identical blocks. Set $V_r = \{A(P) : P \text{ is a path with } r \text{ vertices}\}$ and set $x_s = (\theta d)^{-s}$. Since each path of length r shares a vertex with not more than rsd^s paths of length s, we may take $\Delta_{rs} = rsd^s$. Finally, we have $p_r \leq (d+1)^{r(1/j-1)}$ and the local lemma applies provided

$$(d+1)^{(1/j-1)r} \leq x_r \prod_{s \geq s_0} (1-x_s)^{rsd^s}.$$

Since $(1-x_s) \geq e^{-2x_s}$ we will be done by showing that

$$(d+1)^{(1/j-1)r} \leq (\theta d)^{-r} \prod_{s \geq s_0} e^{-2rs\theta^{-s}}.$$

This follows readily by our initial choice of j, s_0 and θ, as the inequality

$$(d+1)^{1-1/j} \geq \theta d \prod_{s \geq s_0} e^{2s\theta^{-s}}$$

is equivalent to (*). To complete the proof note only that any coloring without j identical blocks on paths of length at least s_0 must be k-nonrepetitive for $k = js_0$. ∎

For the lower bound of $t(d)$ we use regular graphs of large girth.

Theorem 4. *For every $d \geq 2$ we have $t(d) \geq \frac{1}{2}(d+1)$.*

Proof. Let $m = \lfloor d/2 \rfloor$ and let $G = (V, E)$ be a d-regular graph of girth at least $2k+1$. Given a coloring f of V by m colors $\{1, 2, ..., m\}$, partition the set of vertices of G into m disjoint sets $V_i = f^{-1}(i)$, $i = 1, ..., m$. Since G has at least $m|V|$ edges, either the induced subgraph on V_i has at least $|V_i|$ edges for some i, or the bipartite graph consisting of all edges of G between V_a and V_b has at least $|V_a| + |V_b|$ edges for some pair of indices a, b. In the first case, we get a monochromatic cycle of length at least $2k+1$, and hence a monochromatic path of length at least $2k > k$. In the second case, we get an alternating cycle of length at least $2k+2 > 2k$, and hence an alternating path of length $2k$. Thus $\pi_k(G) > m$. ∎

4. When is the rhythm threshold finite?

Let $\mathcal{F}$ be a class of graphs. Clearly the rhythm threshold $t(\mathcal{F})$ may be infinite. The following result of Erdős and Gallai [**11**], stated below as a lemma, implies that this happens when $\mathcal{F}$ contains graphs of arbitrarily large minimum (or average) degree.

LEMMA 2. (Erdős and Gallai [**11**]) *If a graph G has n vertices and more than $(k-2)n/2$ edges then there is a path on k vertices in G.*

THEOREM 5. *For every two integers $k > 1$ and r, any graph $G = (V, E)$ of average degree exceeding $(k-1)(2r-1)-1$ satisfies $\pi_k(G) > r$.*

PROOF. Let $f : V \mapsto \{1, 2, \ldots, r\}$ be a coloring, and define $V_i = f^{-1}(i)$. If the number of edges in the induced subgraph of G on V_i exceeds $(k-2)|V_i|/2$ then, by Lemma 2, it contains a path on k vertices (which is monochromatic). Hence we may assume this is not the case. Similarly, if the bipartite subgraph of G consisting of all its edges that connect V_i and V_j contains more than $(2k-2)(|V_i| + |V_j|)/2$ edges, then it contains a path of length $2k$. Therefore, if G has a k-nonrepetitive r-coloring it contains at most $((k-1)(r-1) + \frac{k-2}{2})|V|$ edges. ∎

On the other hand, bounded degeneracy is not sufficient for finiteness of $t(\mathcal{F})$. Recall that a graph is d-degenerate if every subgraph of it contains a vertex of degree at most d. The result below shows that the rhythm threshold for 2-degenerate graphs is infinite.

THEOREM 6. *For every k and r there is a 2-degenerate graph $G = G(k, r)$ such that $\pi_k(G) > r$.*

PROOF. Take the graph obtained from a large complete graph K_n by replacing each edge by a path of length 2. In any r-coloring of G we get, by the pigeonhole principle, a set V' of $m \geq n/r$ vertices of the same color. Applying again the pigeonhole principle, we get at least $\binom{m}{2}/r$ of the middle vertices among those on the 2-paths connecting the vertices of V' that have the same color. By Lemma 2 this will give us an alternating path of length $2k$ in case $\binom{m}{2}/r > m(k-1)$. ∎

The *acyclic chromatic number* $a(G)$ of a graph G is the minimum number of colors in a proper vertex coloring of the graph in which every cycle has at least 3 colors. Let $t'(\mathcal{F})$ be the edge version of the rhythm threshold $t(\mathcal{F})$, defined exactly the same way, but for edge colorings. We show that $t(\mathcal{F})$ is finite provided $t'(\mathcal{F})$ is finite and $\mathcal{F}$ has bounded acyclic chromatic number. We apply the following result of [**3**] on homomorphisms of edge colored graphs.

LEMMA 3. (Alon and Marshall [**3**]) *Let $\mathcal{F}_r$ be the family of graphs with acyclic chromatic number at most r. Let n be an odd integer. Then there exists a graph H_n on at most rn^{r-1} vertices whose edges are colored with n colors such that any graph $G \in \mathcal{F}_r$ whose edges are n-colored embeds homomorphically into H_n (in a color-preserving manner).*

THEOREM 7. *If $\mathcal{F}$ has bounded acyclic chromatic number and $t'(\mathcal{F})$ is finite, then $t(\mathcal{F})$ is finite.*

PROOF. Let $t'(\mathcal{F}) \leq n$, where n is odd. Let $G \in \mathcal{F}$ and let f be a k-nonrepetitive n-coloring of the edges of G, where k is an absolute constant depending on $\mathcal{F}$, but not on G. Let H_n be a graph from Lemma 3 and let h be a homomorphism from the vertex set of G to the vertex set of H_n, preserving the colors of all edges. We claim that h is a $(k+1)$-nonrepetitive coloring of the graph G (using $|V(H_n)|$ colors). Indeed, since h is a homomorphism of edge colored graphs, for every edge $e = uv$ in G its color $f(uv)$ is uniquely determined by the colors of its ends, that is, by the values $h(u)$ and $h(v)$. Hence, a vertex periodic path with $k+1$ blocks would give k identical blocks on its edges, contradicting the assumption on the coloring f. ∎

5. Another four color problem?

At present it is not known if the rhythm threshold is finite for planar graphs. By the results of Kündgen and Pelsmajer [**12**], or Barát and Varjú [**5**], $t(\mathcal{F})$ is finite if $\mathcal{F}$ has bounded treewidth. This implies, by a deep theorem of Robertson and Seymour [**15**], that $t(\mathcal{F})$ is finite if $\mathcal{F}$ consists of graphs not containing a fixed *planar* graph as a minor. Therefore planar graphs form the smallest minor closed class of graphs for which the situation is not clear.

CONJECTURE 1. *The rhythm threshold of planar graphs is finite.*

Curiously, the least possible candidate number is four. Indeed, the class of triangular graphs (obtained iteratively from the triangle by inserting a new vertex into a face and joining it to the three vertices of that face) shows that three colors would not suffice. Are four colors enough to break the rhythm on planar graphs?

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