The strong chromatic number of a graph

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Abstract

It is shown that there is an absolute constant c with the following property: For any two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on the same set of vertices, where G_1 has maximum degree at most d and G_2 is a vertex disjoint union of cliques of size cd each, the chromatic number of the graph $G = (V, E_1 \cup E_2)$ is precisely cd. The proof is based on probabilistic arguments.

1 Introduction

Let $G = (V, E)$ be a graph on n vertices. If k divides n we say that G is *strongly k-colorable* if for any partition of V into pairwise disjoint sets V_i, each of cardinality k precisely, there is a proper k-vertex coloring of G in which each color class intersects each V_i by exactly one vertex. Notice that G is strongly k-colorable if and only if the chromatic number of any graph obtained from G by adding to it a union of vertex disjoint k-cliques (on the set V) is k. If k does not divide n , we say that G is strongly k-colorable if the graph obtained from G by adding to it $k\lceil n/k \rceil - n$ isolated vertices is strongly k-colorable. The *strong chromatic number* of a graph G , denoted by $s\chi(G)$, is the minimum k such that G is strongly k-colorable. As observed in [6] if G is strongly k-colorable then it is strongly $k+1$- colorable as well, and hence $s\chi(G)$ is in fact the smallest k such that G is strongly s-colorable for all $s \geq k$.

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Motivated by a problem of F. Hsu, J. Schonheim and others (see [6]) conjectured that for any cycle C_n of length $3n$, $s\chi(C_{3n}) \leq 3$. This conjecture is still open , although , as observed by various researchers including F. de la Vega, M. Fellows and the present author it is true that for all n $s\chi(C_{4n}) \leq 4$, (see [1] and [6]).

It appears interesting to study the strong chromatic numbers of more complicated graphs. It is easy to see that any graph G with maximum degree d has strong chromatic number $s\chi(G) > d$. Define $s\chi(d) = max(s\chi(G))$, where G ranges over all graphs with maximum degree at most d. It is easy to see that $s\chi(1) = 2$. As noted in [1] $s\chi(d) > 3\lfloor d/2 \rfloor$ for every d. This simple fact is proved in the beginning of the next section. On the other hand, in [6] it is proved that $s\chi(d) \leq 2(6^{d-1})$. A better result is mentioned in [1]. It asserts that for any graph G with chromatic index f, $s\chi(G) \leq 2^f$. This statement, whose proof is presented in the next section, implies, by Vizing's Theorem (see, e.g. [4]) , that $s\chi(d) \leq 2^{d+1}$ for every d.

Our main result here is the following improvement for these estimates, which shows that in fact $s\chi(d)$ grows only linearly with d.

Theorem 1.1 *There is a constant c such that for every d, $s\chi(d) \leq cd$.*

It would be interesting to find the best possible value of c in this theorem. By the above remark, this value is larger than 3/2, whereas our proof shows that it is smaller than some huge number, possibly about $2^{10^{10}}$. By being more careful this estimate can be reduced to about 10^8, but since it is clear that our approach cannot give any realistic estimate for the best possible c we make no attempt to obtain the best possible constant and merely show it exists.

2 Simple bounds on strong chromatic numbers

The following simple fact is mentioned without a proof in [1].

Proposition 2.1 *For every d, $s\chi(d) > 3\lfloor d/2 \rfloor$.*

Proof Construct a graph G with $12r$ vertices, partitioned into 12 classes of cardinality r each, as follows. Let these classes be $A_0, \ldots, A_3, B_0, \ldots, B_3, C_0, \ldots, C_3$. Each vertex in A_i is joined by edges to each member of A_{i-1} and each member of A_{i+1}, where the indices are reduced modulo 4. Similarly, each member of B_i is adjacent to each member of B_{i-1} and B_{i+1} and each member of C_i is adjacent to each member of $C_{i-1} \cup C_{i+1}$. Consider the following partition of the set of vertices of G into four classes of cardinality $3r$ each;

$V_1 = A_0 \cup A_2 \cup B_0$, $V_2 = A_1 \cup A_3 \cup B_2$,

$V_3 = B_3 \cup C_0 \cup C_2$, $V_4 = B_1 \cup C_1 \cup C_3$.

We claim that there is no proper $3r$-vertex coloring of G in which each color class intersects each set V_i. Indeed , any color class containing a vertex of B_3 cannot contain any vertex of B_0 or of B_2, and since this color class must have a vertex in V_1 and in V_2 it must contain a vertex in $A_0 \cup A_2$ and a vertex in $A_1 \cup A_3$. But this is impossible as each vertex in the first union is adjacent to each one in the second union, completing the proof of the claim.

Thus $s\chi(G) > 3r$ and as the maximum degree in G is $2r$ this shows that $s\chi(2r) > 3r$, completing the proof. $\Box$

Next we prove the following statement, which will be needed later, and which is also mentioned without a proof in [1].

Proposition 2.2 *For any two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on the same set of vertices, where G_1 is a union of r matchings and G_2 is a vertex disjoint union of cliques of size 2^r each, the chromatic number of the graph $G = (V, E_1 \cup E_2)$ is 2^r.*

Proof We apply induction on r. For $r = 1$, G is just a union of two matchings and hence its chromatic number is 2, as claimed. Assuming the result holds for $r - 1$ let us prove it for r. Let G_1 be a union of the r edge disjoint matchings $M_1, \ldots, M_r$. Let M be a matching in G_2 containing precisely 2^{r-1} edges in each 2^r-clique from those in G_2. The graph $(V, M_r \cup M)$ is a union of two matchings and is hence two colorable. Let $c : V \mapsto \{0, 1\}$ be a proper 2-vertex coloring of this graph. Note that exactly half of the vertices in each 2^r-clique in G_2 are colored 0 (and exactly half are colored 1) in this coloring. Let G_2' be the graph obtained from G_2 by splitting each clique of G_2 into two disjoint cliques according to the coloring c as follows; if K is the set of vertices of such a clique then define $K_i = K \cap c^{-1}(i)$ for $i = 0, 1$ and take the two cliques on K_0 and on K_1. Let G_1' be the union of the $r-1$ matchings $M_1, \ldots, M_{r-1}$. By the induction hypothesis the graph whose edges are all edges of G'_1 and all edges of G'_2 is 2^{r-1}-colorable. Let d be a proper 2^{r-1} vertex-coloring of this graph. One can easily check that the assignment of the ordered pair $(c(v), d(v))$ for each vertex v of G is a proper 2^r- vertex coloring of G. This completes the proof. □

Corollary 2.3 *For every d, $s\chi(d) \leq 2^{d+1}$.* □

3 Even Splittings of graphs

In this section we prove the following theorem, which may be interesting in its own right.

Theorem 3.1 *For every $\epsilon > 0$ there exists a constant $c_1 = c_1(\epsilon)$ such that the following holds. For any graph $G = (V, E)$ with maximum degree at most d and for any partition $V = V_1 \cup \ldots \cup V_r$ of V into sets of size $c_2 2^j$ each, where $d/(2^j) \geq c_1$ there is a partition of each set V_i into $J = 2^j$ subsets $V_{i,1}, \ldots, V_{i,J}$, such that each $V_{i,l}$ has precisely c_2 elements and for every l, $1 \leq l \leq J$, the maximum degree of the induced subgraph of G on the set $V_{1,l} \cup V_{2,l} \cup \ldots \cup V_{r,l}$ is at most $(1+\epsilon)d/(2^j)$.*

The proof of the above theorem is probabilistic, and applies the Lovász Local Lemma, proved in [5], which is the following.

Lemma 3.2 (The local lemma [5], see also [8]) *Let $A_1, \ldots, A_n$ be events in an arbitrary probability space. Suppose each A_i is mutually independent of all but at most b other events A_j and suppose the probability of each A_i is at most p. If $ep(b+1) < 1$ then with positive probability none of the events A_i holds.* □

There are two difficulties in trying to prove theorem 3.1 by applying the local lemma. The first one is that we cannot partition the set of vertices of G into J classes by letting each vertex choose randomly and independently its class, since we need to partition each set V_i into equal classes. This may cause more dependencies than we may allow. The second difficulty is that we cannot obtain the desired partition in one step since, again, this would cause too many dependencies. We overcome the latter difficulty by obtaining the partition in j halving steps, and the former one by choosing the random partition in each step in a special manner. This is done in the following lemma.

Lemma 3.3 *For any graph $G = (V, E)$ with maximum degree at most d, where $d \geq 2$, and for any partition $V = V_1 \cup \ldots \cup V_r$ of V into sets of size $2s$ each, there is a partition of each set V_i into 2 subsets $V_{i,1}$ and $V_{i,2}$, such that each $V_{i,l}$ has precisely s elements and for every l, $1 \leq l \leq 2$, the maximum degree of the induced subgraph of G on the set $V_{1,l} \cup V_{2,l} \cup \ldots \cup V_{r,l}$ is at most $d/2 + 2\sqrt{d \log d}$. (Here, and from now on, all logarithms are in the natural base e).*

Proof Let us choose an arbitrary perfect matching in each of the sets V_i, i.e., an arbitrary set of s vertex disjoint edges in each V_i, and let M denote the perfect matching consisting of all these matchings. (Note that M does not have to contain edges of G; it is simply a matching in the complete graph on V which matches the vertices of each set V_i among themselves.) We define a random coloring $f : V \mapsto \{1, 2\}$ by choosing, for each edge u, v of M, randomly and independently one of the following two possibilities, taken with equal probability: Either $f(u) = 1$ and $f(v) = 2$ or $f(u) = 2$ and $f(v) = 1$. For each i, $1 \leq i \leq r$, define $V_{i,1} = V_i \cap f^{-1}(1)$ and $V_{i,2} = V_i \cap f^{-1}(2)$. Clearly each of the sets $V_{i,l}$ has precisely s elements. For $l = 1, 2$, let G_l be the induced subgraph of G on $V_{1,l} \cup \ldots \cup V_{r,l}$. Each vertex v of G belongs to G_l for some $l \in \{1, 2\}$. Let A_v be the event that the degree of v in G_l is greater than $d/2 + 2\sqrt{d \log d}$. Observe that if none of the events A_v holds then our partition satisfies the assertion of the lemma. Hence, in order to complete the proof it suffices to show, using the local lemma, that with positive probabilty no event A_v holds. Fix a vertex v of G and consider the event A_v. Suppose $f(v) = l$, i.e., v is in G_l. If v is matched by the matching M to a neighbor u of v, then $f(u)$ is not l and hence u is not a neighbor of v in G_l. Similarly, if two neighbours of v are matched to each other by M then exactly one of them is a neighbor of v in G_l. Let T be the set of all neighbors of v in G which are matched by M to vertices which are neither v nor one of its neighbors. Let t be the cardinality of T. Clearly $t \leq d$ and by the last few sentences the degree of v in G_l is at most $(d - t)/2$ plus the number of members of T that belong to G_l.However, by our random choice, this number is a binomial random variable with parameters t and $1/2$.By the standard estimates for Binomial distributions (see, e.g. [8], page 29), it follows that for every v

$Prob(A_v) \leq e^{-2(2\sqrt{d\log d})^2/t} \leq d^{-8}$.

Clearly each event A_v is mutually independent of all the events A_u but those for which either v or one of its neighbors is incident with the same edge of M as either u or one of its neighbors. Since there are less than $2(d+1)^2$ such vertices u and since $ed^{-8}2(d+1)^2 < 1$ we conclude, by lemma 3.2, that with positive probability no evet A_v holds. Hence, there is a coloring f for which no A_v holds, completing the proof of the lemma. □.

Proof of Theorem 3.1 Given $\epsilon > 0$ let $c_1 = c_1(\epsilon)$ satisfy

$$c_1 \geq \frac{512}{((1+\epsilon)^{1/3} - 1)^3} \tag{1}$$

and

$$\forall x \geq c_1, \ 2\sqrt{x \log x} \leq x^{2/3} \tag{2}$$

We prove the assertion of Theorem 3.1 with this c_1. Given a graph $G = (V, E)$ with maximum degree at most d and a partition of V into r pairwise disjoint subsets $V_1, \ldots, V_r$ of cardinality $c_2 2^j$ each, as in the hypotheses of the theorem, we apply Lemma 3.3 to G and split it into two induced subgraphs G_1 and G_2, each containing exactly half of the vertices of each V_i. By Lemma 3.3 there is such a splitting in which the maximum degree in each G_i does not exceed $d_1 = d/2 + 2\sqrt{d \log d}$. The set of vertices of each of the two graphs $G_l = (V^l, E^l)$ is partitioned into the r pairwise disjoint sets of equal cardinality $V_i \cap V^l$. By applying Lemma 3.3 again to each of these two graphs we obtain a splitting of G into four induced subgraphs. Continuing in this manner we obtain, after j such halving steps, a partition of G into $2^j = J$ induced subgraphs, each containing exactly c_2 vertices from each set V_i. Define a sequence d_q, $(0 \leq q \leq j)$ as follows; $d_0 = d$ and for all $q < j$: $d_{q+1} = d_q/2 + 2\sqrt{d_q \log d_q}$. Clearly $d_q \geq d/(2^q)$, and hence, by (2), $d_{q+1} \leq d_q/2 + d_q^{2/3}$ for all $q < j$. Moreover, by Lemma 3.3, d_q is an upper bound for the maximum degree in any of the 2^q induced subgraphs of G obtained after q halving steps.

In order to complete the proof it thus remains to show that $d_j \leq (1+\epsilon)d/(2^j)$.

Clearly $d_{q+1} \leq d_q/2 + d_q^{2/3} \leq \frac{1}{2}(d_q^{1/3} + 2)^3$. Hence, by taking cube roots and subtracting $\frac{2}{2^{1/3}-1}$ from both sides

$d_{q+1}^{1/3} - \frac{2}{2^{1/3}-1} \leq \frac{1}{2^{1/3}}(d_q^{1/3} + 2) - \frac{2}{2^{1/3}-1} = \frac{1}{2^{1/3}}(d_q^{1/3} - \frac{2}{2^{1/3}-1})$.

Therefore

$d_j^{1/3} - \frac{2}{2^{1/3}-1} \le \frac{1}{2^{j/3}}(d_0^{1/3} - \frac{2}{2^{1/3}-1})$,

and, since $d_0 = d$ and $2^{1/3} - 1 > 1/4$,

$d_j^{1/3} \le \frac{d^{1/3}}{2^{j/3}} + 8 \le (1+\epsilon)^{1/3}\frac{d^{1/3}}{2^{j/3}}$.

The last inequality follows from (1) and the assumption that $d/(2^j) \ge c_1$.

Thus $d_j \le (1+\epsilon)d/(2^j)$, completing the proof of the theorem. □

4 The proof of the main result

In this section we combine Corollary 2.3 and Theorem 3.1 to establish our main result.

Proof of Theorem 1.1 Let $c_1 \ge 1$ be a number for which the assertion of Theorem 3.1 with $\epsilon = 1$ holds. We prove the assertion of Theorem 1.1 with $c = 2^{4c_1+1}$. Let $G = (V, E)$ be a graph whose maximum degree is at most d. We must show that $s\chi(G) \le cd$. Let j be the maximum integer such that $d/(2^j) \ge c_1$. Observe that $2^j \le d$ and $d/2^j \le 2c_1$. Define also $c_2 = c = 2^{4c_1+1}$. To complete the proof we show that $s\chi(G) \le c_2 2^j \le cd$. Clearly we may assume that the number of vertices of G is divisible by $c_2 2^j$, since otherwise we simply add to G an appropriate number of isolated vertices. Let $V_1, \ldots, V_r$ be a partition of the set V of vertices of G into pairwise disjoint sets each of size $c_2 2^j$. To complete the proof it suffices to show that there is a proper vertex coloring of G in which each color class contains exactly one vertex in each V_i. By Theorem 3.1 there is a partition of the set of vertices of G into $J = 2^j$ pairwise disjoint classes $V^1, \ldots, V^J$, each containing exactly c_2 vertices of each V_i, such that for each l, $1 \le l \le J$, the maximum degree of the induced subgraph of G on V^l is at most $(1+\epsilon)d/(2^j) = \frac{2d}{2^j} \le 4c_1$. For each l, $1 \le l \le J$, let G^l be the induced subgraph of G on V^l. Since $c_2 = 2^{4c_1+1}$, Corollary 2.3 implies that $s\chi(G^l) \le c_2$ for each l. For $1 \le i \le r$ and $1 \le l \le J = 2^j$, define $V_{i,l} = V_i \cap V^l$. For each l, $1 \le l \le J$, the sets $V_{1,l}, \ldots, V_{r,l}$ form a partition of the vertex set V^l of G^l into pairwise disjoint sets of cardinality c_2 each. Since $s\chi(G^l) \le c_2$, there is a proper coloring of G^l in which every color class contains exactly one vertex from each of the sets $V_{i,l}$. Combining these $J = 2^j$ colorings, where the J sets of colors used are pairwise disjoint, we obtain a $c_2 2^j$-proper vertex coloring of G in which every color class contains exactly one vertex from each of the sets V_i. Thus $s\chi(G) \leq c_2 2^j \leq cd$, completing the proof of the theorem. $\Box$.

5 Concluding remarks

1). In [1] and, independently in [6] it is shown that there is a constant c such that for any graph G with maximum degree d and every partition of the set of its vertices into pairwise disjoint subsets each of size at least cd, there is an independent set of G containing a vertex from each of these subsets. Theorem 1.1 is clearly a strengthening of this result.

2). A theorem of Hajnal and Szemerédi [7] asserts that any graph G with n vertices and with maximum degree d has a proper $d+1$-vertex coloring with almost equal color classes, i.e., a coloring in which each color class has either $\lfloor n/(d+1) \rfloor$ or $\lceil n/(d+1) \rceil$ vertices. Theorem 1.1 shows that if we allow to increase the number of colors by a constant factor we can obtain a coloring with almost equal color classes satisfying several additional severe restrictions.

3). It would be interesting to determine the best possible constant c in Theorem 1.1. This constant is probably much closer to 3/2 than to the huge upper bound that can be derived using our approach. It is worth noting that our approach suffices to prove, e.g., that for every d , $s\chi(d) \leq bd^2$ for a rather small constant b.

Another interesting problem is that of exhibiting a polynomial time (deterministic or randomized) algorithm that gets as an input a graph G with maximum degree d and a partition of its set of vertices into pairwise disjoint subsets of cardinality cd each, and produces a proper vertex coloring of this graph in which every color class contains exactly one element of each of these subsets. This problem has been open when the present paper has been submitted, but as is the case in several other known proofs in which the local lemma is used, it can be solved by applying the recent technique of J. Beck [3] (see also [2]).

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