ON SUMS AND PRODUCTS ALONG THE EDGES, II

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Abstract. This note is an addendum to an earlier paper of the authors [1]. We describe improved constructions addressing a question of Erdős and Szemerédi on sums and products of real numbers along the edges of a graph. We also add a few observations about related versions of the problem.

1. Introduction

In this note we describe an improved construction addressing a question of Erdős and Szemerédi about sums and products along the edges of a graph. We also mention some related problems. The main improvement is obtained by a simple modification of the construction in [1] which works for real numbers, instead of the integers considered there.

In their original paper Erdős and Szemerédi [4] considered sum and product along the edges of graphs. Let G_n be a graph on n vertices, $v_1, v_2, \ldots, v_n$, with n^{1+c} edges for some real $c > 0$. Let $\mathcal{A}$ be an n-element set of real numbers, $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$. The *sumset of* $\mathcal{A}$ *along* G_n, denoted by $\mathcal{A} +_{G_n} \mathcal{A}$, is the set $\{a_i + a_j | (i,j) \in E(G_n)\}$. The product set along G_n is defined similarly,

$$\mathcal{A} \cdot_{G_n} \mathcal{A} = \{a_i \cdot a_j | (i,j) \in E(G_n)\}.$$

The Strong Erdős-Szemerédi Conjecture, which was proved in [5] for the special case of n positive integers of size at most $n^{O(1)}$, but was refuted in its original form in [1], is the following.

Conjecture 1. [4] *For every $c >$ and $\varepsilon > 0$, there is a threshold, n_0, such that if $n \geq n_0$ then for any n-element subset of reals $\mathcal{A} \subset \mathbb{R}$ and any graph G_n with n vertices and at least n^{1+c} edges*

$$|\mathcal{A} +_{G_n} \mathcal{A}| + |\mathcal{A} \cdot_{G_n} \mathcal{A}| \geq |\mathcal{A}|^{1+c-\varepsilon}.$$

Now the question is to find dense graphs with small sumset and product set along the edges. Here we extend the construction in [1]. The improvement follows by considering real numbers, instead of integers only.

2. Constructions

2.1. **Sum-product along edges with real numbers.** Here we extend our earlier construction so that we get better bounds in a range of edge densities. In our previous paper for arbitrary large m_0, we constructed a set of integers, $\mathcal{A}$, and a graph on $|\mathcal{A}| = m \geq m_0$ vertices, G_m, with $\Omega(m^{5/3}/\log^{1/3} m)$ edges such that

$$|\mathcal{A} +_{G_m} \mathcal{A}| + |\mathcal{A} \cdot_{G_m} \mathcal{A}| = O\left((|\mathcal{A}| \log |\mathcal{A}|)^{4/3}\right).$$

Thus we had a graph on m vertices and roughly m^{2-c} edges with roughly m^{2-2c} sums and products along the edges for $c = 1/3$. In the following construction we show a similar bound in a range covering all $1/3 \leq c \leq 2/5$. In what follows it is convenient to ignore the logarithmic terms. We thus use from now on the common notation $f = \tilde{O}(g)$ for two functions $f(n)$ and $g(n)$ to denote that there are absolute positive constants c_1, c_2 so that $f(n) \leq c_1 g(n) (\log g(n))^{c_2}$ for all admissible values of n. The notation $f = \tilde{\Omega}(g)$ means that $g = \tilde{O}(f)$ and $f = \tilde{\Theta}(g)$ denotes that $f = \tilde{\Omega}(g)$ and $g = \tilde{O}(f)$.

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Theorem 2. *For arbitrary large m_0, and parameter α, where $0 \le \alpha \le 1/6$, there is a set of reals, $\mathcal{A}$, and a graph on $|\mathcal{A}| = m \ge m_0$ vertices, G_m, with*

$$\tilde{\Omega}\left(m^{2-\frac{2-2\alpha}{5}}\right)$$

edges such that

$$|\mathcal{A} +_{G_m} \mathcal{A}| + |\mathcal{A} \cdot_{G_m} \mathcal{A}| = \tilde{O}\left(|\mathcal{A}|^{2-\frac{4-4\alpha}{5}}\right).$$

Proof: It is easier to describe the construction using prime numbers only. We get a slightly larger exponent in the hidden logarithmic factor, but we are anyway ignoring these factors here. The set of primes is denoted by $\mathbb{P}$ here. We define the set $\mathcal{A}$ first and then the graph using the parameter α.

$$\mathcal{A} := \left\{ \frac{su\sqrt{w}}{t\sqrt{v}} \mid u, v, w, s, t \in \mathbb{P} \text{ distinct and } s, t \le n^{\alpha}, v, w \le n^{\frac{1-6\alpha}{5}}, u \le n^{\frac{3+2\alpha}{5}} \right\}.$$

It is clear that distinct choices of 5-tuples u, v, w, s, t lead to distinct reals. Thus with this choice of parameters the size of $\mathcal{A}$ is $\tilde{\Theta}(n)$. We are going to define a graph G_m with vertex set $\mathcal{A}$, where $|\mathcal{A}| = m = \tilde{\Theta}(n)$. Two elements, $a, b \in \mathcal{A}$ are connected by an edge if in the definition of $\mathcal{A}$ above $a = \frac{su\sqrt{w}}{t\sqrt{v}}$ and $b = \frac{tz\sqrt{v}}{s\sqrt{w}}$. Since the degree of every vertex here is $\tilde{\Theta}(n^{\frac{3+2\alpha}{5}})$ the number of edges is

$$\tilde{\Omega}\left(m^{\frac{8+2\alpha}{5}}\right) = \tilde{\Omega}\left(m^{2-\frac{2-2\alpha}{5}}\right).$$

The products of pairs of elements of $\mathcal{A}$ along an edge of G_m are integers of size at most

$$n^{2\frac{3+2\alpha}{5}} = n^{2-\frac{4-4\alpha}{5}} = \tilde{O}\left(m^{2-\frac{4-4\alpha}{5}}\right).$$

The sums along the edges are of the form

$$\frac{su\sqrt{w}}{t\sqrt{v}} + \frac{tz\sqrt{v}}{s\sqrt{w}} = \frac{s^2wu + t^2vz}{st\sqrt{vw}}.$$

The number of possibilities for the denominator is at most $n^{\frac{2-2\alpha}{5}}$ and the numerator is a positive integer of size at most $2n^{\frac{4+6\alpha}{5}}$, hence the number of sums is at most

$$O(n^{\frac{6+4\alpha}{5}}) = O(n^{2-\frac{4-4\alpha}{5}}) = \tilde{O}\left(m^{2-\frac{4-4\alpha}{5}}\right).$$

□

Based on this construction one can easily get examples for sparser graphs, simply taking smaller copies of G_m and leaving other vertices isolated.

Theorem 3. *For every parameters $0 \le \nu \le 3/5$ and n_0 there are $n > n_0$, an n-element set of reals, $\mathcal{A} \subset \mathbb{R}$, and a graph H_n with $\tilde{\Omega}(n^{1+\nu})$ edges such that*

$$|\mathcal{A} +_{H_n} \mathcal{A}| + |\mathcal{A} \cdot_{H_n} \mathcal{A}| = \tilde{O}\left(|\mathcal{A}|^{3(1+\nu)/4}\right).$$

Proof: The construction of Theorem 2 with $\alpha = 0$ supplies a set of m reals and a graph with $\tilde{\Omega}(m^{8/5})$ edges so that the number of sums and products along the edges is at most $\tilde{O}(m^{6/5})$. Take this construction with $m = n^{5(1+\nu)/8} (\le n)$ and add to it $n - m$ isolated vertices assigning to them arbitrary distinct reals that differ from the ones used already. □

A similar statement holds for integers too.

Theorem 4. *For every parameters $0 \le \nu \le 2/3$ and n_0 there are $n > n_0$, an n-element set of integers $\mathcal{A}$, and a graph H_n with $\tilde{\Omega}(n^{1+\nu})$ edges such that*

$$|\mathcal{A} +_{H_n} \mathcal{A}| + |\mathcal{A} \cdot_{H_n} \mathcal{A}| = \tilde{O}\left(|\mathcal{A}|^{4(1+\nu)/5}\right).$$

This follows as in the real case by starting with the construction of [1] that gives a set of m integers and a graph with $\tilde{\Omega}(m^{5/3})$ edges so that the number of sums and products along the edges is at most $\tilde{O}(m^{4/3})$. This construction with $m = n^{3(1+\nu)/5} \le n$ together with $n - m$ isolated vertices with arbitrary $n - m$ new integers implies the statement above.

2.2. **Matchings.** A particular variant of the sum-product problem for integers is the following:

Problem 5. *Given two n-element sets of integers, $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ let us define a sumset and a product set as*

$$S = \{a_i + b_i | 1 \le i \le n\} \text{ and } P = \{a_i \cdot b_i | 1 \le i \le n\}.$$

Erdős and Szemerédi conjectured that

$$|P| + |S| = \Omega(n^{1/2+c}) \tag{1}$$

for some constant $c > 0$.

The best known lower bound is due to Chang [3], who proved that

$$|P| + |S| \ge n^{1/2} \log^{1/48} n.$$

It was shown recently in [7] that under the assumption of a special case of the Bombieri-Lang conjecture [2], one can take $c = 1/10$ in equation (1), i.e. $|P| + |S| = \Omega(n^{3/5})$, even for multi-sets.

Theorem 6. [7] *Let $M = \{(a_i, b_i) | 1 \le i \le n\}$ be a set of distinct pairs of integers. If P and S are defined as above, then under the hypothesis of the Bombieri-Lang conjecture $|P| + |S| = \Omega(n^{1/2+c})$ with $c = 1/10$.*

If multisets are allowed and the only requirement is that the pairs assigned to distinct edges of the matching are distinct, then any construction of a graph with n edges yields a construction of a matching of size n. It thus follows from [1, Theorem 3] (or from Theorem 4 here) that for the multi-set version there is, for arbitrarily large n, an example of a matching M of size n as above, with n distinct pairs of integers (a_i, b_i), so that $|P| + |S| = \tilde{O}(n^{4/5})$. This shows that the statement of Theorem 6 cannot be improved beyond an extra $1/5$ in the exponent.

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