Poker, Chance and Skill

Noga Alon *

1 Introduction

The question if poker is a game of skill or a game of chance received a considerable amount of attention mainly because of its potential legal implications. See, for example, [3] and its many references. Most of the material dealing with the subject focuses on legal issues, and only briefly touches the question from a purely scientific point of view. In the present article we address the question as a scientific one. To do so, we provide a detailed analysis of several simplified models of poker, which can be viewed as toy models of Texas Hold'em, the most popular variant of poker. The advantage of considering these simplified models is that unlike the real game, they are simple enough to allow a precise mathematical analysis, and yet there is every reason to believe that this analysis captures many of the main properties of the far more complicated real game, and enables us to estimate the advantage of skilled players over less skilled ones. The analysis suggests that skill plays an important role in poker. As explained in the second half of the article, this fact, together with the Central Limit Theorem, imply that skill is the major component in deciding the results of a long sequence of hands. As the common practice is to play many hands, the conclusion is that poker is predominantly a game of skill.

The article is organized as follows. In Section 2 we describe the rules of Texas Hold'em which is probably the most popular poker game played in casinos and card-rooms throughout the world, as well as in online poker sites. Section 3 contains the basic probabilistic information regarding the odds of the main possibilities in the game. In Section 4 we give a detailed analysis of several simplified versions of poker. Section 5 contains a discussion of

*Schools of Mathematics and Computer Science, Tel Aviv University, Tel Aviv 69978, Israel. Email: nogaa@tau.ac.il.

the relevance of the Law of Large Numbers, or more specifically, the Central Limit Theorem, to the determination of the success of skilled and less skilled players in a sequence of games. This is illustrated by considering the simplified versions introduced in Section 4. A summary and concluding remarks appear in the final section 6.

2 The Game

There are many versions of poker, here we focus on Texas Hold'em (often called Hold'em, for short). The game is usually played with at most 10 (and at least 2) players. This is the most popular member of a class of poker games known as community card games, which all bear some similarity to each other. Like most variants of poker, the objective in hold'em is to win pots, where a pot is the sum of the money bet by all players in a hand. A pot is won either at the showdown by forming the best five card poker hand out of the seven cards available, or by betting to cause other players to fold and abandon their claim to the pot. The objective of a player is not to win the maximum number of individual pots, but rather to make mathematically correct decisions in order to maximize the expected net amount won in the long run.

Here is a rough brief description of the game: Each player is dealt two cards and this is followed by a round of betting. Then the dealer spreads three cards face up (called the flop) in the middle, and this is followed by a second round of betting. The dealer places a fourth card (called the turn) face up and another round of betting follows. Finally, the dealer places a fifth card (called the river) face up and the last round of betting takes place. Each player who has not folded during the betting rounds gets the best hand of five cards among his own two cards plus the five community cards in the center.

A more detailed account follows. See, e.g., [9] for several variants and further details. Hold'em is often played using small and big blind bets. A dealer button is used to represent the player in the dealer position; the dealer button rotates clockwise after each hand, changing the position of the dealer and blinds. The small blind is posted by the player to the left of the dealer and is usually equal to half of the big blind. The big blind, posted by the player to the left of the small blind, is equal to the minimum bet.

There are several variations on the betting structure, here we describe limit hold'em. In this version bets and raises during the first two rounds of betting (pre-flop and flop) must be equal to the big blind; this amount is called the small bet. In the next two rounds of betting (turn and river), bets and raises must be equal to twice the big blind; this amount is called the big bet.

A play of a hand begins with each player being dealt two cards face down from a standard deck of 52 cards. These cards are the player's hole or pocket cards, they are the only cards each player will receive individually, and they will only (possibly) be revealed at the showdown, making hold'em a closed poker game. After the pocket cards are dealt, there is a "pre-flop" betting round, beginning with the player to the left of the big blind (or the player to the left of the dealer, if no blinds are used) and continuing clockwise. A round of betting continues until every player has either folded, put in all of their chips, or matched the amount put in by each other active player.

After the pre-flop betting round, assuming there remain at least two players taking part in the hand, the dealer deals a flop; three face-up community cards. The flop is followed by a second betting round. This and all subsequent betting rounds begin with the player to the dealer's left and continue clockwise.

After the flop betting round ends a single community card (called the turn) is dealt, followed by a third betting round. A final single community card (called the river) is then dealt, followed by a fourth betting round and the showdown, if necessary.

If a player bets and all other players fold, then the remaining player is awarded the pot and is not required to show his hole cards. If two or more players remain after the final betting round, a showdown occurs. On the showdown, each player plays the best five-card hand he can make from the seven cards comprising his two pocket cards and the five community cards. A player may use both of his own two pocket cards, only one, or none at all, to form his final five-card hand. If the five community cards form the player's best hand, then the player is said to be playing the board and can only hope to split the pot, since each other active player can also use the same five cards to construct the same hand.

If the best hand is shared by more than one player, then the pot is split equally among them. The best hand is determined according to the ranking described below. If the significant part of the hand involves fewer than five cards, (such as two pair or three of a kind), then the additional cards (called kickers) are used to settle ties. Note that only the card's numerical rank matters; suit values are irrelevant in Hold'em.

The ranking of the hands is as follows:

- Royal Flush (the top hand): The five highest cards, the 10 through the Ace, all five of the same suit. A royal flush is also an ace-high straight flush.
- Straight Flush: Any five cards of the same suit in consecutive numerical order.
- Four of a Kind: Four cards of the same denomination.
- Full House: Any three cards of the same denomination, plus any pair of a different denomination. Ties are broken first by the three of a kind, then the pair.
- Flush: Any five non-consecutive cards of the same suit.
- Straight: Any five consecutive cards of mixed suits. Ace can be high or low.
- Three of a Kind: Three cards of the same denomination.
- Two Pair: Any two cards of the same denomination, plus any other two cards of the same denomination. If both hands have the same high pair, the second pair wins. If both pairs tie, the high (fifth) card wins.
- Pair: Any two cards of the same denomination. In a tie, the high card wins.
- High Card: If no other hand is achieved, the highest card held wins.

Texas hold'em (usually with a no-limit betting structure) is played as the main event in many of the famous tournaments, including the World Series of Poker's Main Event. Traditionally, a poker tournament is played with chips that represent a player's stake in the tournament. Standard play allows all entrants to "buy-in" a fixed amount and all players begin with an equal value of chips. Play proceeds until one player has accumulated all the chips in play. The money pool from the players "buy-ins" are redistributed to the players in relation to the place they finished in the tournament. Usually only a small percentage of the players receive any money, with the majority receiving nothing. As a result the strategy in poker tournaments can be different from that in a cash game.

3 Odds and Probabilities

Some familiarity with the odds of the various possible combinations in poker is necessary, though certainly not sufficient, for skilled poker play. The ranking of hands in poker is determined according to their frequencies as 5-card poker hands. These frequencies can be easily computed. There are $\binom{52}{5} = 2,598,960$ different poker hands. Among these 4 are Royal Flush and 36 are non-royal Straight Flush. These and the numbers of the other hands are given below.

The numbers of 5-card poker hands:

- Royal Flush: $\binom{4}{1} = 4$
- Straight (non-royal) Flush: $\binom{9}{1}\binom{4}{1} = 36$
- Four of a Kind: $\binom{13}{1}\binom{4}{4}\binom{48}{1} = 624$
- Full House: $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 3,744$
- Flush: $\binom{13}{5}\binom{4}{1} - 40 = 5,108$
- Straight: $\binom{10}{1}\binom{4}{1}^5 - 40 = 10,200$
- Three of a Kind: $\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2 = 54,912$
- Two Pair: $\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1} = 123,552$
- Pair: $\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3 = 1,098,240$
- High Card: $[\binom{13}{5} - 10](4^5 - 4) = 1,302,540$

Thus, for example, a fraction of $\frac{123,552}{2,598,960} = 0.047539$ of all 5 card hands form a Two Pair.

More relevant to Hold'em is the corresponding information for 7-card hands. Their total number is $\binom{52}{7} = 133,784,560$. The number of hands for each possibility of the best 5 card subset is also not difficult to compute. This is done in [1] , and appears below, together with the probability of each possibility in a random 7-card hand.

The numbers and frequencies of 7-card poker hands:

Royal Flush	$4,324$	.0000323
Straight (non-royal) Flush	$37,260$	.000278
Four of a Kind	$224,848$	.0017
Full House	$3,473,184$	.026
Flush	$4,047,644$	.030
Straight	$6,180,020$	.046
Three of a Kind	$6,461,620$	.048
Two Pair	$31,433,400$	.235
Pair	$58,627,800$	.438
High Card	$23,294,460$	.174

Hence, when playing Hold'em a player should expect to get Three of a Kind or higher once in about 20 hands, and Four of a Kind once in about 600 hands.

During the game, a player should be capable of estimating the probability of improving his hand when the turn or river community cards will be dealt. If, for example, the player holds two diamonds, and the flop contains two other diamonds, then there are 9 additional diamonds in the deck, implying that the probability that the next community card will be a diamond is $9/47$, and in case it will not, the probability that the last community card will be a diamond is $9/46$. A player should also always be aware of the expected winning amount in a game; in general one should bet when the expected value of the gain (which is the amount in the pot after the bet, times the probability of winning) is greater than the wager. Of course, even if the player knows the precise probability, this should be modified from time to time in order not to reveal the strategy of the player; bluffing is a crucial part of the game as will be clear from the analysis of the simplified versions considered in the next section.

4 Simple Variants

There is a significant amount of literature on various toy models of poker, starting with the variants discussed in the classical book of Von Neumann and Morgenstern [8]. See, for example, [7], [5], [6]. In most of these articles, however, the authors try to find the best strategy of the players assuming they play optimally. Our treatment here is different, as the main intention is to assess the significance of skill in the game. We therefore investigate the case in which one player is more skilled than the other(s). Although the models we suggest are vast simplifications of the real game, they do seem to capture many of the properties of real poker.

4.1 The Basic Variant

Consider a version of Hold'em in which each player gets two face down pocket cards, the flop, turn and river community cards are spread face up in the middle, and only then there is one round of betting. Suppose, further, that in this round each player is allowed to either fold, or bet 1 chip, and these decisions are made simultaneously by all players. If all players fold then nothing happens, if at least one player bets, then the active player with the highest hand wins the pot. Given the 5 community cards, there are $m = \binom{47}{2} = 1081$ possibilities for the two pocket cards of each player, and ignoring equalities, there is a linear order among them. Therefore, a perfect player that sees the community cards and his hole cards, knows precisely the rank of his hole cards among the 1081 possibilities, and hence can compute, in principle, the precise probability that his hand is the highest among all hands of the participants. It is worth noting that knowing these precise probabilities in all cases is not an easy matter, and is probably beyond the ability of a human being, as this requires to memorize a huge table of ranks representing all possible values of the community cards and the player's hole cards. Yet, it seems that skilled poker players can estimate well the probability in each case. Ignoring the (rather negligible) effect of the fact that the pairs forming the pocket cards of all players should be disjoint, one can model this situation by a game in which the players are dealt random distinct numbers between $m = 1081$ (the strongest possibility for the pocket cards given the community cards) and 1 (the weakest possibility). As m is a large number this can be further simplified by considering the case in which each player is dealt his hole number; a uniformly chosen random real number in the unit interval $[0, 1]$, where a higher number is considered better than a lower one. In what follows we refer to this game as the *basic game*.

We start with the simplest case, in which there are two players, A (Alice) and B (Bob). In this case, Alice gets a uniform random number $x_A \in [0, 1]$, and Bob gets a uniform random number $x_B \in [0, 1]$, where the choices of x_A, x_B are independent. Each player knows his/her own number, but not the one of the other player, and they have to choose between folding and betting 1 chip.

Suppose that Bob is an unskilled player, who plays randomly. That is, for any value of x_B, Bob decides to fold with probability 1/2, and decides to bet with probability 1/2. Alice, who is a skilled player, suspects that this is Bob's strategy, and chooses her strategy in order to ensure maximum expected gain in the game against Bob. To determine the strategy of Alice, let us consider how she should behave when her pocket number is $x_A = x$. If she decides to bet, then the expected number of chips she wins (including her own chip) is

$$\frac{1}{2}1 + \frac{1}{2}x2.$$

Indeed, with probability 1/2 Bob will fold, and in this case Alice will win her single chip, giving the first term above. With probability 1/2 Bob will decide to bet, in this case with probability x his number x_B lies in $[0, x)$ and is thus smaller than Alice's number, and if so Alice will win two chips. This gives the second term. Alice should bet if and only if her expected win exceeds her cost, which is the 1 chip she bets. Thus, she should choose to bet if and only if $\frac{1}{2}1 + \frac{1}{2}x2 \geq 1$, that is, if her hole number $x = x_A$ is at least 1/2.

If, indeed, Bob and Alice follow the above strategies, then at least one of them folds with probability $1 - \frac{1}{2} \cdot \frac{1}{2} = 3/4$, and thus, with probability 3/4 the expected net gain of Alice is 0. The probability that Alice's net gain is 1 is

$$\frac{1}{2}\int_{1/2}^{1} x dx = \frac{3}{16},$$

and the probability that Alice's net gain is -1 is

$$\frac{1}{2}\int_{1/2}^{1} (1 - x) dx = \frac{1}{16}.$$

Altogether, in a single hand, the expected value of the random variable X describing Alice's net gain is

$$E(X) = \frac{3}{16} \cdot 1 + \frac{1}{16} \cdot (-1) = \frac{1}{8},$$

and its variance is

$$Var[X] = E(X^2) - (E(X))^2 = \frac{3}{16} + \frac{1}{16} - (\frac{1}{8})^2 = \frac{15}{64}.$$

We have thus proved the following, where here and in what follows we refer to the player playing randomly as the unskilled player.

Proposition 4.1 *In a single hand of the basic game with two players, a skilled one and an unskilled one, the expected value of the net gain of the skilled player is* $1/8$ *and the variance of this net gain is* $15/64$.

Note that, not surprisingly, the skilled player has a significant advantage over the unskilled one.

4.2 The Importance of Being Unpredictable

Suppose that Bob and Alice play a sequence of hands of the the basic game described above. Bob is likely to realize that Alice's strategy is better than his random one, and he is also likely to observe that she is betting if and only if her hole number x_A is at least $1/2$. He can thus decide to adopt Alice's winning strategy, and bet if and only if his number x_B is at least $1/2$. However, when he starts doing so, Alice, who is more skilled, realizes that this is the case. She can thus adjust her strategy and choose the optimal response to the new strategy of Bob. It is not difficult to modify the previous computation to this case. Observe, first, that if $x_A < 1/2$, then Alice should not bet, as with the new strategy of Bob this can never lead to any winning. If Alice hole number is $x \geq 1/2$, and she decides to bet, then the expected amount she wins is

$$\frac{1}{2} \cdot 1 + (x - \frac{1}{2})2 = 2x - \frac{1}{2}.$$

Indeed, with probability $\frac{1}{2}$ Bob's number x_B will lie in $[0, 1/2]$, he will not bet, and Alice will get her chip back. Similarly, with probability $x - \frac{1}{2}$ Bob's number will lie in $[\frac{1}{2}, x)$ and in this case Alice's win will be 2. Therefore, Alice should bet if and only if $2x - \frac{1}{2} \geq 1$, that is, if $x \geq \frac{3}{4}$. In case Bob and Alice play according to these new strategies, then the random variable describing Alice's net gain is 0 with probability $1 - \frac{1}{2} \cdot \frac{1}{4} = \frac{7}{8}$, it is $+1$ with probability $\int_{3/4}^{1} (x - \frac{1}{2}) dx = \frac{3}{32}$ and it is -1 with probability $\int_{3/4}^{1} (1 - x) dx = \frac{1}{32}$. This gives the following.

Proposition 4.2 *In a single play of the basic game with two players* A *and* B*, where* A *bets if and only if* $x_A \geq 3/4$ *and* B *bets if and only if* $x_B \geq 1/2$*, the expected value of the net gain of* A *is* $1/16$ *and the variance of this net gain is* $31/256$.

Note that here the losing player is using exactly the same strategy used by the winning player in the previous subsection. This shows that already in this simplified version of the game, a winning player should adjust her strategy to those of the other players. It also shows the importance of bluffing; once your strategy is revealed, the other players can exploit it. These principles hold (in a far more sophisticated way) in real poker; it is crucial for a winning player to stay unpredictable, and to take into account the strategy of the other players.

4.3 More Players

In real poker the number of players is often larger than 2. Consider the basic game in which there are $n+1$ players denoted by $P_0, P_1, \ldots, P_n$. As our objective is to measure the significance of skill, assume that the first player, P_0, is skilled, and all other players are unskilled and play randomly. Therefore, the players are dealt $n+1$ uniform, independent random numbers in $[0,1]$, where x_i is the hole number of P_i, then each of them decides to fold or bet one chip, where all these decisions are taken simultaneously, and finally the active player with the largest number wins the pot. Let us compute the optimal strategy for P_0, assuming all other players play randomly. If $x_0 = x$ and P_0 decides to bet, then the expected amount of chips he wins is

$$\frac{1}{2^n}\sum_{k=0}^{n}(k+1)\binom{n}{k}x^k.$$

Indeed, the probability that exactly k players among the n unskilled ones decide to bet is

$$\frac{\binom{n}{k}}{2^n}.$$

If so, then the probability that all their hole numbers will lie in $[0,x)$ is x^k, and in this case P_0 will win the pot, whose size will be $k+1$. Therefore, P_0 should bet if and only if

$$\frac{1}{2^n}\sum_{k=0}^{n}(k+1)\binom{n}{k}x^k \geq 1.$$

Since

$$\sum_{k=0}^{n}(k+1)\binom{n}{k}x^k = \frac{d}{dx}\Big(\ x(1+x)^n\ \Big) = (1+x)^n + nx(1+x)^{n-1},$$

it follows that P_0 should bet when $x_0 = x$ if and only if

$$(1+x)^n + nx(1+x)^{n-1} \geq 2^n.$$

In particular, for $n = 1$ (two players, one skilled and one unskilled), the skilled player should bet if and only if $(1 + x) + x \geq 2$, that is, if and only if $x \geq 1/2$, as we have already seen in subsection 4.1. If $n = 2$ (three players), the skilled player should bet if and only if $(1+x)^2 + 2x(1+x) \geq 4$, that is, if and only if $x \geq \frac{\sqrt{13}-2}{3} = 0.535..$, and if $n = 9$ (10 players, 9 of whom are unskilled), the skilled player should bet if and only if his hole number x satisfies $(1 + x)^8(10x + 1) \geq 512$, that is, whenever x exceeds 0.685...

Here, too, the mathematical analysis of the simplified model reveals a crucial feature of real poker: a skilled player should adjust his strategy to the number of players. In general, when this number grows, the player should fold more often and bet mostly with stronger hands.

4.4 Blinds and Position

In the basic model considered in subsection 4.1, there is no nontrivial optimal strategy in the sense of Game Theory, that is, if both players play optimally, then their best (mixed) strategy is to keep folding and never bet. Indeed, as a uniformly chosen random number in $[0, 1]$ is strictly smaller than 1 with probability 1, one can show that for any nontrivial betting policy of one of the players, there is a strategy that beats it. The reason for this is that this simplified version of the game ignores the cost of playing and, more crucially, contains no forced bets (called blinds, or ante in real poker) which are necessary to create an initial stake for the players to contest. We thus discuss here a slightly more realistic model of the game, containing a forced blind bet. In order to enable a rigorous analysis, this model is still far from the real game, and yet its analysis illustrates nicely the fact that in real poker the strategy has to be adjusted to the position and the order in which players have to act. Consider, thus, a model in which there are two players. The game starts with a blind bet of 1 chip by the first player, then the 5 community cards as well as the two pocket cards of each player are dealt. The second player can now either fold or bet 3 chips, and the first player can also either fold or raise his bet to 3, where both players make their decisions simultaneously. If both players fold nothing happens, if one player folds and the other bets, then the active player wins the pot, and if both players bet, the higher hand wins the pot. The choice of the numbers 1 and 3 here is arbitrary, and the analysis can be carried out for different numbers in a similar manner. By the discussion in subsection 4.1, assuming the players can memorize a substantial table of possibilities, the game is well approximated by a version in which the first player makes a blind bet of 1, then the players get uniform, independent, random pocket numbers in $[0,1]$, and then the second player either folds or bets 3, and the first either folds or increases his bet to 3. The blind bet alternates between the players, as obviously having to start with it is a disadvantage. We call this version of the game the *basic game with a blind bet*, and analyze it as in subsection 4.1 for two players, a skilled one (Alice) and an unskilled one playing randomly (Bob). There are two cases to consider, depending on the identity of the player posting the blind bet.

Assume, first, that Alice is making the blind bet. If her number is x and she decides to bet, then her expected win is $\frac{1}{2}\cdot 3+\frac{1}{2}x\cdot 6=\frac{3}{2}+3x$. Indeed, with probability $1/2$ Bob folds and then Alice gets back her 3 chips, and with probability $\frac{1}{2}x$ Bob bets and his number is smaller than x, and if so Alice wins 6 chips. Alice should bet if and only if she expects to win at least the cost of increasing her bet. As this cost is 2, she should bet if and only if $\frac{3}{2}+3x\geq 2$, that is, if and only if $x\geq\frac{1}{6}$. If she uses this strategy, then her net gain is -3 with probability $\frac{1}{2}\int_{1/6}^{1}(1-x)dx=\frac{25}{144}$. It is -1 with probability $\frac{1}{2}\cdot\frac{1}{6}=\frac{1}{12}$, 0 with probability $1/2$, and $+3$ with probability $\frac{1}{2}\int_{1/6}^{1}xdx=\frac{35}{144}$.

A similar analysis shows that when Bob is posting the blind bet Alice should bet if and only if her number $x=x_A$ satisfies $\frac{1}{2}\cdot 4+\frac{1}{2}x\cdot 6=2+3x\geq 3$, that is, if and only if $x\geq 1/3$. With this strategy the expected net gain of Alice is -3 with probability $\frac{1}{2}\int_{1/3}^{1}(1-x)dx=\frac{1}{9}$, it is 0 with probability $1/3$, it is $+1$ with probability $\frac{1}{2}\cdot\frac{2}{3}=\frac{1}{3}$, and it is $+3$ with probability $\frac{1}{2}\int_{1/3}^{1}xdx=\frac{2}{9}$. We summarize these facts in the following.

Proposition 4.3 *Suppose a skilled player is playing one basic game with a blind bet against an unskilled player.*
(i) If the skilled player posts the blind bet, then her expected net gain is

$$\frac{25}{144}\cdot(-3)+\frac{1}{12}\cdot(-1)+\frac{35}{144}\cdot 3=\frac{1}{8}$$

and the variance is

$$\frac{25}{144}\cdot(-3)^2+\frac{1}{12}\cdot 1^2+\frac{35}{144}\cdot 3^2-(\frac{1}{8})^2=\frac{733}{192}.$$

(ii) If the unskilled player posts the blind bet, then the expected gain of the skilled player is

$$\frac{1}{9}\cdot(-3)+\frac{1}{3}\cdot 1+\frac{2}{9}\cdot 3=\frac{2}{3}$$

with variance

$$\frac{1}{9}\cdot(-3)^2+\frac{1}{3}\cdot 1^2+\frac{2}{9}\cdot 3^2-(\frac{2}{3})^2=\frac{26}{9}.$$

Note that the skilled player has to use one strategy when posting the blind bet and another one when the second player is posting the blind bet. Indeed, in real poker the strategy has to take the position into account.

5 The Effect of the Central Limit Theorem

The analysis of the simplified models of poker discussed in the previous section shows that skilled players have a rather significant advantage over unskilled ones; this advantage becomes more and more prominent as the number of hands played increases. Intuitively that's a clear fact, as in the long run the cards dealt to all player are similar on average. A rigorous explanation with precise quantitative estimates can be given using the Central Limit Theorem.

By (a special case of) the Central Limit Theorem (see, e.g., [4]), the normalized sum of independent uniformly bounded random variables is converging to a normal distribution. A precise version follows.

Theorem 5.1 *Let M be a positive real, and let $X_1, X_2, \ldots$ be a sequence of independent random variables, where each X_i satisfies $|X_i| \leq M$, the expectation of X_i is μ_i and its variance is σ_i^2. Define*

$$Z_n = \frac{\sum_{i=1}^n X_i - \sum_{i=1}^n \mu_i}{\sqrt{\sum_{i=1}^n \sigma_i^2}}.$$

Then, for every real z,

$$\lim_{n \to \infty} Prob[Z_n \leq z] = \Phi(z)$$

where

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt, \tag{1}$$

is the cumulative distribution function of a standard Normal Random Variable.

Applying this theorem to the basic game between a skilled and an unskilled player in the basic game discussed in subsection 4.1, we get the following.

Proposition 5.2 *In a sequence of n hands of the basic game between a skilled and an unskilled player, the probability that the skilled player will not lead at the end is approximately $\Phi(-\sqrt{n/15})$, where $\Phi(z)$ is given in (1).*

The proof is simple. For each i, $1 \leq i \leq n$, let X_i denote the net gain of the skilled player in the i-th hand. By Proposition 4.1 the expected value of each X_i is $\mu_i = \frac{1}{8}$ and its variance is $\sigma_i^2 = 15/64$. Using the notation of Theorem 5.1, put

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n/8}{\sqrt{15n/64}}.$$

Since the random variables X_i are independent (and bounded), the theorem applies and shows that for large n, the probability that $\sum_{i=1}^{n} X_i$ is at most some real number y, which is precisely the probability that

$$Z_n \leq \frac{y - n/8}{\sqrt{15n/64}}$$

is approximately

$$\Phi(\frac{y - n/8}{\sqrt{15n/64}}).$$

As $\sum_{i=1}^{n} X_i$ is the total net gain of the skilled player, the probability he will not lead at the end is precisely the probability that $\sum_{i=1}^{n} X_i \leq 0$. The desired result follows by substituting $y = 0$ in the last displayed equation.

The above approximation is very accurate already for modest values of n, and certainly for all $n > 50$. Taking the values of the function Φ from a table of Normal Distribution we conclude that, for example, for $n = 60$ this probability is $\Phi(-2) = 0.0227..$ and for $n = 240$ the probability is $\Phi(-4) = 0.00003167..$, that is, smaller than $1/30,000$. For $n = 350$ the probability the unskilled player wins is already smaller than one in a million. Note that by the same reasoning one can bound the probability that after n games the skilled player will have a net gain of at most y chips. Thus, for example, the probability that after $n = 240$ hands the skilled player will have a net gain of at most $n/16 = 15$ chips is roughly $\Phi((15 - 30)/\sqrt{15 \cdot 240/64}) = \Phi(-2) = 0.0227..$

A similar computation for the case of the simple game with a blind bet can be carried out using Proposition 4.3.

Proposition 5.3 *Suppose a skilled and an unskilled player are playing* $2n$ *hands of the basic game with a blind bet, where each player posts the blind bet* n *times. Then the probability that the skilled player will not lead at the end is approximately*

$$\Phi(-\frac{19\sqrt{n}}{\sqrt{3863}}).$$

We omit the detailed computation and only give two examples. If $n = 90$ then the probability that at the end the skilled player will not be ahead is about $\Phi(-19\sqrt{90}/\sqrt{3863}) = 0.00187$.. For $n = 140$ this probability drops down to less than 0.00016.

The discussion above shows that the skill component in poker (at least in the simplified models considered here), which gives some advantage in a single hand, provides a major advantage in a sequence of games. In fact, when the sequence becomes long, as is usually the case in poker games, a skilled player wins against an unskilled one with overwhelming probability. It is instructive to compare the situation here to other games, without restricting the discussion to card games. Consider, for example, tennis. There is certainly an important skill component in tennis, but there is surely also some influence of chance in the game, arising from the impact of lots of random elements, like the wind, the sun, balls hitting hidden bumps in the court, etc. Indeed, without these, a stronger player would beat a weaker one in every point (while serving, say), and this is certainly not the case. In reality, a top-ten player probably wins about 55% of the points in a match against a player ranked 100, that is, the stronger player has an advantage of about 0.1 in a single point. However, since a match consists of $3, 4$ or 5 sets, each set consists of at least 6 (and usually more) games, and each game consists of aleast 4 points, in a typical match there are at least 72 points, and often at least twice that number. The Central Limit Theorem thus kicks in, and implies that even a relatively small advantage in a single point becomes a major factor in deciding the final result of the game. The situation in poker is similar. Indeed, poker is different than tennis as it has an inherent element of chance in it, but the influence of this is not necessarily larger, and in fact appears to be smaller, than the influence of chance elements in tennis. The repeated nature of the game reduces considerably the effect of chance, making poker almost entirely a game of skill.

6 Summary and Concluding Remarks

By analyzing simplified versions of poker we have seen that although like in essentially almost any other game there is some influence of chance in poker, the game is predominantly a game of skill. Indeed, the discussion in Section 4 shows that in the simplified one-round version of the game, a good player should first be able to master the probabilities in the game sufficiently well in order to be able to translate his pocket cards and the community cards to an accurate rank of his cards among the available possibilities. He should then be able to use this information to estimate the probability of winning. We have seen that the strategy of a wining player should be adjusted to that of the other payers, as a strategy that is winning against some player may well be losing against another. The number of players and the position at the table should also be taken into account, and bluffing is important in order not to reveal one's strategy. Therefore, a significant amount of skill is required to play well any of the simplified versions of the game discussed in Section 4. The real game, is, of course, far more complicated, and there is every reason to believe that skill plays a dominant role in the real version as well.

The Central Limit Theorem discussed in Section 5 implies that the significance of skill increases dramatically as the number of hands played grows. As usually the number of hands played is rather large, this fact implies that the end result in a long sequence of hands is determined with near certainty by the skill of the players.

The real game is far more complicated than the simplified versions analyzed here, and playing it well requires a lot of skill. A skilled player should be able to assess the strength of his hand as a function of his hole cards, the community cards, the number of players still in the game, their betting strategy and the position at the table. He should be able to assess the model of play of the other players, estimate the probability of improving his hand once the next community cards are revealed, and should be able to hide his strategy by bluffing and leaving his behavior unpredictable. It is not surprising that there is no software that plays poker as well as a good human player, although, for comparison, there are computer programs that play chess at least as well as the very best human chess players. Indeed, in many ways poker requires more human skill than chess, as an optimal strategy depends so crucially on the behavior of the opponents. The challenges of poker have been investigated in papers in Game Theory like [8], [7], [6], and in Artificial Intelligence (see, e.g., [2]), and there are still many intriguing questions concerning the analysis of optimal strategies for the game.

In almost every existing game there is an element of skill and an element of chance. As a matter of fact, the principles of Statistical Physics and Quantum Mechanics imply that some influence of chance appears in essentially every phenomenon in our life, not only in games. Despite the inherent element of chance in poker, our analysis of the simplified models suggests that the result of a soccer match, and probably even that of a tennis match, are influenced by chance more than the results in poker played over a long sequence of hands. The main reason some people may feel otherwise is psychological- one tends to associate randomness with cards or dice more than with weather, wind or bumps in a court, even when the latter have a greater effect on the end result. The fact that a significant number of players excel repeatedly in poker tournaments is a further indication that poker is mainly a game of skill.

Practice and study do help to improve in poker, and although luck may well play an essential role in a single hand, we believe that skill is the major component, by far, in deciding the results of a long sequence of hands. As the common practice is to play many hands, this strongly supports the conclusion that skill is far more dominant than luck, and that poker is predominantly a game of skill.

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