

SEPARATION DIMENSION OF BOUNDED DEGREE GRAPHS*

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Abstract. The *separation dimension* of a graph G is the smallest natural number k for which the vertices of G can be embedded in \mathbb{R}^k such that any pair of disjoint edges in G can be separated by a hyperplane normal to one of the axes. Equivalently, it is the smallest possible cardinality of a family \mathcal{F} of total orders of the vertices of G such that for any two disjoint edges of G , there exists at least one total order in \mathcal{F} in which all the vertices in one edge precede those in the other. In general, the maximum separation dimension of a graph on n vertices is $\Theta(\log n)$. In this article, we focus on bounded degree graphs and show that the separation dimension of a graph with maximum degree d is at most $2^{9 \log^* d} d$. We also demonstrate that the above bound is nearly tight by showing that, for every d , almost all d -regular graphs have separation dimension at least $\lceil d/2 \rceil$.

Key words. separation dimension, boxicity, linegraph, bounded degree

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1. Introduction. Let $\sigma : U \rightarrow [n]$ be a permutation of elements of an n -set U . For two disjoint subsets A, B of U , we say $A \prec_\sigma B$ when every element of A precedes every element of B in σ , i.e., $\sigma(a) < \sigma(b) \forall (a, b) \in A \times B$. We say that σ *separates* A and B if either $A \prec_\sigma B$ or $B \prec_\sigma A$. We use $a \prec_\sigma b$ to denote $\{a\} \prec_\sigma \{b\}$. For two subsets A, B of U , we say $A \preceq_\sigma B$ when $A \setminus B \prec_\sigma A \cap B \prec_\sigma B \setminus A$.

Families of permutations which satisfy some type of “separation” properties have been long studied in combinatorics. One of the early examples of it is seen in the work of Ben Dushnik in 1950, in which he introduced the notion of *k-suitability* [9]. A family \mathcal{F} of permutations of $[n]$ is *k-suitable* if, for every k -set $A \subseteq [n]$ and for every $a \in A$, there exists a $\sigma \in \mathcal{F}$ such that $A \preceq_\sigma \{a\}$. Fishburn and Trotter, in 1992, defined the *dimension* of a hypergraph on the vertex set $[n]$ to be the minimum size of a family \mathcal{F} of permutations of $[n]$ such that every edge of the hypergraph is an intersection of *initial segments* of \mathcal{F} [11]. It is easy to see that an edge e is an intersection of initial segments of \mathcal{F} if and only if for every $v \in [n] \setminus e$, there exists a permutation $\sigma \in \mathcal{F}$ such that $e \prec_\sigma \{v\}$. Such families of permutations with small sizes have found applications in showing upper bounds for many combinatorial parameters like poset dimension [15], product dimension [12], boxicity [7], etc. Small families of permutations with certain separation and covering properties have found applications

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in event sequence testing [8]. This paper is a part of our broad investigation¹ on a similar class of permutations which we make precise next.

DEFINITION 1. A family \mathcal{F} of permutations of $V(H)$ is pairwise suitable for a hypergraph H if, for every two disjoint edges $e, f \in E(H)$, there exists a permutation $\sigma \in \mathcal{F}$ which separates e and f . The cardinality of a smallest family of permutations that is pairwise suitable for H is called the separation dimension of H and is denoted by $\pi(H)$.

A family $\mathcal{F} = \{\sigma_1, \dots, \sigma_k\}$ of permutations of a set V can be seen as an embedding of V into \mathbb{R}^k with the i th coordinate of $v \in V$ being the rank of v in σ_i . Similarly, given any embedding of V in \mathbb{R}^k , we can construct k permutations by projecting the points onto each of the k axes and then reading them along the axis, breaking the ties arbitrarily. From this, it is easy to see that $\pi(H)$ is the smallest natural number k so that the vertices of H can be embedded into \mathbb{R}^k such that any two disjoint edges of H can be separated by a hyperplane normal to one of the axes. This prompts us to call such an embedding a *separating embedding* of H and $\pi(H)$ the *separation dimension* of H .

A major motivation to study this notion of separation is its interesting connection with a certain well-studied geometric representation of graphs. The *boxicity* of a graph G is the minimum natural number k for which G can be represented as an intersection graph of axis-parallel boxes in \mathbb{R}^k . It is established in [4] that the separation dimension of a hypergraph H is equal to the boxicity of the intersection graph of the edge set of H , i.e., the line graph of H .

It is easy to check that separation dimension is a monotone property, i.e., adding more edges to a graph cannot decrease its separation dimension. The separation dimension of a complete graph on n vertices and hence the maximum separation dimension of any graph on n vertices is $\Theta(\log n)$ [4]. The separation dimension of sparse graphs, i.e., graphs with linear number of edges, has also been studied. It is known that the maximum separation dimension of a k -degenerate graph on n vertices is $O(k \log \log n)$ and there exists a family of 2-degenerate graphs with separation dimension $\Omega(\log \log n)$ [6]. This tells us that even graphs in which every subgraph has a linear number of edges (a 2-degenerate graph on m vertices has at most $2m$ edges) can have unbounded separation dimension. Hence, it is interesting to see what additional sparsity condition(s) can ensure that the separation dimension remains bounded.

Along that line of thought, here we investigate how large the separation dimension of bounded degree graphs can be. To be precise, we study the order of growth of the function $f : \mathbb{N} \rightarrow \mathbb{N}$ where $f(d)$ is the maximum separation dimension of a d -regular graph. Since any graph G with maximum degree $\Delta(G)$ at most d is a subgraph (not necessarily spanning) of a d -regular graph, and since separation dimension is a monotone property, $\max\{\pi(G) : \Delta(G) \leq d\} = f(d)$. In this note, we show that for any d ,

$$\left\lceil \frac{d}{2} \right\rceil \leq f(d) \leq 2^{9 \log^* d} d.$$

It is not difficult to improve the constant 9 in the upper estimate above; we make no attempt to optimize it here.

We arrive at the above upper bound using probabilistic methods, and it improves the $O(d \log \log d)$ bound which follows from [7], once we note the connection between separation dimension and boxicity established in [4]. The upper bound in [7] was

¹Many of our initial results on this topic are available as a preprint [5].

proved using 3-suitability. The lower bound here is established by showing that, for every d , almost all d -regular graphs have separation dimension at least $\lceil d/2 \rceil$. A critical ingredient of our proof is the small set expansion property of random regular graphs. Prior to this, the best lower bound known for $f(d)$ was $\log(d)$, which is the separation dimension of $K_{d,d}$, the d -regular complete bipartite graph. Since it is known (see [4]) that $\pi(G) \in O(\chi_a(G))$, where $\chi_a(G)$ is the acyclic chromatic number of G , which is the minimum number of colors in a proper vertex coloring so that the union of any two color classes contains no cycle, and it is also known [1] that, for every d , almost all d -regular graphs have acyclic chromatic number $O(d)$, it follows that, for every d , almost all d -regular graphs have separation dimension $\Theta(d)$. It seems plausible to conjecture that in fact $f(d) = \Theta(d)$, but at the moment we are unable to prove or disprove this conjecture.

2. Upper bound. In order to establish the upper bound on $f(d)$ (Theorem 5), we need two technical lemmata; the first (Lemma 2) we established in [4], and the second (Lemma 4), which is similar to Lemma 4.2 in [13], is established using the local lemma.

LEMMA 2 ([4]; also Lemma 7 in [5]). *Let $P_G = \{V_1, \dots, V_r\}$ be a partitioning of the vertices of a graph G , i.e., $V(G) = V_1 \uplus \dots \uplus V_r$. Let $\hat{\pi}(P_G) = \max_{i,j \in [r]} \pi(G[V_i \cup V_j])$. Then, $\pi(G) \leq 13.68 \log r + \hat{\pi}(P_G)r$.*

A useful consequence of Lemma 2 is that if we can somehow partition the vertices of a graph G into r parts such that the separation dimension of the union of any two parts is bounded, then $\pi(G)$ is $O(r)$. For example, consider P_G to be the partition of $V(G)$ corresponding to the color classes in a distance-two coloring of G , i.e, a vertex coloring of G in which no two vertices of G that are at a distance at most 2 from each other are given the same color. Then the subgraphs induced by any pair of color classes is a collection of disjoint edges, and hence $\hat{\pi}(P_G) \leq 1$. It is easy to see that a distance-two coloring of a d -regular graph can be done using $d^2 + 1$ colors, and hence $f(d)$ is $O(d^2)$. Similarly, taking the parts to be color classes in an acyclic vertex coloring of G , it follows that $\pi(G)$ is $O(\chi_a(G))$, where $\chi_a(G)$ denotes the acyclic chromatic number of G , and hence by [1], $f(d)$ is $O(d^{4/3})$. It was shown in [7] that if G is a linegraph of a multigraph, then the boxicity of G is at most $2\Delta(\lceil \log \log \Delta \rceil + 3) + 1$, where Δ is the maximum degree of G . Since the separation dimension of a graph is equal to the boxicity of its linegraph [4], it follows that $f(d)$ is $O(d \log \log d)$. Theorem 5 improves this bound.

LEMMA 3 (the Lovász local lemma [10]). *Let G be a graph on vertex set $[n]$ with maximum degree d and let A_1, \dots, A_n be events defined on some probability space such that for each i , $\Pr[A_i] \leq 1/4d$. Suppose further that each A_i is jointly independent of the events A_j for which $\{i, j\} \notin E(G)$. Then $\Pr[\overline{A_1} \cap \dots \cap \overline{A_n}] > 0$.*

LEMMA 4. *For a graph G with maximum degree $\Delta \geq 2^{64}$, there exists a partition of $V(G)$ into $\lceil 400\Delta/\log \Delta \rceil$ parts such that for every vertex $v \in V(G)$ and for every part V_i , $i \in [\lceil 400\Delta/\log \Delta \rceil]$, $|N_G(v) \cap V_i| \leq \frac{1}{2} \log \Delta$.*

Proof. Since we can have a Δ -regular supergraph (with possibly more vertices) of G , we can as well assume that G is Δ -regular. Let $r = \lceil \frac{400\Delta}{\log \Delta} \rceil \leq \frac{401\Delta}{\log \Delta}$. Partition $V(G)$ into V_1, \dots, V_r using the following procedure: for each $v \in V(G)$, independently assign v to a set V_i uniformly at random from V_1, \dots, V_r .

We use the following well-known multiplicative form of the Chernoff bound (see, e.g., Theorem A.1.15 in [2]). Let X be a sum of mutually independent indicator random variables with $\mu = E[X]$. Then for any $\delta > 0$, $\Pr[X \geq (1 + \delta)\mu] \leq c_\delta^\mu$, where $c_\delta = e^\delta / (1 + \delta)^{(1+\delta)}$.

Let $d_i(v)$ be a random variable that denotes the number of neighbors of v in V_i . Then $\mu_{i,v} = E[d_i(v)] = \frac{\Delta}{r} \leq \frac{1}{400} \log \Delta$. For each $v \in V(G), i \in [r]$, let $E_{i,v}$ denote the event $d_i(v) \geq \frac{1}{2} \log \Delta$. Then applying the above Chernoff bound with $\delta = 199$, we have $Pr[E_{i,v}] = Pr[d_i(v) \geq 200 \frac{\log \Delta}{400}] \leq 2^{-3.1 \log \Delta} = \Delta^{-3.1}$. In order to apply Lemma 3, we construct a graph H whose vertex set is the collection of “bad” events $E_{i,v}, i \in [r], v \in V(G)$, and two vertices $E_{i,v}$ and $E_{i',v'}$ are adjacent if and only if the distance between v and v' in G is at most 2. Since for each $i \in [r]$ and $v \in V(G)$ the event $E_{i,v}$ depends only on where the neighbors of v went to in the random partitioning, it is jointly independent of all the events $E_{i',v'}$ which are nonadjacent to it in H . It is easy to see that the maximum degree of H , denoted by d_H , is at most $(1 + \Delta + \Delta(\Delta - 1))r = (1 + \Delta^2)r \leq \frac{402\Delta^3}{\log \Delta}$. For each $i \in [r], v \in V(G)$, $Pr[E_{i,v}] \leq \frac{1}{\Delta^{3.1}} \leq \frac{\log \Delta}{1608\Delta^3} \leq \frac{1}{4d_H}$. Therefore, by Lemma 3, we have $Pr[\bigcap_{i \in [r], v \in V(G)} \overline{E_{i,v}}] > 0$. Hence, there exists a partition satisfying our requirements. \square

THEOREM 5. *For every positive integer d , $f(d) \leq 2^{9 \log^* d}$.*

Proof. If $d \leq 1$, then G is a collection of matching edges and disjoint vertices and therefore $f(1) = 1$. When $d > 1$, it follows from Theorem 10 in [7] that $\pi(d) \leq (4d - 4)(\lceil \log \log(2d - 2) \rceil + 3) + 1$. For every $1 < d < 2^{64}$, it can be verified that

$$(4d - 4)(\lceil \log \log(2d - 2) \rceil + 3) + 1 \leq 2^{9 \log^* d}.$$

Therefore, the statement of the theorem is true for every $d < 2^{64}$.

For $d \geq 2^{64}$, let P_G be a partition of $V(G)$ into $V_1 \uplus \dots \uplus V_r$, where $r = \lceil 400d / \log d \rceil$ and $|N_G(v) \cap V_i| \leq \frac{1}{2} \log d \forall v \in V(G), i \in [r]$. Existence of such a partition is guaranteed by Lemma 4. From Lemma 2, we have $\pi(G) \leq 13.68 \log r + \hat{\pi}(P_G)r$, where $\hat{\pi}(P_G) = \max_{i,j \in [r]} \pi(G[V_i \cup V_j])$. Since $|N_G(v) \cap V_i| \leq \frac{1}{2} \log d$ for every $v \in V(G), i \in [r]$, the maximum degree of the graph $G[V_i \cup V_j]$ is at most $\log d$ for every $i, j \in [r]$. Therefore, $\hat{\pi}(P_G) \leq f(\log d)$. Thus we have, for every $d \geq 2^{64}$,

$$\begin{aligned} f(d) &\leq \left\lceil \frac{400d}{\log d} \right\rceil f(\log d) + 13.68 \log \left\lceil \frac{400d}{\log d} \right\rceil \\ (1) \quad &\leq 2^9 \frac{d}{\log d} f(\log d). \end{aligned}$$

Now we complete the proof by using induction on d . The statement is true for all value of $d < 2^{64}$, and we have the recurrence relation of (1) for larger values of d . For an arbitrary $d \geq 2^{64}$, we assume inductively that the bound in the statement of the theorem is true for all smaller values of d . Now, since $d \geq 2^{64}$, we can apply the recurrence in (1). Therefore,

$$\begin{aligned} f(d) &\leq 2^9 \frac{d}{\log d} f(\log d) \\ &\leq 2^9 \frac{d}{\log d} 2^{9 \log^*(\log d)} \log d \quad (\text{by induction}) \\ &= 2^{9 \log^* d} d. \quad \square \end{aligned}$$

3. Lower bound. To simplify the presentation, we omit the floor and ceiling signs throughout the proof whenever these are not crucial. Consider the probability space of all labelled d -regular graphs on n vertices with uniform distribution. We say that almost all d -regular graphs have some property P if the probability that P holds

tends to 1 as n tends to ∞ . We need the following well-known fact about expansion of small sets in d -regular graphs.

THEOREM 6 (c.f., e.g., Theorem 4.16 in [14]). *Let $d \geq 3$ be a fixed integer. Then for every $\delta > 0$, there exists $\epsilon > 0$ such that for almost every d -regular graph there are at most $(1 + \delta)|S|$ edges inside any $S \subset V(G)$ of size $\epsilon|V(G)|$ or less.*

THEOREM 7. *For every positive integer d , almost all d -regular graphs have separation dimension at least $\lceil d/2 \rceil$.*

Proof. The claim is easy to verify for $d = 1$ and $d = 2$, and hence we assume $d \geq 3$. Let $\delta \in (0, 1)$ be arbitrary and $\epsilon > 0$ chosen so as to satisfy the small set expansion guaranteed by Theorem 6. Choose n larger than $4(d + 1)(1/\delta\epsilon)^{d/2}$ and let G be a d -regular graph on n vertices which satisfies the small set expansion property of Theorem 6. Note that almost all d -regular graphs qualify.

For a permutation $\sigma = (v_0, \dots, v_{n-1})$ of $V(G)$, we call an edge $\{v_i, v_j\}$ of G σ -short if $|i - j| \leq \delta\epsilon n$. We claim that for any permutation σ , the number of σ -short edges of G is at most $(\frac{1+\delta}{1-\delta})n$. To see this, cover the permutation σ with overlapping blocks of size $b = \epsilon n$ and amount of overlap $s = \delta b = \delta\epsilon n$. To be precise, the blocks are

$$B_i = \{v_j : i(1 - \delta)b \leq j < (i + 1)(1 - \delta)b + b\} \quad \forall 0 \leq i < \frac{1}{\epsilon(1 - \delta)}.$$

Due to the overlap between the blocks, every σ -short edge is inside some block B_i . Since each block B_i has at most ϵn vertices, there are at most $(1 + \delta)\epsilon n$ edges in each of them and hence at most $(\frac{1+\delta}{1-\delta})n$ σ -short edges in total.

Suppose now for contradiction that $\pi(G) < \lceil d/2 \rceil$ and hence, being an integer, $\pi(G) \leq (\frac{d}{2} - \frac{1}{2})$. Let \mathcal{F} be a set of $\pi(G)$ permutations that is pairwise suitable for G . Then by the above discussion, at most $(\frac{d}{2} - \frac{1}{2})(\frac{1+\delta}{1-\delta})n$ edges of G are σ -short for some permutation $\sigma \in \mathcal{F}$. Let us call the remaining edges, that is, those which are not short in any permutation in \mathcal{F} , *long*. Since $\delta \in (0, 1)$ was arbitrary, we can choose it small enough so that $(\frac{d}{2} - \frac{1}{2})(\frac{1+\delta}{1-\delta}) \leq (\frac{d}{2} - \frac{1}{4})$. Thus, we have at least $n/4$ long edges in G .

Since G is $(d + 1)$ -edge colorable (by Vizing's theorem), the edges of G can be partitioned into $d + 1$ matchings. By averaging, at least one of the matchings has at least $n/4(d + 1)$ long edges $L = \{e_1, \dots, e_{\lceil n/4(d+1) \rceil}\}$. We complete the proof by arguing that \mathcal{F} cannot separate all the edges in L .

For each permutation $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathcal{F}$, construct a graph H_σ on the set of vertices L in which each e_i is adjacent to e_j if and only if e_i and e_j are separated by σ . We claim that the chromatic number of each H_σ is at most $1/\delta\epsilon$. Indeed, for $s = \delta\epsilon n$, every long edge goes over at least one of the points $\sigma_s, \sigma_{2s}, \dots, \sigma_{s/\delta\epsilon}$, and all the edges e_i that go over the point js form an independent set in H_σ . Since all the pairs of edges e_i have to be separated by some σ , we conclude that the union of the graphs H_σ over all $\sigma \in \mathcal{F}$ is the complete graph on L . However, since the complete graph on L has chromatic number $|L| = \lceil n/4(d + 1) \rceil$, we need $n/4(d + 1) \leq (1/\delta\epsilon)^{d/2}$ contradicting the choice of n . Therefore, $\pi(G) \geq \lceil d/2 \rceil$. \square

Remark 8. The *linear arboricity* $\text{la}(G)$ of a graph G is the minimum number of linear forests (disjoint union of paths) that can cover the edges of G . We cannot hope for a lower bound on $\pi(G)$ that is bigger than $\text{la}(G)$ using the above technique. This is because we can write down one permutation for each linear forest so that every edge of G appears as an edge of length 1 in one of these permutations. It is known that the linear arboricity of a d -regular graph is at most $d/2 + cd^{2/3}(\log d)^{1/3}$ for some absolute constant c ([2, p. 78]; see also [3]), and it is equal to $\lceil (d + 1)/2 \rceil$ for almost all random d -regular graphs [16].

REFERENCES

- [1] N. ALON, C. MCDIARMID, AND B. REED, *Acyclic coloring of graphs*, Random Structures Algorithms, 2 (1991), pp. 277–288.
- [2] N. ALON AND J. H. SPENCER, *The Probabilistic Method*, 3rd ed., John Wiley & Sons, New York, 2008.
- [3] N. ALON, V. J. TEAGUE, AND N. WORMALD, *Linear arboricity and linear k -arboricity of regular graphs*, Graphs Combin., 17 (2001), pp. 11–16.
- [4] M. BASAVARAJU, L. S. CHANDRAN, M. C. GOLUMBIC, R. MATHEW, AND D. RAJENDRAPRASAD, *Boxicity and separation dimension*, in Proceedings of the 40th International Workshop on Graph-Theoretic Concepts in Computer Science (WG 2014), to appear; also available online from <http://arXiv.org/1404.4486>.
- [5] M. BASAVARAJU, L. S. CHANDRAN, R. MATHEW, AND D. RAJENDRAPRASAD, *Pairwise Suitable Family of Permutations and Boxicity*, preprint; <http://arXiv.org/abs/1212.6756>.
- [6] M. BASAVARAJU, L. S. CHANDRAN, R. MATHEW, AND D. RAJENDRAPRASAD, *Separation dimension of sparse graphs*, in Proceedings of the 9th International Colloquium on Graph Theory and Combinatorics (ICGT 2014), to appear; <http://arXiv.org/abs/1404.4484>.
- [7] L. S. CHANDRAN, R. MATHEW, AND N. SIVADASAN, *Boxicity of line graphs*, Discrete Math., 311 (2011), pp. 2359–2367.
- [8] Y. M. CHEE, C. J. COLBOURN, D. HORSLEY, AND J. ZHOU, *Sequence covering arrays*, SIAM J. Discrete Math., 27 (2013), pp. 1844–1861.
- [9] B. DUSHNIK, *Concerning a certain set of arrangements*, Proc. Amer. Math. Soc., 1 (1950), pp. 788–796.
- [10] P. ERDŐS AND L. LOVÁSZ, *Problems and results on 3-chromatic hypergraphs and some related questions*, Infinite and Finite Sets, 10 (1975), pp. 609–627.
- [11] P. C. FISHBURN AND W. T. TROTTER, *Dimensions of hypergraphs*, J. Combin. Theory Ser. B, 56 (1992), pp. 278–295.
- [12] Z. FÜREDI, *On the Prague dimension of Kneser graphs*, in Numbers, Information and Complexity, Springer, New York, 2000, pp. 125–128.
- [13] Z. FÜREDI AND J. KAHN, *On the dimensions of ordered sets of bounded degree*, Order, 3 (1986), pp. 15–20.
- [14] S. HOORY, N. LINIAL, AND A. WIGDERSON, *Expander graphs and their applications*, Bull. Amer. Math. Soc., 43 (2006), pp. 439–561.
- [15] H. A. KIERSTEAD, *On the order dimension of 1-sets versus k -sets*, J. Combin. Theory Ser. A, 73 (1996), pp. 219–228.
- [16] C. MCDIARMID AND B. REED, *Linear arboricity of random regular graphs*, Random Structures Algorithms, 1 (1990), pp. 443–445.