Polychromatic Colorings of Plane Graphs

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Abstract

We show that the vertices of any plane graph in which every face is of length at least g can be colored by $\lfloor (3g-5)/4 \rfloor$ colors so that every color appears in every face. This is nearly tight, as there are plane graphs that admit no vertex coloring of this type with more than $\lfloor (3g+1)/4 \rfloor$ colors. We further show that the problem of determining whether a plane graph admits a vertex coloring by 3 colors in which all colors appear in every face is $\mathcal{NP}$-complete even for graphs in which all faces are of length 3 or 4 only. If all faces are of length 3 this can be decided in polynomial time. The investigation of this problem is motivated by its connection to a variant of the art gallery problem in computational geometry.

1 Introduction

A *plane graph* is a graph G together with an embedding of G into the plane with no crossing edges. Let $F(G)$ denote the set of faces of G. The *length* of a face $f \in F(G)$ is the number of vertices on its boundary. For a plane graph G, let $g(G)$ denote the length of the shortest face in G. A face of length k in a plane graph is sometimes also called a k-face. A vertex k-coloring is a map $\chi : V(G) \to \{1, \ldots, k\}$. It is *proper* if for every edge $uv \in E(G)$, $\chi(u) \neq \chi(v)$. For a (not necessarily proper) vertex k-coloring $\chi : V(G) \to \{1, \ldots, k\}$ of G we say that a face

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$f \in F(G)$ is *polychromatic* if all k colors appear on its vertices. A vertex k-coloring of G is called *polychromatic* if every face of G is polychromatic. The *polychromatic number* of G, denoted by $p(G)$, is the largest number of colors k such that there is a polychromatic vertex k-coloring of G. Define $p(g) = \min\{p(G) \mid g(G) = g\}$.

It is clear that for every plane graph G, $p(G) \leq g(G)$. On the other hand $p(g) \geq 2$ for all plane graphs G with $g(G) \geq 3$. This was first proved by Bose et al. [4] (see also [11]) by using the Four-Color Theorem, and afterwards by Bose et al. [3] without using it. The following sketch of the second proof is similar in spirit to some of the ideas in the present paper.

Triangulate the graph G by adding edges, resulting in a new graph H where each face has length 3. The dual graph H^* of H is then 3-regular and 2-edge connected. By Petersen's Theorem (see, e.g., [16]), there exists a perfect matching M in H^*. After deleting the edges of H corresponding to those of M, the remaining graph H' has only faces of length 4. Therefore there is no odd cycle in H' and hence H' is bipartite. Thus, there is a proper vertex 2-coloring of H', which is a polychromatic vertex 2-coloring of H and hence also of G.

By these arguments $2 \leq p(G) \leq 3$ for every triangulation G. The following simple characterization of triangulations G with $p(G) = 3$ is an immediate consequence of an old result of Heawood.

Theorem 1. *Let G be a triangulation. The following two statements are equivalent:*
(i) $p(G) = 3$, and
(ii) G is Eulerian, i.e., every vertex degree in G is even.

Our main result bounds the minimum possible polychromatic number for plane graphs G with $g(G) = g$.

Theorem 2. $p(1) = p(2) = 1, p(3) = p(4) = 2$ *and for* $g \geq 5$,

$$\left\lfloor \frac{3g-5}{4} \right\rfloor \leq p(g) \leq \left\lfloor \frac{3g+1}{4} \right\rfloor.$$

This settles a question raised in [9]. Note that the set $\left\{ \left\lfloor \frac{3g-5}{4} \right\rfloor, \ldots, \left\lfloor \frac{3g+1}{4} \right\rfloor \right\}$ contains two or three integers. On the other hand the exact determination of $p(G)$ is in general hard.

Theorem 3. *For a plane graph G it is $\mathcal{NP}$-complete to decide whether $p(G) \geq 3$. This holds even for graphs with faces of length 3 or 4 only.*

Hence we do not expect a nice characterization of plane polychromatically 3-colorable graphs, like the one for triangulations given in Theorem 1.

1.1 Related work

In a polychromatic coloring of a graph G the vertices of each color-class hit every face of G. This property can be used for the so called *vertex-guard* problem. A vertex-guard is located at a vertex of the plane graph G and it *guards* all incident faces (for our purposes it can also guard the unbounded face if it lies on its boundary). A set of vertex-guards *guards* a given plane graph if all faces are guarded by at least one guard. What is the minimum number of guards needed? In [4] it is shown that one can guard any plane graph on n vertices with no faces of length 1 or 2 by $\lfloor \frac{n}{2} \rfloor$ many guards. This clearly follows from the fact that $p(G) \geq 2$ for any such graph. They also construct graphs on n vertices for which $\lfloor \frac{n}{2} \rfloor$ many guards are necessary. Similarly, a simple consequence of Theorem 2 is the following:

Corollary 4. *Every plane graph G with $g(G) = g$ can be guarded with at most $\frac{n}{\lfloor (3g-5)/4 \rfloor} \leq \frac{4n}{3g-8}$ many guards.*

Proof. By Theorem 2, G admits a polychromatic $\lfloor \frac{3g-5}{4} \rfloor$-coloring. Set guards on the vertices of the smallest color class which is of size at most $\frac{n}{\lfloor \frac{3g-5}{4} \rfloor} \leq \frac{4n}{3g-8}$. Because the coloring is polychromatic each face is incident to at least one guard and the statement follows. $\square$

Hoffmann and Kriegel [8] proved that the polychromatic number of any plane, bipartite, 2-connected simple graph is at least 3. Horev and Krakovski [9] showed that any connected plane multigraph G with $g(G) \geq 3$ and maximum degree at most 3, which is not K_4, can be colored with 3 colors such that every bounded face of G is polychromatic.

Dinitz et al. [5] consider graphs that they call rectangular partition graphs, and show that for every such graph there is a vertex 3-coloring in which all bounded faces are polychromatic. On the other hand every rectangular partition G contains a 4-face, hence $p(G) \leq 4$.

Two variants of polychromatic colorings for the n-dimensional hypercube Q_n are considered by Alon et al. [2]. They observe that the vertices of Q_n can be colored by $d+1$ colors so that every d-dimensional subcube Q_d of Q_n is polychromatic, that is, Q_d contains a vertex of each color. Indeed, color each vertex in level i of Q_n with color $i \bmod (d+1)$. Moreover this coloring with $d+1$ many colors is best possible for all d and n sufficiently large as shown in [2]. A similar problem is considered for edges as well. It is shown that there is an edge-coloring of Q_n with $\lfloor \frac{(d+1)^2}{4} \rfloor$ many colors such that every d-dimensional subcube is polychromatic, that is, contains an edge of each color. This is tight as shown by Offner [13].

1.2 Notation

We distinguish between (simple) graphs and multigraphs, the former having at most one edge between the same pair of vertices, whereas the latter possibly have more edges between a single pair of vertices. A loop of a graph is an edge with the same two endpoints. The induced subgraph of $G = (V, E)$ on a set U of vertices is denoted by $G[U]$.

The neighborhood of a vertex v in the graph G is denoted by $\Gamma_G(v)$ and the vertex degree of v is $d_G(v) = |\Gamma_G(v)|$. When the graph G is clear from the context, we sometimes denote $d_G(v)$ by $d(v)$. The minimum degree of a vertex in G is denoted by $\delta(G)$, and the maximum degree by $\Delta(G)$. For a directed graph G, the in- and out-neighborhood of a vertex v are denoted by

$\Gamma_G^-(v)$ and $\Gamma_G^+(v)$, respectively. Similarly we define the in- and out-degree, $d_G^-(v) = |\Gamma_G^-(v)|$ and $d_G^+(v) = |\Gamma_G^+(v)|$, respectively.

2 Simple cases

Lemma 5 (Heawood 1898 [7, 14]). *The vertices of a triangulation G are properly 3-colorable if and only if G is Eulerian.*

Lemma 6. *Let G be a triangulation. Then the following are equivalent*

(i) G is polychromatically 3-colorable.

(ii) G is properly 3-colorable.

Proof. A triangle is properly 3-colored if and only if its three vertices have the three different colors. Also, a triangle is polychromatically 3-colored if and only if its three vertices have the three different colors. Thus these two notions are equivalent for triangulations. □

These two lemmas together prove Theorem 1. Furthermore we have

Proposition 7. *Every outerplanar graph G with $g(G) \geq 3$ is polychromatically 3-colorable.*

Proof. Triangulate all bounded faces of G to get a new outerplanar graph G'. It is easy and well known that every outerplanar graph is 3-colorable (see, for example, [16], page 283). In a proper 3-coloring of G' every triangular face is polychromatic. The outerface is clearly polychromatic as well, since all vertices lie on its boundary. Therefore, this is also a polychromatic coloring of G with 3 colors. □

3 The proof of Theorem 2

In this section we prove Theorem 2. In the course of the proof, vertices are made responsible for certain faces by assigning them to these faces. This is done in the first subsection. Using such an assignment we can relate our problem to the existence of certain edge-colorings. In the second subsection we establish the required results for these edge-colorings. We prove the lower and the upper bound of the theorem in the last two subsections.

3.1 Assigning vertices to faces

Lemma 8. *Let G be a plane graph, let $\emptyset \neq F' \subseteq F(G), \emptyset \neq V' \subseteq V(G)$ and let $i(F', V')$ denote the number of incidences between F' and V'. Then $i(F', V') \leq 2|F'| + 2|V'| - 3$.*

Proof. Define the incidence graph H of $V' \subseteq V(G)$ and $F' \subseteq F(G)$ by $V(H) = F' \cup V'$ and $fv \in E(H)$ for $v \in V', f \in F'$ if and only if v is contained in the boundary of f in G. It is easy to see that H is planar, simple and bipartite. From Euler's Formula and the fact that H is simple and triangle-free it follows that H contains at most $2V(H)-4$ many edges, provided that H contains at least three vertices. In this case we conclude that $i(V',F') = |E(H)| \leq 2(|V'|+|F'|)-4$. On the other hand if $|V(H)| = 2$ and H contains one edge, then $i(V',F') = 2(|V'|+|F'|)-3 = 1$. $\square$

The following result is well known (see, for example, [10], Theorem 2.4.2). For completeness, we include a short proof.

Lemma 9. *Let $A \in \{0,1\}^{m\times n}$ be a matrix, $A = (a_{i,j})_{i\in\{1,\dots,m\},j\in\{1,\dots,n\}}$. The following two statements are equivalent.*

- *(i) There is a matrix $C \in \{0,1\}^{m\times n}$, $C \leq A$ (that is $c_{i,j} \leq a_{i,j}$ for all $i \in \{1,\dots,m\}$ and all $j \in \{1,\dots,n\}$) such that every row in C contains at least q 1's and every column in C contains at most r 1's.*
- *(ii) For every $M \subseteq \{1,\dots,m\}$ and every $N \subseteq \{1,\dots,n\}$, $\sum_{i\in M, j\in\{1,\dots,n\}\setminus N} a_{i,j} \geq q|M| - r|N|$.*

Proof. Define a network with vertices $s, t, r_1, \dots, r_m, c_1, \dots, c_n$ as follows. Connect the source s with all vertices r_i with edges having capacity q, connect r_i with c_j with edges having capacity a_{ij}, and connect all c_j to the sink t with edges having capacity r. If condition (i) holds, then we can also assume that there exists such a matrix C where in every row there are exactly q 1's. Thus there exists a flow of value mq if and only if (i) holds. It is easy to show that all cuts have size at least qm if and only if condition (ii) holds. This implies the statement by using the MaxFlow-MinCut Theorem. $\square$

Corollary 10. *Let G be a plane graph with $g(G) = g$. For each face $f \in F(G)$ we can assign $g-2$ vertices that lie on its boundary such that no vertex is assigned to more than two faces.*

Proof. Let $A = (a_{f,v})_{f\in F, v\in V} \in \{0,1\}^{|F|\times|V|}$ be the face-vertex incidence matrix of G. That is $a_{f,v} = 1$ if and only if vertex v is contained in face f. We want to show that there is a matrix $C \in \{0,1\}^{|F|\times|V|}$ such that $C \leq A$, in every row of C there are at least $g-2$ 1's, and in every column of C there are at most two 1's.

By Lemma 9 with $q = g-2$ and $r = 2$ it is enough to show that for every $F' \subseteq F, V' \subseteq V$, $\sum_{f\in F', v\in V\setminus V'} a_{f,v} \geq (g-2)|F'| - 2|V'|$.

Henceforth we obtain

$$\begin{aligned}\sum_{f\in F', v\in V\setminus V'} a_{f,v} &= \sum_{f\in F', v\in V} a_{f,v} - \sum_{f\in F', v\in V'} a_{f,v} \\ &\geq g|F'| - \sum_{f\in F', v\in V'} a_{f,v} \\ &\geq g|F'| - 2|F'| - 2|V'|,\end{aligned}$$

where the last inequality follows from Lemma 8 in case both V' and F' are nonempty, and is trivial if at least one of them is empty. $\square$

3.2 Polychromatic edge-colorings

Similar to the case of vertex-colorings, we define a polychromatic edge k-coloring $\chi : E(G) \to \{1, \ldots, k\}$ of G. A vertex $v \in V(G)$ is called *polychromatic* if all k-colors appear on the edges incident to v. An edge-coloring of G is *polychromatic* if every vertex $v \in V(G)$ is polychromatic.

Proposition 11. *For every integer $d > 0$ and every loopless multigraph G with minimum degree at least d there is a polychromatic edge-coloring of G with $p = \lfloor \frac{3d+1}{4} \rfloor$ colors.*

Before we prove Proposition 11 we state three useful lemmas. For completeness, we include their short proofs.

Lemma 12 ([6, 1])**.** *It is possible to color the edges of any loopless bipartite multigraph G by r colors $\{1, \ldots, r\}$, such that for every vertex v of G, the number of edges of each color incident with v are nearly equal. That is for every $i \in \{1, \ldots, r\}$, the number of edges of color i incident with v is either $\lfloor d(v)/r \rfloor$ or $\lceil d(v)/r \rceil$.*

Proof. First split vertices of G, if needed, to make its maximum degree at most r. This is done as follows. As long as there is a vertex v of G of degree $d > r$, modify it using the following procedure. Define $k = \lceil d/r \rceil$ and replace v by k new vertices $v_1, v_2, \ldots, v_k$, called its descendants. Let $vu_1, vu_2, \ldots vu_d$ be an arbitrary enumeration of all edges of G incident with v. For each $i, 1 \leq i \leq k$, let the edges incident with the new vertex v_i be the edges $v_i u_j$ for all j satisfying $(i-1)r < j \leq \min(d, ir)$. This process terminates with a bipartite graph in which all degrees are at most r. By König's Theorem (cf., e.g., [16]) the edges of this graph can be properly colored by the r colors $\{1, \ldots, r\}$. By collapsing all descendants of each vertex v back, keeping the colors of the edges, we obtain a coloring $f : E(G) \to \{1, \ldots, r\}$ of the edges of the original graph G by r colors satisfying the assertion of the claim. □

The following two lemmas are well known.

Lemma 13. *Every loopless multigraph G contains a spanning bipartite graph $B \subseteq G$ with $d_B(v) \geq \left\lceil \frac{d_G(v)}{2} \right\rceil$ for every $v \in V(G)$.*

Proof. Let B be a maximum edge-cut in G with respect to the number of edges. Assume that there is a vertex $v \in V(G)$ with $d_B(v) < \left\lceil \frac{d_G(v)}{2} \right\rceil$. If we then swap v to the other bipartite set, this would yield another edge-cut with more edges, contradicting the maximality. □

Lemma 14. *Every loopless multigraph G has an orientation of its edges such that $d^+(v) \geq \left\lfloor \frac{d(v)}{2} \right\rfloor$ for all $v \in V(G)$.*

Proof. We may assume that G is connected. If all degrees in G are even we simply orient it along an Eulerian cycle. Otherwise, define a new graph G' which consists of all vertices of G and a new vertex x and connect all odd degree vertices of G to x. Then all vertices in G' have even degrees and therefore there is an Eulerian cycle in G'. Orient the edges along such an Eulerian cycle and delete the vertex x. Every vertex $v \in V(G)$ with even degree has then exactly $d(v)/2$ many outgoing edges. Each vertex $v \in V(G)$ with odd degree has either $(d(v)+1)/2$ or $(d(v)-1)/2$ many outgoing edges. □

Proof of Proposition 11. By Lemma 13 there is a spanning bipartite subgraph H of G satisfying $\delta(H) \geq \lceil \frac{d}{2} \rceil$. Let A_1 and A_2 denote its vertex classes.

Applying Lemma 12 to H with $r = p = \lfloor \frac{3d+1}{4} \rfloor$ results in an edge coloring χ with the following two properties.

(i) Every vertex v with $d_H(v) \geq p$ is polychromatic. Indeed v is incident with at least $\lfloor d_H(v)/p \rfloor \geq 1$ many edges of each of the p colors.

(ii) For every vertex u with $d_H(u) < p$ each color appears at most once on edges incident to u since $\lceil d_H(u)/p \rceil = 1$. In other words all edges incident with u have distinct colors.

Extend the edge-coloring χ to G as follows: orient the edges of both $G[A_1]$ and $G[A_2]$ according to Lemma 14. Hence $d^+_{G[A_i]}(v) \geq \lfloor \frac{d-d_H(v)}{2} \rfloor$, for $i = 1, 2$ and all $v \in A_i \subset V(G)$. For each vertex $v \in A_i$, color the edges oriented from v to its outneighbors in $G[A_i]$ with the colors not appearing at the edges of H incident to v (if there are any such colors). Thus, the edges incident with any vertex $v \in V(G)$ are finally colored with $\min \left\{ d_H(v) + \left\lfloor \frac{d-d_H(v)}{2} \right\rfloor, p \right\} \geq \lceil \frac{d}{2} \rceil + \left\lfloor \frac{\lfloor \frac{d}{2} \rfloor}{2} \right\rfloor = \lfloor \frac{3d+1}{4} \rfloor = p$ distinct colors, where the inequality follows from the fact that $d_H(v) \geq \lceil \frac{d}{2} \rceil$. This completes the proof. □

3.3 The lower bound of Theorem 2

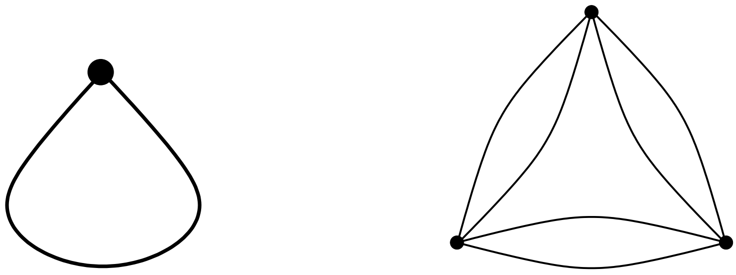
We now prove that $p(g) \geq \left\lfloor \frac{3g-5}{4} \right\rfloor$ for all $g \geq 5$.

Let $G = (V, E)$ be a plane graph with $g(G) = g$. By Corollary 10 we can assign to every face of G $g-2$ vertices from its boundary such that no vertex is assigned to more than two faces of G. Define an auxiliary multigraph H, with $V(H) = F(G) \cup \{x, y\}$, where $F(G)$ is the set of faces of G and x, y are two additional vertices. For every vertex $v \in V(G)$ define an edge of H, which we call the v-edge, as follows. If v is assigned to the two distinct faces f_1 and f_2 then the v-edge is $f_1 f_2$. If it is assigned only to one face f, the v-edge is fx, and if it is not assigned to any face, then the v-edge is xy. In addition, add to H $g-2$ parallel edges connecting x and y to ensure that all degrees in H are at least $g-2$. Thus, H is a loopless multigraph with minimum degree at least $g-2$. By Proposition 11 with $d = g-2$ we can color the edges of H with $p = \lfloor \frac{3(g-2)+1}{4} \rfloor = \lfloor \frac{3g-5}{4} \rfloor$ colors such that every vertex $f \in V(H)$ is incident with edges of all p colors.

Define a vertex-coloring of G by coloring every vertex $v \in V(G)$ by the same color as that of the v-edge. This clearly gives a coloring in which every face $f \in F(G)$ is polychromatic, as needed. □

3.4 The upper bound of Theorem 2

Obviously if $g(G) = 1$, then G contains a loop. The plane graph G_1 with one single loop shows that $p(1) = 1$ and G_2 shows that $p(2) = 1$ as well (see Figure 1).



(a) $p(G_1) = 1$ (b) $p(G_2) = 1$

Figure 1: Graphs G_i with $p(G_i) = 1$ and $g(G_i) = i$, for $i = 1, 2$.

For all graphs G with $g(G) \geq 3$ we have already seen in Section 1 that $p(G) \geq 2$. A planar embedding of K_4 is a non-Eulerian triangulation and has therefore $g(K_4) = 3$ and $p(K_4) = 2$ (this is a trivial special case of Theorem 1).

For $g = 4$ we look at Figure 2(a) and (b) which shows a construction of a graph G. The graph G equals the forcing graph where each of the six bold edges $v_{k_1}v_{k_2}$ is replaced by one of six copies of the base graph by identifying the vertices v_i and v_j with the vertices v_{k_1} and v_{k_2}. Clearly, $g(G) = 4$. It is easy to check that the following holds.

Claim 15. *In every polychromatic 3-coloring of a base graph (see Figure 2(a)) the vertices v_i and v_j are colored with distinct colors.*

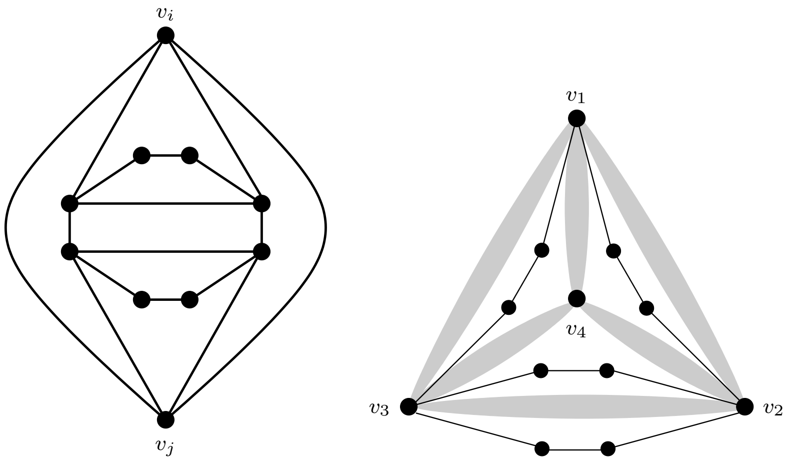
Thus from the fact that K_4, the graph underlying the forcing graph, is not properly 3-colorable, it follows that $p(G) = 2$. We next show that $p(g) \leq \lfloor \frac{3g+1}{4} \rfloor$ for all $g \geq 5$. Define the graph G_g as depicted in Figure 3. For g even set $k = l = \frac{g}{2}$ and for g odd set $k = \frac{g+1}{2}$ and $l = k - 1$. Inside the small triangle and outside the big triangle add a path of $g - 2$ new vertices as indicated by the dashed arcs. Then $g(G_g) = g$. Note that the vertices of the three faces of G_g that contain no dashed arcs are $\{u_1, u_2, \ldots, u_k, w_1, w_2, \ldots, w_k, v_1, v_2 \ldots, v_l\}$, and none of these vertices lies in all three faces. This implies the following:

Claim 16. *In any polychromatic coloring of G_g, every color appears on at least two vertices in the set $\{u_1, u_2, \ldots, u_k, w_1, w_2, \ldots, w_k, v_1, v_2 \ldots, v_l\}$.*

Using this claim we see that

$$2p(G_g) \leq 2k + l = \begin{cases} 3k, & \text{if } g \text{ is even.} \\ 3k - 1, & \text{if } g \text{ is odd.} \end{cases} = \begin{cases} \frac{3g}{2} & \text{if } g \text{ is even.} \\ \frac{3g+1}{2} & \text{if } g \text{ is odd.} \end{cases}$$

In both cases we thus have $p(G_g) \leq \left\lfloor \frac{3g+1}{4} \right\rfloor$. □



(a) Base graph for $g = 4$ (b) Forcing graph

Figure 2: Graph G with $p(G) = 2$ and $g(G) = 4$.

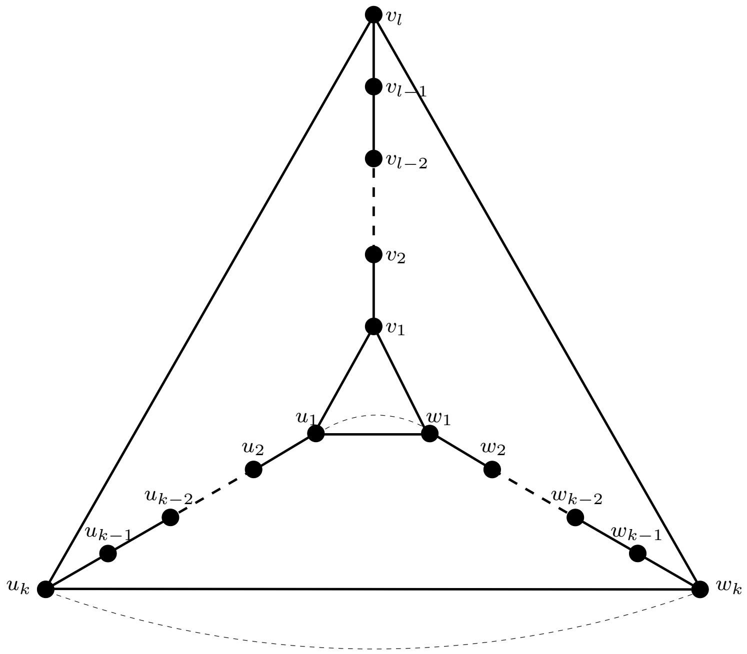


Figure 3: Graph G_g with $g(G_g) = g$ and $p(G_g) \leq \lfloor \frac{3g+1}{4} \rfloor$.

4 The complexity of polychromatic colorability

In this section we study the complexity of determining $p(G)$ for a given plane graph G. Let L denote some set of non-negative integers. We define the following two decision problems.

L-PLANE-PROPER-k-COLORABILITY:
Given: A plane graph G where the length of each of its faces is in L.
Question: Does there exist a proper k-coloring of $V(G)$?

L-PLANE-POLY-k-COLORABILITY:
Given: A plane graph G where the length of each of its faces is in L.
Question: Does there exist a polychromatic k-coloring of $V(G)$?

In case we do not impose any restriction on the lengths of the faces in G we omit the set L.

Since it can be checked in polynomial time whether a k-coloring is polychromatic, L-PLANE-POLY-k-COLORABILITY is in $\mathcal{NP}$. Obviously a similar observation is true for proper k-coloring as well.

Clearly every plane graph is polychromatically 1-colorable. Thus the decision problem PLANE-POLY-1-COLORABILITY is trivial in the sense that for every plane graph the answer to the problem is "Yes".

In the following three subsections we investigate the complexity of polychromatic 2-colorings, polychromatic 3-colorings and polychromatic 4-colorings, respectively. For $k \geq 5$, deciding whether or not PLANE-POLY-k-COLORABILITY is $\mathcal{NP}$-complete remains open.

4.1 Polychromatic 2-colorability

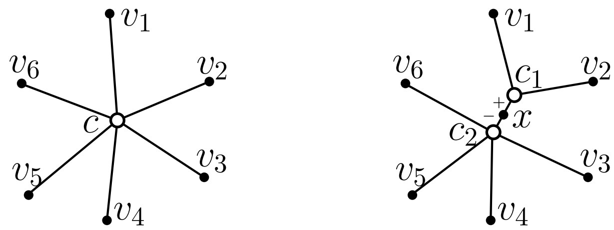
Theorem 17. *PLANE-POLY-2-COLORABILITY is in $\mathcal{P}$.*

Proof. (sketch) We call a CNF-formula F planar* if its vertex-clause incidence graph H is planar. Note that this differs from the common notion of a planar CNF-formula F', where one assumes that the vertex-clause incidence graph H together with a cycle $l_1, l_2, \ldots, l_n, l_1$ connecting the positive literals $l_1, \ldots, l_n$ of F' and together with edges between the corresponding positive and negative literals is required to be planar.

A vertex-coloring of a plane graph is 2-polychromatic if no face is monochromatic. We can associate with one color the logic predicate 'true' and with the other color 'false' and interpret the vertices as variables. Then we add a clause-vertex to each face and connect it to its incident variable-vertices. By this we get a planar* CNF-formula (where all variables occur only as positive ones).

Deciding whether the plane graph is 2-polychromatic is equivalent to deciding whether the corresponding planar* CNF-formula is not-all-equal satisfiable (PLANAR*-NAE-SAT).

In [12] it is shown that PLANAR-NAE-3-SAT is in $\mathcal{P}$ by a reduction to PLANAR-MAX-CUT. The reduction in fact holds also for PLANAR*-NAE-3-SAT. A well known reduction works to shorten the clauses of a planar (and planar*) formula to length 3, whilst preserving not-all-equal satisfiability and planarity. We briefly sketch this reduction which is illustrated in Figure 4. A clause c of length $k > 3$ is replaced by two clauses c_1, c_2 of length 3 and $k-1$, respectively. A new variable x occurs positive in c_1 and negative in c_2. Placing the new variable and clauses as in Figure 4 preserves planarity and not-all-equal satisfiability. □



(a) Clause c (b) Clauses c_1 and c_2

Figure 4: Reducing PLANAR-NAE-SAT to PLANAR-NAE-3-SAT.

4.2 Polychromatic 3-colorability

We start with a lemma about the complexity of proper colorings.

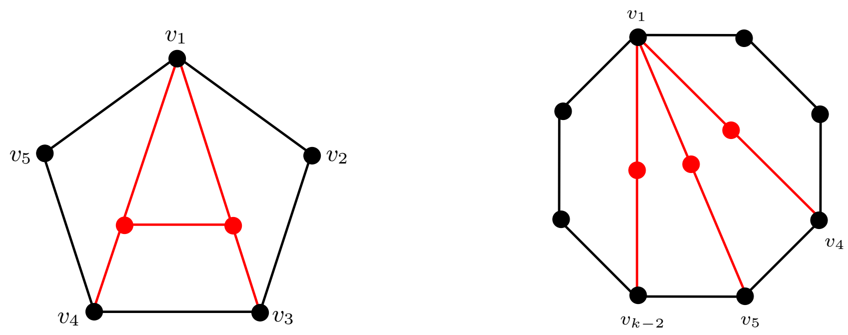
Lemma 18. *$\{3,4\}$-PLANE-PROPER-3-COLORABILITY is $\mathcal{NP}$-complete.*

Proof. It has been shown in [15] that PLANE-PROPER-3-COLORABILITY is $\mathcal{NP}$-complete. We construct in polynomial time from a plane graph G another plane graph G' with faces of length 3 and 4 only such that G is properly 3-colorable if and only if G' is properly 3-colorable.

First we iteratively process every k-face f of G with $k \geq 6$ and subdivide f such that every newly introduced face has length 5, as follows. Let $v_1, v_2, \ldots, v_k$ be the vertices of f. We add $k-3$ copies of P_2, a path of length two with vertices u, v, w to f by identifying the ith copy of u with v_1 and the ith copy of w with v_k, for $i = 4, \ldots, k-2$ (see Figure 5(b)). After subdividing all faces of length at least 6 every face has length $3, 4$ or 5. Every proper 3-coloring of the old vertices can be extended to a proper coloring of the new graph, since the new vertices, the copies of v, have degree 2 only.

Next we subdivide each 5-face f by the construction shown in Figure 5(a). In more detail, let $v_1, v_2, \ldots, v_5$ be the five vertices of f. We add two copies of P_2 with vertices u, v, w by identifying both copies of u with v_1, the first copy of w with v_3 and the second copy of w with v_4. Further we connect the first copy of v with the second copy of v. Again it is easy to check that every proper 3-coloring of the 5-face has an extension to a proper 3-coloring of this subdivision.

This implies that if G is properly 3-colorable then G' is also properly 3-colorable. Since $G \subseteq G'$, the other direction is trivial. □



(a) 5-face

(b) k-face, $k \geq 6$

Figure 5: Fill graphs for k-faces, $k \geq 5$.

Let us now prove Theorem 3, that is, show that $\{3,4\}$-PLANE-POLY-3-COLORABILITY is $\mathcal{NP}$-complete.

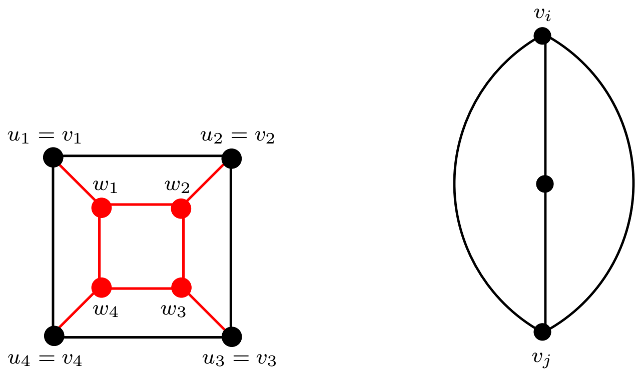
Proof of Theorem 3. We present a reduction from $\{3,4\}$-PLANE-PROPER-3-COLORABILITY discussed above to $\{3,4\}$-PLANE-POLY-3-COLORABILITY. Let G be a plane graph with faces of length 3 and 4 only. We construct in polynomial time a plane graph G' with faces of length 3 and 4 only such that G is properly 3-colorable if and only if G' is polychromatic 3-colorable.

In a first step we "fill" every 4-face of G by substituting it with a copy of the graph in Figure 6(a). Let f be a 4-face of G containing the four consecutive vertices v_1, v_2, v_3, v_4. We identify v_i with the copy of the vertex u_i for $i \in \{1, \ldots, 4\}$. We show that the resulting subgraph is polychromatically 3-colorable if f is properly 3-colorable. For this, we fix a proper 3-coloring χ of f. Suppose first that all three colors appear on the vertices of f. Without loss of generality we can assume that $\chi(v_1) = 1, \chi(v_2) = 2, \chi(v_3) = 3$ and $\chi(v_4) = 2$. Then, for instance, coloring the copies of w_1 by 3, w_2 by 2, w_3 by 1 and w_4 by 2 extends χ to a 3-coloring of the new vertices in f such that each of the five new faces inside f is polychromatic. Suppose now that only two distinct colors appear on the vertices of f, say $\chi(v_1) = \chi(v_3) = 1$ and $\chi(v_2) = \chi(v_4) = 2$. We can extend χ to a polychromatic 3-coloring including the new vertices in f as follows. Color w_1 by 3, w_2 by 2, w_3 by 3 and w_4 by 1. Again the five new faces inside f are polychromatic.

In a second step we construct G' by substituting every edge uv that was already present in G with a copy of the base graph from Figure 6(b). For this, we identify the copy of vertex v_i with u and the copy of vertex v_j with v. It is easy to see that similarly to Claim 15 in every polychromatic 3-coloring of G' the vertices v_i and v_j are colored with distinct colors. So G' can only be polychromatically 3-colorable if G is properly 3-colorable. Furthermore, G' contains only faces of length 3 and 4. This concludes the proof.

$\square$

Similarly one can show the hardness of determining the (poly-)chromatic number for graphs G with larger faces. We omit the detailed proofs here.



(a) Fill gadget for 4-faces

(b) Base graph

Figure 6: Gadgets for the reduction

Proposition 19. *L-PLANE-PROPER-3-COLORABILITY is $\mathcal{NP}$-complete for*

(i) $L = \{3, s\}, s \geq 4$;

(ii) $L = \{4, t\}, t \geq 5$ *odd; or*

(iii) $L = \{t\}, t \geq 5$ *odd.*

Proposition 20. *L-PLANE-POLY-3-COLORABILITY is $\mathcal{NP}$-complete for*

(i) $L = \{3, s\}, s \geq 4$; *or*

(ii) $L = \{4, t\}, t \geq 5$ *odd.*

Recall that $2 \leq p(5) \leq 4$ and $p(6) \geq 3$ by Theorem 2. Hence it may be that $p(5) \geq 3$, that is, every plane graph G with $g(G) \geq 5$ is polychromatically 3-colorable. Nevertheless we can show the following conditional hardness result.

Proposition 21. $\{5, 6, 7, \ldots\}$*-PLANE-POLY-3-COLORABILITY is $\mathcal{NP}$-complete provided that* $p(5) = 2$.

Proof. (sketch) The proof proceeds along the same lines as the proof for Theorem 3. This time we reduce from $\{5\}$-PLANE-PROPER-3-COLORABILITY, see Proposition 19.

Now we construct a base graph H with $g(H) \geq 5$ that is polychromatically 3-colorable. Moreover H has exactly two outer vertices u and x and these are colored with two distinct colors in every polychromatic 3-coloring of H. In the reduction every original edge of G is substituted by a copy of H. For constructing H we use the assumption $p(5) = 2$.

Assume that there are graphs with smallest face length 5 and having no polychromatic 3-coloring, that is $p(5) = 2$. Fix such a graph G which is minimal with respect to the number of edges. Deleting any edge e' from G that is incident to two distinct faces of G results in a graph $G - e'$ with $p(G - e') \geq 3$ and $g(G - e') \geq 5$.

Let $e = xy \in E(G)$ be an edge which lies on the boundary of the outer face f_1 and is incident to another face f_2 of G. Such an edge e has to exist, otherwise G contains only one single face (G

forms a tree), and thus $p(G) = |V(G)|$. Note here that e might form a loop and in that case $x = y$. As shown, $G - e$ is polychromatically 3-colorable. Denote by V_i the set of vertices contained in f_i, for $i \in \{1, 2\}$ and define $\chi(V_i)$ to be the set of colors appearing on V_i, $\chi(V_i) = \{\chi(v) \mid v \in V_i\}$.

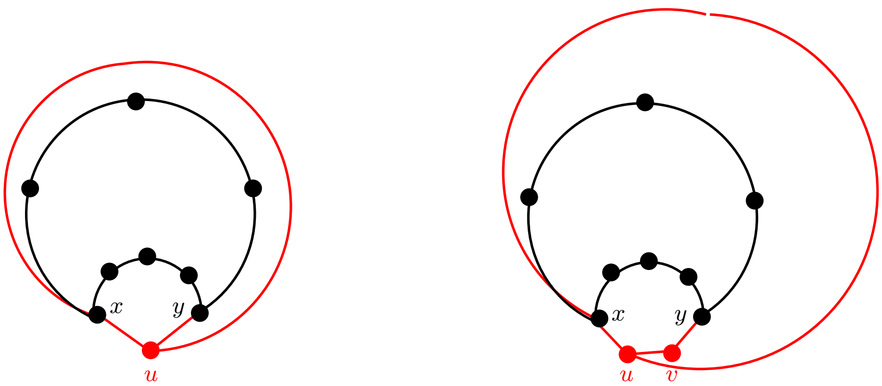
For each polychromatic 3-coloring χ of $G-e$ the vertices of at least one of the two sets V_1 or V_2 are colored with less than 3 distinct colors, that is $|\chi(V_1) \cap \chi(V_2)| < 3$. Indeed, otherwise χ is also a polychromatic 3-coloring of G. On the other hand $|\chi(V_1) \cap \chi(V_2)| \geq 1$, since $V_1 \cap V_2 = \{x, y\}$.

We distinguish two types of polychromatic 3-colorings χ of $G - e$:

type I: $\chi(V_1) \cap \chi(V_2)$ contains two distinct colors.

type II: $\chi(V_1) \cap \chi(V_2)$ contains only one color.

If there exists a polychromatic 3-coloring of $G-e$ of *type I* then we add one new vertex u to $G-e$ and connect it to x and y (see Figure 7(a)). Otherwise, if every polychromatic 3-coloring of $G-e$ is of *type II*, we add two new vertices u, v and connect the vertices x, u, v, y with a path of length three such that x and y are its endpoints and u and v are its inner vertices (see Figure 7(b)). Clearly the newly constructed graph H is polychromatically 3-colorable. In each polychromatic 3-coloring of H, u (and possibly v) is colored with a color not appearing on any other vertex of either V_1 or V_2. Hence the color of x and u have to be distinct. In a last step we add a second edge to the vertex-pair x and u in H, such that the face formed by the two edges with endpoints x and u contains all other vertices of the new graph H. We again call H a base graph, see for instance Subsection 3.4.



(a) *type I*   (b) *type II*

Figure 7: Construction of the base graph H.

□

Let us note that the method for constructing the base graph in the above proof can be used to obtain base graphs for the case $p(G) = 3$ and $g(G) < 5$ as well.

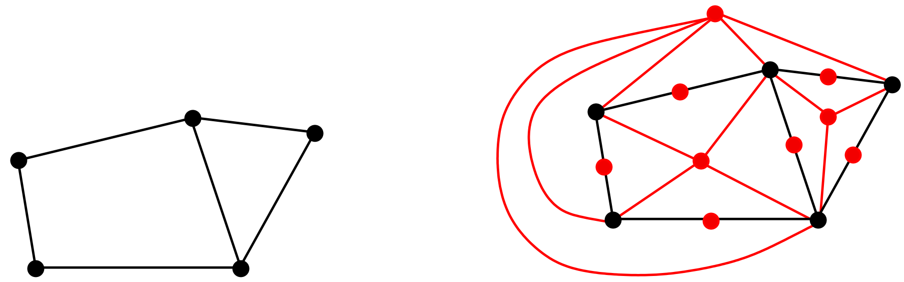
4.3 Polychromatic 4-colorability

Theorem 22. $\{4\}$-*PLANE-POLY-4-COLORABILITY is* $\mathcal{NP}$*-complete.*

Proof. We reduce from PLANE-PROPER-3-COLORABILITY.

Let G be a plane graph. On each edge $uv \in E(G)$ we add a new vertex x_{uv} and replace the edge uv by a path of length two with vertices u, x_{uv}, v. For each face $f \in F(G)$ we add a vertex v_f, place it into the interior of f, and connect v_f with all vertices of G contained in f. This yields a new plane graph G' with $|V(G')| = |V(G)| + |E(G) + |F(G)|$ and all faces having length exactly 4. See Figure 8 for an example.

We claim that G is properly 3-colorable if and only if G' is polychromatically 4-colorable.



(a) original graph G

(b) new graph G'

Figure 8: Example of the construction of the graph G'.

If G is properly 3-colorable with colors $1, 2$ and 3, then we extend this coloring χ in G' such that each vertex v_f corresponding to a face f in G gets color 4 and the vertex x_{uv} with neighbors u and v gets the color $\{1,2,3\} \setminus \{\chi(u), \chi(v)\}$. In this way each face of G' is polychromatic and therefore the whole coloring χ is polychromatic.

Now let us fix a polychromatic 4-coloring χ' of G'. Let v_f be any vertex of G' corresponding to a face f of G. Without loss of generality suppose that v_f has color 4. Then for each edge $uv \in E(G)$ which is incident to f the vertices $u, x_{uv}, v \in V(G')$ have to get the colors $1, 2$ or 3. Henceforth for every face g of G that is incident to f, the vertex v_g gets color 4 as well. We can assume that G is connected and thus also G^* (the dual graph of G) is connected. As a consequence the color 4 "propagates" from face to face and $\chi'(v_{f'}) = 4$ for every face f' of G'. Also color 4 appears at no other vertex of G'. Now the coloring restricted to the vertices in G has to be proper because every 4-face f with vertices u, x_{uv}, v, v_f of G' can only be polychromatic if all of its four vertices are colored with distinct colors, and in particular u and v get distinct colors. □

5 Concluding remarks and open problems

One can consider polychromatic edge-colorings of plane graphs, rather than polychromatic vertex-colorings. Here the situation is simpler, and a direct application of Proposition 11 to the dual of a given plane graph in which every face contains at least g edges implies that the edges of any such plane graph can be colored by $\lfloor \frac{3g+1}{4} \rfloor$ colors so that every color appears in every face. This is tight, as shown by the plane graph consisting of two vertices with three internally vertex disjoint paths P_1, P_2, P_3 between them, with P_1, P_2 of length $\lceil g/2 \rceil$ and P_3 of length $\lfloor g/2 \rfloor$.

The assertion of the main combinatorial result can be (slightly) strengthened; it actualy implies that the vertices of any plane graph G (with no assumption on $g(G)$) can be colored by $\lfloor \frac{3g-5}{4} \rfloor$ colors, so that every face of length at least g contains vertices of all colors. This follows, for example, by applying Theorem 2 to the graph G' obtained from G as follows: for every face f of G of length smaller than g, add a path containing, say, g internal vertices between an arbitrarily chosen pair of vertices of f, where this path lies inside the face (and hence replaces it by two faces of length at least g each).

The proof of Theorem 2 can be easily extended to colorings of graphs embedded on surfaces of higher genus. In fact, one can consider polychromatic colorings of general hypergraphs. Call a vertex-coloring of a hypergraph $H = (V, E)$ polychromatic if all colors appear in every hyperedge. The polychromatic number of H is the maximum k such that there is a polychromatic vertex-coloring of H with k colors. A close look at the proof of Theorem 2 shows that it actually gives the following.

Theorem 23. *For every constant c there is a constant $b(c)$ so that the following holds: Let $H = (V, E)$ be a hypergraph in which the number of incidences between any set $V' \subseteq V$ and $E' \subseteq E$ is at most $2(|V'| + |E'|) + c$. Suppose, further, that each hyperedge of H is of cardinality at least g. Then, the polychromatic number of H is at least $\frac{3}{4}g - b(c)$.*

Indeed, the assumption about the incidences replaces the assertion of Lemma 8, and the rest of the proof is essentially identical to that of Theorem 2.

If G is a graph embedded on a surface of Euler characteristic χ, then the argument in the proof of Lemma 8 and Euler's formula for the corresponding surface imply that the number of incidences between any set V' of vertices of G and any set of faces of it F', with $|F'| \geq 2$ is at most $2(|V'| + |F'| - \chi)$ (it is $|V'|$ if $|F'| = 1$, and 0 if $F' = \emptyset$). We can thus apply the theorem above to the hypergraph whose vertices are the vertices of G and whose edges are the sets of vertices of the faces of G, and conclude that if every face has at least g vertices then there is a vertex coloring with $\frac{3}{4}g - b(\chi)$ colors so that every color appears in every face.

We close with a few open problems.

Open Problem 1. *Determine $p(g)$ exactly for every positive integer g. The first open case is $g = 5$ where we know that $2 \leq p(5) \leq 4$.*

Open Problem 2. *Is the problem PLANE-POLY-k-COLORABILITY $\mathcal{NP}$-complete, for every fixed $k \geq 5$?*

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