High-girth near-Ramanujan graphs with localized eigenvectors

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Abstract

We show that for every prime d and $\alpha \in (0, 1/6)$, there is an infinite sequence of $(d+1)$-regular graphs $G = (V, E)$ with girth at least $2\alpha \log_d(|V|)(1 - o_d(1))$, second adjacency matrix eigenvalue bounded by $(3/\sqrt{2})\sqrt{d}$, and many eigenvectors fully localized on small sets of size $O(|V|^\alpha)$. This strengthens the results of [GS18], who constructed high girth (but not expanding) graphs with similar properties, and may be viewed as a discrete analogue of the "scarring" phenomenon observed in the study of quantum ergodicity on manifolds. Key ingredients in the proof are a technique of Kahale [Kah92] for bounding the growth rate of eigenfunctions of graphs, discovered in the context of vertex expansion and a method of Erdős and Sachs for constructing high girth regular graphs.

1 Introduction

We study the relationship between geometric properties of finite regular graphs, such as girth and expansion, and localization properties of their Laplacian / adjacency matrix eigenvectors. This line of work was initiated by Brooks and Lindenstrauss, who proved that the eigenvectors of high girth graphs cannot be too localized in the following sense (in fact, they studied graphs with few short cycles, but we will state the restriction of their results for high girth graphs for simplicity).

Theorem 1.1 ([BL13]). *Suppose $G = (V, E)$ is a $(d+1)-$regular graph with adjacency matrix A. Then for any normalized ℓ_2 eigenvector $v \in \mathbb{R}^V$ of A and $S \subset V$ with $\|v_S\|_2^2 := \sum_{x\in S} v_x^2 \geq \varepsilon$,*

$$|S| \geq \Omega_d(\varepsilon^2 d^{2^{-7}\varepsilon^2 \mathrm{girth}(G)}), \tag{1}$$

where girth(G) *denotes the length of the shortest cycle in G.*

The recent work [GS18] improved (1) to

$$|S| \geq \frac{\varepsilon d^{\varepsilon \mathrm{girth}(G)/4}}{2d^2}, \tag{2}$$

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under the same assumptions on G. Moreover, given any $\varepsilon \in (0,1)$, they proved that for infinitely many $m \in \mathbb{N}$ there is a $(d+1)-$regular graph G_m with m vertices, $\mathrm{girth}(G_m) \geq (1/8)\log_d(m)$, and a localized eigenvector satisfying $\|v_S\|_2^2 = \varepsilon$ and $|S| \leq O(d^{4\varepsilon \mathrm{girth}(G_m)})$. This shows that (2) is sharp up to constants and a factor of εd^{-2} in the regime where the girth is logarithmic in the number of vertices.

In this work, we construct examples which improve on the above in three ways: (1) the graphs we construct are expanders with near-optimal spectral gap, (2) we improve the bounds on $\mathrm{girth}(G_m)$ as well as the localization size $|S|$ by constant factors, (3) our constructions are explicit whereas [GS18] used the probabilistic method to show existence non-constructively.

Theorem 1.2. *For every $d = p+1$, p prime, and parameter $\alpha \in (0, 1/6)$ there are infinitely many integers m such that there exists a $(d+1)-$regular graph $G_m = (V_m, E_m)$ on m vertices with the following properties,*

1. $|\lambda_i(A_m)| \leq (3/\sqrt{2})\sqrt{d}$ *for all nontrivial eigenvalues $i \neq 1$ of the adjacency matrices A_m.*
2. $\mathrm{girth}(G_m) \geq 2\alpha \log_{2d-1}(m) - O(1) = 2\alpha\log_d(m) \cdot (1 - O(\log^{-1}(d)))$.
3. *There is a set $S_m \subset V_m$ of size $O(m^\alpha)$ such that A_m has at least $\ell_m := \lfloor \alpha \log_d(m) \rfloor$ eigenvalues $\lambda \in (-2\sqrt{d}, 2\sqrt{d})$ with corresponding eigenvectors $v : A_m v = \lambda v$ supported entirely on S_m.*

Moreover, the set of eigenvalues λ realized by the localized eigenvectors of A_m, over all such m is dense in the interval $(-2\sqrt{d}, 2\sqrt{d})$.

Note that the number $3/\sqrt{2} \cdot \sqrt{d} \approx 2.121\sqrt{d}$ above is quite close to the best possible bound of $2\sqrt{d}$ for an infinite sequence of regular graphs [Nil91].

Remark 1.1. (Partial Localization on Smaller Sets). In fact the proof of Theorem 1.2 produces eigenvectors v, with the additional property that for $\varepsilon \in (0,1)$, there exists a subset S of vertices with $|S| = O(m^{\Theta(\varepsilon)\alpha})$, and $\|v_S\|_2^2 \geq \Theta(\varepsilon)$.

Finally, we show how to modify our construction to produce many localized eigenvectors corresponding to eigenvalues with very high multiplicity.

Theorem 1.3 (Many Localized Eigenvectors)**.** *Theorem 1.2 is true with the last property replaced by: there are $\ell_m := \alpha \log_d(m)$ eigenvalues $\lambda_1, \ldots, \lambda_{\ell_m}$, each of multiplicity at least $\Omega(m^{1-4\alpha})$, such that each eigenspace has a basis of orthogonal eigenvectors supported on sets of size $O(m^\alpha)$.*

1.1 Implications for Quantum Ergodicity on Graphs

The additional property of expansion in our examples is relevant to the study of quantum ergodicity on graphs. Anantharaman and Le Masson proved that if a sequence of graphs has few short cycles *and* a spectral gap, then the eigenvectors must be equidistributed on average in a sense stronger than Theorem 1.1.

Theorem 1.4. *[ALM15, BLML15] Suppose $G_m = (V_m, E_m)$ is a sequence of $(d+1)$-regular graphs on m vertices with adjacency matrices A_m satisfying:*

(BST) The sequence of graphs converges to a tree in the sense of Benjamini-Schramm, i.e., there exist $R_m \to \infty$ and $\alpha_m \to 0$, such that

$$\frac{1}{m}|\{v \in V_m : N_m(v, R_m) \text{ contains a cycle}\}| \leq \alpha_m,$$

where $N_m(v, R)$ is the set of vertices at distance at most R from v, in G_m. Note that this condition is implied by $\mathrm{girth}(G_m) \to \infty$.

(EXP) There is a constant $\beta > 0$ such that

$$|\lambda_i(A_m)| < (d+1)(1-\beta),$$

for all nontrivial eigenvalues $i \neq 1$.

Then for any sequence of test functions $a_m : V_m \to \mathbb{R}$ with $\sum_{x \in V_m} a_m(x) = 0, \|a_m\|_\infty \leq 1$:

$$\frac{1}{m}\sum_{i \leq m} |\langle \psi_i^{(m)}, a_m \psi_i^{(m)} \rangle|^2 \lesssim \beta^{-2} \min\{R_m, \log(1/\alpha_m)\}^{-1} \|a_m\|_2^2 + \alpha_m^{1/2} \|a_m\|_\infty^2 \longrightarrow 0, \tag{3}$$

where $\psi_1^{(m)}, \ldots \psi_m^{(m)}$ is any eigenbasis of A_m.

The above may be viewed as a discrete analogue of the quantum ergodicity theorem of Shnirelman, Zelditch, and Colin de Verdiére [Shn74, DV85, Z+87], which states that if the geodesic flow on a compact manifold is ergodic, then it must have a dense subsequence of Laplacian eigenfunctions whose mass distribution converges weakly to the volume measure as the energy goes to infinity. In Theorem 1.4, the manifold has been replaced by a sequence of graphs, the condition of ergodic geodesic flow has been replaced by BST and EXP, and the notion of weak convergence involves a sequence of test functions on the graphs rather than a single test function on the manifold.

An even stronger notion of delocalization for the Laplacian on a manifold is Quantum Unique Ergodicity (QUE) (see e.g. [Sar12] for a detailed discussion), where instead of a dense subsequence of eigenfunctions, one requires that *every* subsequence of eigenfunctions becomes equidistributed. It is not completely clear what the correct analogous notion should be for a sequence of finite graphs. There are various proposals; one definition which appears in Anantaraman's ICM survey [Ana18] and in [LMS+17, Question 1.3] in the context of sequences of manifolds is: for every sequence of test functions a_m as in Theorem 1.4, and *every* sequence of eigenfunctions $\psi_{i_m}^{(m)}$, one has

$$|\langle \psi_{i_m}^{(m)}, a_m \psi_{i_m}^{(m)} \rangle| \longrightarrow 0. \tag{4}$$

Since the graphs constructed in Theorem 1.2 satisfy BST and EXP, the theorem shows that these properties cannot imply unique ergodicity in the above sense: take the $\psi_{i_m}^{(m)}$ to be the localized eigenvectors of G_m, and let a_m be the indicator functions of the sets S_m on which they are localized, translated by a constant to have mean zero. It is then immediate that $\langle \psi_{i_m}^{(m)}, a_m \psi_{i_m}^{(m)} \rangle = 1 - o(1)$ for the entire sequence.

The presence of localized eigenvectors is sometimes referred to as "scarring" (see e.g. [Ana18, HH10]), which may be partial or complete depending on whether a large fraction or all of the mass is localized on a small set. Theorem 1.2 and Remark 1.1 may be interpreted as saying that scarring can occur even under strong expansion and girth assumptions.

Remark 1.2 (QE over intervals)**.** The works [ALM15, BLML15] also study a more refined version of quantum ergodicity on graphs, where the average (3) is taken over a spectral window $I \subset (-2\sqrt{d}, 2\sqrt{d})$ rather than the entire spectrum. These results hold on intervals I of width roughly $1/\log(m)$, and it would be interesting to see whether our examples can prove a lower bound on the length of the smallest window that is possible. While Theorem 1.3 does produce many localized eigenvectors in a very small window (due to high multiplicity), the problem of controlling the other eigenvectors well enough to say that the average in a small window is not equidistributed is not pursued here and remains open.

1.2 Techniques and Vertex Expansion

The starting point of the proofs of Theorems 1.2 and 1.3 is a construction in the proof of [GS18, Theorem 1.6] which has the following ingredients:

1. (Pairing trees [GS18, Lemma 3.4]): A pair of trees is glued by randomly identifying their leaves, ensuring that the final graph has high girth.

2. (Degree-Correction [GS18, Lemma 3.5]): The above gluing yields an irregular graph where the identified leaf vertices have degree two. Each such vertex is identified with a particular vertex of degree $d-1$ in a degree-correcting gadget whose remaining vertices have degree $d+1$ thereby yielding a $d+1$ regular graph.

We modify this proof in two ways. First, we replace the random pairing in step (1) by a more efficient, simpler, and deterministic method. Second, in order to obtain the additional property of expansion, we replace the degree-correcting gadget in step (2) by a high girth Ramanujan graph [LPS88, Mur03]. To analyze the spectrum of the resulting graph, we must argue that its largest nontrivial eigenvector cannot have too much mass on the interface between the trees and the Ramanujan graph — once this is established, it is easy to analyze the contributions from the two pieces separately. We do this by employing a lemma of N. Kahale, which supplies a way to control the mass of eigenvectors on certain highly symmetric sets (such as our interface) by exhibiting certain appropriate super-harmonic test functions, and by a careful construction of such a function.

Kahale's lemma originally appeared in the influential paper [Kah92] which showed that a $(d+1)$-regular graph $G=(V,E)$ with all nontrivial eigenvalues bounded by $2\sqrt{d}+o_n(1)$ must have linear expansion at least $(d+1)/2-o_n(1)$, where linear expansion is defined as:

$$\max_{S\subset V, |S|=\gamma|V|} \frac{N(S)}{|S|},$$

for a small constant $\gamma>0$ (in fact, he showed a more general inequality relating the parameters). As we discuss in Remark 4.1, this implies that our examples cannot have $|\lambda_i|\leq 2\sqrt{d}+o_n(1)$ since our gluing procedure produces a set with significantly smaller linear vertex expansion than $(d+1)/2$.

Note that it is possible to prove Theorem 1.2 with a weaker bound of $3\sqrt{d}$ without using Kahale's lemma; however, since the bound we attain is quite close to optimal and we have not seen this technique appear in the quantum ergodicity literature, we believe it is valuable to present it.

2 Pairing trees

Our goal is to construct high girth almost-Ramanujan expanders with one or many localized eigenvectors. The starting point of the construction is the following lemma, improving the one from [GS18] and simplifying its proof. We refer to a finite tree in which all vertices except the leaves have degree $(d+1)$ and every leaf is at distance D from the root as a *d-ary tree of depth D*.

Lemma 2.1 (Pairing of Trees)**.** *Suppose T_1 and T_2 are two $d-$ary trees of depth D, each with $n=(d+1)d^{D-1}$ leaf vertices V_1 and V_2. Then there is a bijection $\pi: V_1\to V_2$ such that the graph obtained from the vertex disjoint union of T_1 and T_2 by identifying v and $\pi(v)$ for all v has girth at least*

$$\lfloor 2\log_{2d-1}(n-1)\rfloor + 2 \;(\; > 2\log_{2d-1} n).$$

Proof. We apply a variant of the method of Erdős and Sachs [ES63], (see also [ABGR18] for a similar argument). Let g be the maximum possible girth of a graph obtained as above, and let $\pi : V_1 \to V_2$ be a bijection for which the girth is g and the number of cycles of length exactly g is minimum. Note that g is even, as the graph is bipartite. Let G be the graph with the identified leaves obtained by π, and let L denote the set of all n vertices of degree 2 in it, that is, all the identified leaves. Obviously every cycle of G must contain vertices of L. Let $x \in L$ be a vertex contained in a shortest cycle C of G.

Claim: For every $k \geq 0$ the number of vertices $y \in L$ of distance at most $2k$ from x is at most $(2d-1)^k$.

Proof of claim: Any shortest path of length precisely $2s \leq 2k$ between x and another vertex $y \in L$ is a concatenation of some number r of paths $P_1, P_2, \ldots, P_r$, where P_i is a path from x_{i-1} to x_i with $x_0, x_1, \ldots, x_r \in L$, $x_0 = x$, $x_r = y$, and either all even paths P_i are in T_1 and all odd ones are in T_2 or vice versa. Let $2k_i$ be the length of P_i, then $\sum_{i=1}^{r} k_i = s \leq k$. Let $m(r,s)$ denote the number of paths as above with these values of r and s. We next show that for all $1 \leq r \leq s$

$$m(r,s) = 2\binom{s-1}{r-1}(d-1)^r d^{s-r} \tag{5}$$

The factor 2 is for deciding if the first path P_1 is in T_1 or in T_2. The factor $\binom{s-1}{r-1}$ is the number of ways to choose the subset of $(r-1)$ elements $\{k_1, k_1+k_2, \ldots k_1+k_2+\cdots+k_{r-1}\}$ in the set $\{1, 2, \ldots, s-1\}$. (This already determines $k_r = s - (k_1 + k_2 + \cdots + k_{r-1})$.) Once these choices are fixed, there is only one way for the edges numbers $1, 2, \ldots, k_i$ of each path P_i, given the previous paths, as these edges go up the tree. There are $d-1$ possibilities for the edge number k_{i+1} of this path, and there are d^{k_i-1} choices for the remaining edges of P_i. The product of all these terms gives the expression in (5) for $m(r,s)$. For each fixed s, summing over all $1 \leq r \leq s$ we conclude that the number $m(s)$ of paths of length exactly $2s$ starting in x is

$$m(s) = \sum_{r=1}^{s} 2\binom{s-1}{r-1}(d-1)^r d^{s-r} = 2(d-1)\sum_{j=0}^{s-1}\binom{s-1}{j}(d-1)^j d^{s-1-j} = 2(d-1)(2d-1)^{s-1}.$$

Adding the trivial path of length 0 from x to itself and summing over all s from 1 to k we conclude that the total number of paths as above of length at most $2k$ starting at x is

$$1 + \sum_{s=1}^{k} 2(d-1)(2d-1)^{s-1} = 1 + 2(d-1)\frac{(2d-1)^k - 1}{2d-2} = (2d-1)^k.$$

The number above provides an upper bound for the number of vertices $y \in L$ that lie within distance $2k$ of x (which may be smaller as several paths may lead to the same vertex). This completes the proof of the claim.

Returning to the proof of the lemma, define $k = \lfloor \log_{2d-1}(n-1) \rfloor$. Then $(2d-1)^k < n$ and hence there is a vertex $y \in L$ whose distance from x in G is larger than $2k$ (and hence at least $2k+2$). Let u be the unique parent of x in T_1 and let u' be the unique parent of x in T_2. Similarly, let v be the unique parent of y in T_1 and let v' be the unique parent of y in T_2. Change the bijection π to the bijection π' obtained by swapping the images of x and y to get a graph G' obtained from G by removing the edges xu and yv and by adding the edges xv and yu. This swapping removes the shortest cycle C of length g in G that contains x, and is not contained in G'. Every new cycle contained in G' and not in G must include at least one of the new edges xv, yu. If it contains exactly one of them, say xv, then it must also contain a path in G from x to v. The length of such a path is at least $2k+1$ (as the distance in G from x to y is at least $2k+2$) showing that in this case the length of the new cycle is at least $2k+2$. If it contains both new edges xv and yu it must also contain either a path in G from x to y (of length at least $2k+2$)) or a path in G from x to u (of length at least $g-1$) and a path in G from y to v (of length at least $g-1$). Therefore, in this case the length of the new cycle is either at least $2k+2+2 = 2k+4$ (in fact larger) or at least $2g-2+2 = 2g > g$. It follows that the only possibility to obtain a new cycle of length at most g is if $g \geq 2k+2$. If the girth g is smaller than G' has girth at least g and the number of its cycles of length g is smaller than that number in G, contradicting the choice of G. This shows that the girth g satisfies $g \geq 2k+2 = 2\lfloor \log_{2d-1}(n-1) \rfloor + 2$, completing the proof of the lemma. □

Remark 2.1. For large fixed d, the above graph has girth close to $2\log_d N$, where N is the number of its vertices. This is strictly larger than the highest known girth of an N vertex $(d+1)$-regular graph, which is roughly $\frac{4}{3}\log_d N$ (for some values of d). However, many of the vertices of our graph here have degree 2, and suppressing them will not give graphs of girth larger than $(1+o(1))\log_d N$.

3 The construction

Let d a prime and $\alpha \in (0, 1/6)$ be given. Let $H = (V, E)$ be a $(d+1)$-regular non-bipartite Ramanujan graph with m vertices and girth larger than $2/3\log_d(m)$; by [LPS88] such graphs exist for infinitely many m. Set $r = \lfloor \alpha \log_d(m) \rfloor$ and note that

$$\frac{2}{3}\log_d m \geq 4r. \tag{6}$$

Fix a vertex u of H. The induced subgraph on all vertices of distance at most r from u is a tree T_1 rooted at u. Let n be the number of its leaves, let the set of leaves be $L_1 = \{u_1, .., u_n\}$ and let V_1 denote the set of all non-leaves of T_1. Take a matching from the set L_1 to the set of vertices $L_2 = \{v_1, .., v_n\}$, all at distance exactly $r+1$ from u, and remove the matching $u_i v_i$. Note that all u_i are far from each other in the graph $H - V_1$ since the girth is significantly larger than $2r$. Similarly, all vertices v_i are far from each other in $H - V_1$ for the same reason. Now take another d-tree T_2 isomorphic to T_1 on new vertices, and let u' denote its root. Identify the leaves of T_1 with these of T_2 using Lemma 2.1. Let V_2 denote the set of all non-leaves of T_2. As the vertices u_i are far from each other in $H - V_1$ the girth stays as large as guaranteed by the lemma. Finally add a third tree T_3 with the same parameters on new vertices, rooted at v', identify its leaves with the vertices v_i and let V_3 denote the set of its non-leaves. Call the resulting graph G.

The next sequence of lemmas will be needed to show that G satisfies the claims of Theorem 1.2.

Lemma 3.1. *The girth of G is at least*

$$2\log_{2d-1}((d+1)d^{r-1}) \; (\geq 2\alpha \log_{2d-1}(m) - O(1)).$$

Proof. Follows from the above definition, Lemma 2.1 and that $\frac{2}{3}\log_d m > 4r$. □

We now discuss the eigenvectors of G. We begin by recording some facts about eigenvalues and eigenvectors of rooted d−ary trees which also appear in [GS18] and will be critical to our construction. Recall that the eigenvalues of a d−ary tree are contained in the interval $(-2\sqrt{d}, 2\sqrt{d})$ [HLW06, Section 5]. For our purposes we will only consider eigenvalues corresponding to eigenvectors which are *radial*, which means that they assign the same value to vertices in a given level. We will refer to such eigenvalues as *radial eigenvalues*.

Lemma 3.2. *(Radial Eigenvalues)[GS18, Lemma 3.1] For any positive integer $D \geq 2$, A_D the adjacency matrix of T_D, a $d-$ary tree of depth D, has exactly $D+1$ radial eigenvalues counting multiplicities.*

Lemma 3.3. *(Eigenvalues of $d-$ary Trees)[GS18, Lemma 3.2] The set of all radial eigenvalues of any infinite sequence of distinct finite $d-$ary trees is dense in the interval $(-2\sqrt{d}, 2\sqrt{d})$.*

Lemma 3.4 (Eigenvectors of $d-$ary Trees)**.** *[GS18, Lemma 3.3] Assume $d \geq 2$ and let T be a $d-$ary tree of depth D with root r. Let $S_0 = \{r\}, S_1, \ldots, S_D \subset T$ be the vertices at levels $0, 1, \ldots, D$ of the tree and let v be a radial eigenvector of its adjacency matrix with eigenvalue $\lambda = 2\sqrt{d}\cos\theta \in (-2\sqrt{d}, 2\sqrt{d})$. Then every pair of adjacent levels has approximately the same total ℓ_2^2 mass as the root:*

$$\Omega(\sin^2\theta) = \frac{\|v_{S_i}\|_2^2 + \|v_{S_{i+1}}\|_2^2}{\|v(r)\|_2^2} = O(1/\sin^2\theta).$$

Given the above we have the following lemma about how radial tree eigenvectors can be used to construct eigenvectors of G.

Lemma 3.5. *For any radial eigenvalue λ of the adjacency matrix of a tree of depth $r-1$, there exists an eigenvector ν supported on $V_1 \cup V_2$ such that*

$$A_G \nu = \lambda \nu.$$

Proof. For completeness we include the arguments that essentially appear in [GS18, Proof of Theorem 1.6]. Consider any such eigenvalue λ and its corresponding radial eigenvector f. Now construct the function ν that equals f on the top $r-1$ levels of T_1, i.e., V_1 and correspondingly $-f$ on V_2, and is zero elsewhere. We claim that ν is an eigenvector of G with eigenvalue λ. To see this, note that the eigenvector equation is trivially satisfied on V_1 and V_2 because all new neighbors of those vertices are assigned a value of 0 in ν. The remaining vertices where the eigenvector equation needs to be checked are the ones obtained by gluing L_1 to the leaves of T_2. Now every such vertex v, satisfies $\nu(v) = 0$ and there exists two neighbors of v, say $u \in V_1$ and $w \in V_2$ with $\nu(u) = -\nu(w)$ and furthermore ν is 0 on every remaining neighbor, clearly implying the eigenvector equation at v. □

We next show that G is nearly Ramanujan.

4 The spectrum of G

Proposition 4.1. *For every fixed $\varepsilon > 0$, if m is sufficiently large then the absolute value of every nontrivial eigenvalue of G is at most*

$$\left(\frac{3d-1}{\sqrt{d(2d-1)}} + \varepsilon\right)\sqrt{d}.$$

Remark 4.1. For every fixed d, the number $\frac{3d-1}{\sqrt{d(2d-1)}}$ is smaller than $\frac{3}{\sqrt{2}} = 2.12132..$ For $d = 2$ (cubic graphs) the number is $5/\sqrt{6} = 2.04124...$ Therefore, the graph G is close to being Ramanujan. However, for all $r > 1$ it is not quite Ramanujan. Indeed, it contains a set of vertices Y, namely the set of all vertices in levels $r-1, r-3, r-5, \ldots$ of the two trees T_1, T_2, that expands by a factor of less than $(d+1)/2$. By Theorem 4.1 in [Kah95] such a graph must contain a nontrivial eigenvalue of absolute value bigger than $2\sqrt{d} + \delta(d, r)$ for some positive $\delta(d, r)$.

To prove Proposition 4.1, we need the following simple lemma about the spectrum of finite d-ary trees.

Lemma 4.1. *Let T be a finite d-ary tree, that is, a tree with a root r of degree $d+1$ in which every non-leaf has d children. Let A_T denote the adjacency matrix of T, let W be the set of its non-leaves and let L be the set of its leaves. Then for any vector f supported on $W \cup L$,*

$$|f^T A_T f| \leq 2\sqrt{d} \sum_{w \in W} f(w)^2 + \sqrt{d} \sum_{v \in L} f(v)^2.$$

Proof. Orient all edges towards the root r. Then

$$\begin{aligned}
|f^T A_T f| &= \sum_{u \to v} 2|f(u)||f(v)| \\
&= \sum_{u \to v} 2t|f(u)|\frac{|f(v)|}{t} \text{ for any choice of } t > 0 \\
&\leq \sum_{u \to v} \left(t^2 f^2(u) + \frac{1}{t^2} f^2(v)\right) \\
&= \frac{d+1}{t^2} f^2(r) + t^2 \sum_{u \in L} f^2(u) + \sum_{u \in W - \{r\}} f^2(u) \left(t^2 + \frac{d}{t^2}\right) \\
&\leq 2\sqrt{d} \sum_{u \in W} f^2(u) + \sqrt{d} \sum_{u \in L} f^2(u) \text{ by choosing } t^2 = \sqrt{d}
\end{aligned}$$

□

We also need the following lemma of Kahale about growth rate of eigenfunctions.

Lemma 4.2. *([Kah95, Lemma 5.1]) Consider a graph on a vertex set U, and let A denote its adjacency matrix. Let X be a set of vertices. Let h be a positive integer and let s be a function on U. Let X_i be the set of all vertices at distance i from X, and assume that the following conditions hold.*

1. *For $h-1 \leq i, j \leq h$ all vertices in X_i have the same number of neighbors in X_j.*
2. *The function s is constant on X_{h-1} and on X_h.*
3. *The function s is positive and $As(v) \leq |\mu| s(v)$ for every v of distance at most $h-1$ from X, where μ is a nonzero real.*

Then for any function g on U which satisfies $|Ag(u)| = |\mu||g(u)|$ for all vertices u of distance at most $h-1$ from X, we have:

$$\frac{\sum_{v \in X_h} g(v)^2}{\sum_{v \in X_h} s(v)^2} \geq \frac{\sum_{v \in X_{h-1}} g(v)^2}{\sum_{v \in X_{h-1}} s(v)^2}.$$

Equipped with the above lemma we now proceed to proving Proposition 4.1.

Proof of Proposition 4.1. The adjacency matrix A_G of G is the sum $A_G = A_H - A_M + A_{T_2} + A_{T_3}$, where A_H is the adjacency matrix of H, A_M is the adjacency matrix of the matching $u_i v_i$ and A_{T_2}, A_{T_3} are the adjacency matrices of the trees T_2 and T_3, respectively. It is convenient to view all the graphs H, T_2, T_3, M as graphs on the set of all vertices of G, which is $U = V \cup V_2 \cup V_3$. Thus all the matrices above have rows and columns indexed by the set U, and each of the corresponding graphs has many isolated vertices.

Put $b = \frac{3d-1}{\sqrt{d(2d-1)}}$. We have to show that every nontrivial eigenvalue μ of G has absolute value at most $(b+\varepsilon)\sqrt{d}$ for any $\varepsilon > 0$ provided m is sufficiently large. Let $g : U \to R$ be an eigenvector of μ

satisfying $\sum_{v\in U} g(u)^2 = 1$. As g is orthogonal to the top eigenvector, $\sum_{v\in U} g(v) = 0$. The total number of vertices in $V_2 \cup V_3$ is smaller than $2n$, and therefore, by Cauchy-Schwartz, $|\sum_{v\in V_2\cup V_3} g(v)| \le \sqrt{2n}$. It thus follows that $|\sum_{v\in V} g(v)| \le \sqrt{2n}$. Considering the projection of the restriction of g to V on the all ones vector and its complement we conclude that

$$|g^T A_H g| \le 2\sqrt{d}\sum_{v\in V} g^2(v) + \frac{(d+1)2n}{m}. \tag{7}$$

Recall that L_1 is the set of leaves of T_2 (and T_1). By Lemma 4.1

$$|g^T A_{T_2} g| \le 2\sqrt{d}\sum_{v\in V_2} g^2(v) + \sqrt{d}\sum_{v\in L_1} g^2(v). \tag{8}$$

Similarly

$$|g^T A_{T_3} g| \le 2\sqrt{d}\sum_{v\in V_3} g^2(v) + \sqrt{d}\sum_{v\in L_2} g^2(v). \tag{9}$$

The contribution of the omitted matching can be bounded as follows

$$|g^T A_M g| = |\sum_{i=1}^{n} 2g(u_i)g(v_i)| \le \sum_{v\in L_1\cup L_2} g^2(v). \tag{10}$$

Combining (7), (8), (9), (10) we conclude that

$$\begin{aligned} |\mu| = |g^T A g| &\le 2\sqrt{d}\sum_{v\in U} g^2(v) + (\sqrt{d}+1)\sum_{u\in L_1\cup L_2} g^2(u) + \frac{(d+1)2n}{m}, \\ &= 2\sqrt{d} + (\sqrt{d}+1)\sum_{u\in L_1\cup L_2} g^2(u) + \frac{(d+1)2n}{m}. \end{aligned} \tag{11}$$

In order to complete the proof, it thus suffices to show that if $|\mu| \ge (b+\varepsilon)\sqrt{d}$, then, as m tends to infinity, the sum $\sum_{u\in L_1\cup L_2} g^2(u)$ tends to zero. This is done using Lemma 4.2, as described next.

1. We first bound the sum $\sum_{u\in L_2} g^2(u)$. This is simple and works even if we only assume that $|\mu| \ge 2\sqrt{d}$. Indeed, starting with $X = \{v'\}$, let X_i be the set of vertices of distance i from X. Define $s(v) = d^{-i/2}$ for all $v \in X_i$ for $0 \le i \le r+t$, where $r+t$ is the largest integer smaller than half the girth of H. It is easy to check that the conditions of Lemma 4.2 hold. Thus by its conclusion the sum $\sum_{v\in X_i} g^2(v)$ is nondecreasing in i for all $i \ge r$. Since this sum for $i = r$ is exactly $\sum_{v\in L_2} g^2(v)$ and t tends to infinity with m, and since the sum over all $r \le i \le r+t$ is at most 1, it follows that the sum for $i = r$ is negligible.

2. Bounding the sum $\sum_{u\in L_1} g^2(u)$ is harder. Here we use the assumption that $|\mu| \ge (b+\varepsilon)\sqrt{d}$ where $b = \frac{3d-1}{\sqrt{d(2d-1)}}$. Define $X = X_0 = \{u, u'\}$ and let X_i denote the set of all vertices of G of distance exactly i from X. Put $c = \sqrt{(2d-1)/d}$ and note that $c + 1/c = b$. Define a sequence of reals $s_0, s_1, s_2, \ldots$ as follows. For $0 \le i \le r$, $s_i = c^i d^{-i/2}$. For $i = r+1$, $s_{r+1} = c^{r-1}d^{-(r+1)/2}$ and for all $i \ge 1$, $s_{r+1+i} = \alpha_i d^{-(r+1+i)/2}$ where the numbers α_i are defined by setting $\alpha_0 = c^{r-1}$ and $\alpha_i/\alpha_{i-1} = x_i$ for $i \ge 1$ with $x_1 = 1/c = \sqrt{d/(2d-1)}$ and for $i \ge 1$,

$$x_{i+1} = \min\{b + \varepsilon - \frac{1}{x_i}, c\}.$$

Using the sequence s_i define a function s on the vertices in the union $\cup_{i\le r+t}X_i$, where $r+t$ is smaller than half the girth of H, by putting $s(v)=s_i$ for all $v\in X_i$. We proceed to show that for every vertex $v\in\cup_{i<r+t}X_i$,

$$As(v)\le|\mu|s(v). \tag{12}$$

For $v\in X=X_0$ this is equivalent to

$$\frac{c}{\sqrt{d}}(d+1)\le|\mu|$$

which is certainly true as

$$|\mu|>b\sqrt{d}=(c+\frac{1}{c})\sqrt{d}\ge\frac{c}{\sqrt{d}}(d+1).$$

For $v\in X_i$ with $1\le i\le r-1$ the required inequality is

$$\frac{\sqrt{d}}{c}s_i+\frac{dcs_i}{\sqrt{d}}\le|\mu|s_i$$

which follows from the fact that $1/c+c=b\le b+\varepsilon$. For $v\in X_r$ the inequality is

$$\frac{2\sqrt{d}}{c}s_i+\frac{d-1}{\sqrt{d}}\frac{s_i}{c}\le|\mu|s_i.$$

For this it suffices to check that

$$\frac{2}{c}+\frac{d-1}{dc}\le b+\varepsilon$$

which holds as the left hand side is equal to b. For $v\in X_{r+1}$ it suffices to check that

$$\frac{\sqrt{d}s_{r+1}}{x_1}+\sqrt{d}x_2s_{r+1}\le|\mu|s_{r+1}$$

which holds as $x_2\le b+\varepsilon-1/x_1$ by its definition. Finally, for $v\in X_{r+1+i}, i\ge 1$ the required inequality is equivalent to $\sqrt{d}\alpha_{i-1}+\sqrt{d}\alpha_{i+1}\le|\mu|\alpha_i$, that is , $\sqrt{d}\frac{1}{x_i}+\sqrt{d}x_{i+1}\le(b+\varepsilon)\sqrt{d}$ which again holds by the definition of x_{i+1}. This completes the proof of (12). Next we observe that $x_1=1/c>1/\sqrt{2}$, that $x_{i+1}\ge x_i$ for all i, and that if $x_i<c-\varepsilon$ then $x_{i+1}\ge x_i+\varepsilon$. Indeed $x_1=\sqrt{d/(2d-1)}>1/\sqrt{2}$, and $x_{i+1}\le c$ for all i by the definition of x_{i+1}. The function $g(x)=b-1/x$ is increasing and concave in the interval $[1/c,c]$ and as $g(x)=x$ at the endpoints of the interval, $g(x)\ge x$ for all x in the interval implying that $x_{i+1}=\min\{g(x_i)+\varepsilon,c\}\ge x_i$ for all $x_i\in[1/c,c]$ and that $x_{i+1}\ge x_i+\varepsilon$ if $x_i\le c-\varepsilon$.

By the above discussion it follows that x_i is at least $c>\sqrt{3/2}$ for all $i\ge 1/\varepsilon$ implying that the sequence $|X_j|s_j^2$ is increasing exponentially for $j>r+1/\varepsilon$. Between levels r and $r+1/\varepsilon$ the terms of this sequence decrease by a factor larger than $1/c$ in each step, and it therefore follows that the term number r of this sequence is negligible compared to any term number $r+\omega(1)$. This together with Lemma 4.2 completes the proof.

□

Using the above ingredients we now verify all the claims in Theorem 1.2.

Proof of Theorem 1.2. By the construction with described in Section 3, there exists a graph $G = (V, E)$ with the following properties (with $r = \lfloor \alpha \log_d(m) \rfloor$):

1. $|V| = M = m + 2n$ where $n = \frac{d^{r+1}+d^r-2}{d-1} = d^r\left(1 + O(\frac{1}{d})\right)$.
2. By Lemma 3.1, and the choice of r, we have

   $$\text{girth}(G) \geq 2\log_{2d-1}((d+1)d^{r-1}) \geq \log_{2d-1}(d^r) \geq 2\alpha \log_{2d-1}(m) = 2\alpha \log_d(m) \cdot (1 - O(\log^{-1}(d))).$$

   Thus the second claim in the theorem is verified by observing that $M = m + O(m^\alpha)$.
3. By Proposition 4.1,

   $$|\lambda_i(A_M)| \leq (3/\sqrt{2})\sqrt{d}$$

   for all nontrivial eigenvalues $i \neq 1$ of the adjacency matrix A_M.
4. The third claim follows by Lemmas 3.2, 3.3, and 3.5, and noticing that $|V_1 \cup V_2| = O(d^r) = O(m^\alpha)$.

The final claim in the statement of the theorem is an easy consequence of Lemma 3.3 which states that the set of finite d−ary tree eigenvalues is a dense subset of $(-2\sqrt{d}, 2\sqrt{d})$. Furthermore, Remark 1.1 follows by choosing S to be the top $\lfloor \varepsilon r \rfloor$ levels of T_1. □

We conclude by explaining how to modify the construction to produce many localized eigenvectors.

Proof of Theorem 1.3. The proof follows from the observation that in the construction described in Section 3, one can glue several trees to H 'far away' from each other to maintain high girth and every other property mentioned in Theorem 1.2. More precisely, in the construction in Section 3, instead of considering a tree T_1 rooted at u, consider trees $T_1^{(1)}, T_1^{(2)}, \ldots, T_1^{(k)}$ rooted at $u_1, u_2, \ldots, u_k$ and repeat the construction k times with the corresponding trees

$$\{T_2^{(1)}, T_2^{(2)}, \ldots, T_2^{(k)}\} \text{ and } \{T_3^{(1)}, T_3^{(2)}, \ldots, T_3^{(k)}\}$$

rooted at $\{u'_1, u'_2, \ldots u'_k\}$ and $\{v_1, v_2, \ldots, v_k\}$ respectively. The same arguments as before imply that the girth condition in Theorem 1.2 is satisfied as long as the graph distance in H between any u_i and u_j is at least $4r$. Lemma 4.3 below shows that one can take k to be as large as $m^{1-4\alpha}$ where $r = \lfloor \alpha \log_d(m) \rfloor$. Finally the proof of Proposition 4.1 in this case follows in exactly the same way after defining the set X to be $\{v_1, v_2, \ldots, v_k\}$ and $\{u_1, u'_1, u_2, u'_2, \ldots, u_k, u'_k\}$ in 1. and 2. respectively instead of $\{v\}$ and $\{u, u'\}$. □

Lemma 4.3. *Given any $d+1$-regular graph $G = (V, E)$ of size m, and any k, there are at least $\frac{m(d-1)}{(d+1)d^k}$ vertices in V, all of whose mutual distances are at least k.*

Proof. Consider a maximal set S of such vertices. Simple considerations imply that every other vertex in V must be at distance at most k from the set S. Now the total number of such vertices is at most $|S|(d+1)d^k/(d-1)$ which finishes the proof. □

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