Asymptotically optimal induced universal graphs

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Abstract

We prove that the minimum number of vertices of a graph that contains every graph on k vertices as an induced subgraph is $(1+o(1))2^{(k-1)/2}$. This improves earlier estimates of Moon, of Bollobás and Thomason and of Alstrup, Kaplan, Thorup and Zwick. The method supplies sharp estimates for the analogous problems for directed graphs, tournaments, bipartite graphs, oriented graphs and more.

1 Introduction

Let $\mathcal{F}$ be a finite family of graphs. A graph G is *induced universal* for $\mathcal{F}$ if every member F of $\mathcal{F}$ is an induced subgraph of G. There is a vast literature about induced universal graphs since their introduction by Rado [13]. Let $\mathcal{F}(k)$ denote the family of k-vertex undirected graphs, and let $f(k)$ denote the smallest possible number of vertices of an induced universal graph for $\mathcal{F}(k)$. Moon [11] observed that a simple counting argument gives $f(k) \geq 2^{(k-1)/2}$ and proved that $f(k) \leq O(k2^{k/2})$. Alstrup, Kaplan, Thorup and Zwick [3] determined $f(k)$ up to a constant factor, showing that $f(k) \leq 16 \cdot 2^{k/2}$. Bollobás and Thomason [6] proved that the random graph $G(n, 0.5)$ on $n = k^2 2^{k/2}$ vertices is induced universal for $\mathcal{F}(k)$ with high probability, that is, with probability that tends to 1 as k tends to infinity. The question of finding tighter bounds for $f(k)$, suggested by the work of Moon, is mentioned by Vizing in [14] and by Alstrup et. al (despite the fact their work determines it up to a constant factor of $16\sqrt{2}$) in [3]. Here we show that the lower bound is tight, up to a lower order additive term.

Theorem 1.1

$$f(k) = (1 + o(1))2^{(k-1)/2}.$$

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The proof combines probabilistic and combinatorial arguments with some simple group theoretical facts about graphs with large automorphism groups. Essentially the same arguments supply asymptotically tight estimates for the analogous questions for directed graphs, oriented graphs, tournaments, bipartite graphs or complete graphs with colored edges, improving results in [12], [9], [3]. Since the proofs in all cases, besides possibly that of bipartite graphs, are similar, we focus here on the undirected case and merely include the statements and sketches of proofs of these variants, with more details about the family of bipartite graphs.

As a byproduct of (a variant of) the first part of the proof we show that the minimum number of vertices n so that the random graph on n vertices is induced universal for $\mathcal{F}(k)$ with high probability is $(1+o(1))\frac{k}{e}2^{(k-1)/2}$, improving the estimate in [6] (which was harder to improve in 1981, when [6] was written, but is simpler now, using some of the more recently developed high deviation inequalities.)

The rest of this paper is organized as follows. Theorem 1.1 is proved in Section 2 and Section 3 deals with the minimum possible n so that the random graph $G(n, 0.5)$ is induced universal for $\mathcal{F}(k)$ with high probability. In Section 4 we describe several variants of Theorem 1.1 that can be proved using a similar approach, and the final Section 5 contains some concluding remarks and open problems.

In order to simplify the presentation we omit all floor and ceiling signs whenever these are not essential, and assume, whenever needed, that k is sufficiently large. Throughout the proof we make no attempt to optimize the absolute constants whenever these are not crucial. All logarithms are in base 2, unless otherwise specified.

2 The main proof

In this section we prove Theorem 1.1. It is convenient to split the proof into two main parts.

2.1 Asymmetric graphs

Call a graph on k vertices *asymmetric* if any induced subgraph of it has at most k^{4m} automorphisms, where $m = 2\sqrt{k \log k}$. Note that, in particular, the number of automorphisms of any such graph is at most k^{4m}. Let $\mathcal{H}(k)$ denote the family of all asymmetric graphs on k vertices.

In this subsection we prove that there are small induced universal graphs for $\mathcal{H}(k)$: in fact, the random graph with the appropriate number of vertices is, with high probability, such a universal graph.

Let n be the smallest integer that satisfies the following inequality

$$\binom{n}{k}\frac{k!}{k^{8m}}2^{-\binom{k}{2}} \geq 1. \qquad (1)$$

Since the ratio between $\binom{n+1}{k}$ and $\binom{n}{k}$ is $\frac{n+1}{n-k+1}$ which is very close to 1 for the relevant parameters, the left-hand-side of (1) for this smallest n is $1+o(1)$ and thus one can solve for n and see that it satisfies

$$n = 2^{(k-1)/2}(1+O(\frac{\log^{3/2} k}{\sqrt{k}})).$$

In particular, $n = (1+o(1))2^{(k-1)/2}$.

Theorem 2.1 *Let n be as above and let $G = (V,E) = G(n, 0.5)$ be the random graph on a set V of n vertices obtained by picking, randomly and independently, each pair of vertices to be an edge with probability $1/2$. Then, with high probability (that is, with probability that tends to 1 as k, and hence n, tend to infinity), G is an induced universal graph for $\mathcal{H}(k)$.*

In the proof of the above theorem we apply (a known consequence of) Talagrand's Inequality, described, for example, in [2], Theorem 7.7.1. The statement follows.

Theorem 2.2 (Talagrand's Inequality) *Let $\Omega = \prod_{i=1}^{p} \Omega_i$, where each Ω_i is a probability space and Ω has the product measure, and let $h : \Omega \to R$ be a function. Assume that h is Lipschitz, that is, $|h(x) - h(y)| \leq 1$ whenever x, y differ in at most one coordinate. For a function $f : N \to N$, h is f-*certifiable *if whenever $h(x) \geq s$ there exists $I \subseteq \{1, \ldots, p\}$ with $|I| \leq f(s)$ so that for every $y \in \Omega$ that agrees with x on the coordinates I we have $h(y) \geq s$. Suppose that h is f-*certifiable *and let Y be the random variable given by $Y(x) = h(x)$ for $x \in \Omega$. Then for every b and t*

$$Prob[Y \leq b - t\sqrt{f(b)}] \cdot Prob[Y \geq b] \leq e^{-t^2/4}.$$

We also need the following simple lemma.

Lemma 2.3 *Let $H \in \mathcal{H}(k)$. Let K and K' be two sets of labelled vertices, each of size k, where $|K \cap K'| = k - i$. Then the number of ways to choose the edges and nonedges among all $2\binom{k}{2} - \binom{k}{k-i}$ pairs that lie in K or in K' so that the induced subgraph on K is isomorphic to H and the induced subgraph on K' is also isomorphic to K is at most*

$$\frac{k!}{|Aut(H)|}k^i k^{4m}.$$

Proof: There are exactly $\frac{k!}{|Aut(H)|}$ copies of H on K. For each such fixed copy, we bound the number of embeddings of H in K'. There are at most $k(k-1)\cdots(k-i+1) < k^i$ ways to choose the vertices of H mapped to the vertices of $K' - K$. Fix a set T of these i vertices and their embedding. In order to complete the embedding, the induced subgraph of the copy of H placed in K on the set of vertices $K \cap K'$ has to be isomorphic to the induced subgraph of H on $V(H) - T$. If so, then the number of ways to embed these $k-i$ vertices is the number of automorphisms of this induced subgraph of H, which is, by the definition of $\mathcal{H}$, at most k^{4m}. □

Proof of Theorem 2.1: Let H be a fixed member of $\mathcal{H}(k)$ and let $s = |Aut(H)|$ be the size of its automorphsim group. Then

$$s \le k^{4m} = k^{8\sqrt{k \log k}}.$$

Let $G = (V, E)$ be the random graph $G(n, 0.5)$, where n is as chosen in (1). For every subset $K \subset V$ of size $|K| = k$ let X_K denote the indicator random variable whose value is 1 if the induced subgraph of G on K is isomorphic to H and let $X = \sum_K X_K$, where the summation is over all subsets $K \subset V$ of cardinality k. Thus X is the number of copies of H in G. The expectation of each X_K is clearly

$$E(X_K) = \frac{k!}{s} 2^{-\binom{k}{2}}.$$

Thus, by linearity of expectation,

$$E(X) = \binom{n}{k} \frac{k!}{s} 2^{-\binom{k}{2}} \ge k^{4m}$$

where in the last inequality we used (1) and the fact that $s \le k^{4m}$. Note also that since

$$\binom{n}{k} \frac{k!}{k^{8m}} 2^{-\binom{k}{2}} = 1 + o(1)$$

it follows that the expectation of X is at most $(1 + o(1))k^{8m} < n^{0.01}$ (even if $s = 1$).

We say that two copies of H in G have a *nontrivial intersection* if they share at least two vertices. Put $\mu = E(X)$. Let Z denote the random variable $Z = \sum_{K,K'} X_K X_{K'}$ where the summation is over all (ordered) pairs of k-subsets K, K' of V that satisfy $2 \le |K \cap K'| \le k - 1$. Thus, Z is the number of pairs of copies of H in G that have a nontrivial intersection. We next compute the expectation of Z and show that it is much smaller than μ (and hence also much smaller than μ^2). Put $\Delta = E(Z)$ and note that

$\Delta = \sum_{j=2}^{k-1} \Delta_j$ where Δ_j is the expected number of pairs K, K' with $X_K = X_{K'} = 1$ and $|K \cap K'| = j$.

Claim: For each $2 \le j \le k-1$

$$\Delta_j \le \mu \frac{1}{n^{0.48}} \tag{2}$$

Proof of Claim: Consider two possible cases, as follows.

Case 1: $2 \le j \le 3k/4$.

In this case

$$\Delta_j \le \binom{n}{k} \frac{k!}{s} 2^{-\binom{k}{2}} \binom{k}{j} \binom{n-k}{k-j} \frac{k!}{s} 2^{-\binom{k}{2}+\binom{j}{2}}.$$

Indeed, there are $\binom{n}{k}$ ways to choose the set K, and $\binom{k}{j}\binom{n-k}{k-j}$ ways to choose K' with $|K \cap K'| = j$. There are $\frac{k!}{s}$ ways to place a copy of H in K and $\frac{k!}{s}$ ways to place a copy of H in K' (this is an overcount, as these two copies have to agree on the edges in their common part). This determines all the edges and nonedges in the induced graph on K and on K', and the probability that G indeed has exactly these edges is

$$2^{-\binom{k}{2}} \cdot 2^{-\binom{k}{2}+\binom{j}{2}}$$

Therefore,

$$\frac{\Delta_j}{\mu^2} \le \frac{\binom{k}{j}\binom{n-k}{k-j} 2^{\binom{j}{2}}}{\binom{n}{k}} \le (\frac{k^2 2^{(j-1)/2}}{n})^j \le (\frac{1}{n^{1/4-0.005}})^j \le \frac{1}{n^{0.49}}.$$

Here we used the fact that $k = (2+o(1))\log_2 n$ and that since $j \le 3k/4$ it follows that $2^{(j-1)/2} \le n^{3/4+o(1)}$. Recall that $\mu \le n^{0.01}$ and thus

$$\frac{\Delta_j}{\mu} \le \mu \frac{\Delta_j}{\mu^2} \le \frac{1}{n^{0.48}}$$

as claimed.

Case 2: $j = k - i,\ i \le k/4$.

In this case we have, by Lemma 2.3

$$\Delta_j \le \binom{n}{k}\binom{k}{j}\binom{n-k}{k-j} \frac{k!}{s} k^i k^{4m} 2^{-2\binom{k}{2}+\binom{j}{2}}.$$

Indeed, there are

$$\binom{n}{k}\binom{k}{j}\binom{n-k}{k-j}$$

ways to choose the sets K, K' having intersection $j = k - i$, and by Lemma 2.3 for each such choice there are at most $\frac{k!}{s} k^i k^{4m}$ ways to place copies of H in each of them. The probability that this coincides with all edges and nonedges of the induced subgraph of G on K and on K' is $2^{-2\binom{k}{2}+\binom{j}{2}}$.

Since $\mu \geq k^{4m} > 1$ this implies that

$$\frac{\Delta_j}{\mu^2} < \frac{\Delta_j}{\mu} \leq \binom{k}{j}\binom{n-k}{k-j} k^i k^{4m} 2^{-\binom{k}{2}+\binom{j}{2}}$$

$$\leq \binom{k}{i}\binom{n-k}{i} k^i k^{4m} 2^{-i(k-i)} \leq (k^2 n 2^{-(k-i)})^i k^{4m} \leq \frac{1}{n^{0.5-o(1)}} \leq \frac{1}{n^{0.48}}.$$

This completes the proof of the claim.

Returning to the proof of the theorem, note that by the claim, the variance of the random variable X satisfies

$$Var(X) \leq E(X) + \sum_{K,K'} Cov(X_K, X_{K'})$$

where the summation is over all ordered pairs K, K' where $2 \leq |K \cap K'| \leq k-1$ (since for all other pairs the covariance is zero). Since

$$Cov(X_K, X_{K'}) = E(X_K X_{K'}) - E(X_K)E(X_{K'}) \leq E(X_K X_{K'})$$

it follows from the claim above that

$$Var(X) \leq \mu + \Delta \leq (1 + o(1))\mu.$$

Therefore, by Chebyshev's Inequality, the probability that $X \geq 3\mu/4$ is (much) bigger than $3/4$. By Markov's inequality, the probability that $\Delta \leq \mu/4$ is also (much) bigger than $3/4$ and hence with probability larger than $1/2$, both events happen simultaneously, that is, the number of copies of H in G is at least $3\mu/4$ and the number of pairs of copies of H with nontrivial intersection is smaller than $\mu/4$. By removing one copy of H from each pair with a nontrivial intersection we conclude that if this is the case, then G contains a family of at least $\mu/2$ copies of H with no two having a nontrivial intersection.

Let $h(G)$ be the maximum cardinality of a family of copies of H in G in which no two members have a nontrivial intersection, and let Y be the random variable $Y = h(G)$. Our objective is to apply Talagrand's Inequality (Theorem 2.2) to deduce that the probability that Y is zero is tiny.

By the above discussion, the probability that Y is at least $\mu/2$ exceeds $1/2$. It is also clear that the value of $Y = h(G)$ can change by at most 1 if we add or delete one edge to

G, and that h is f-certifiable where $f(s) = s\binom{k}{2}$. Therefore, by Theorem 2.2 with $b = \mu/2$ and $t = \sqrt{\mu}/k$ we conclude that the probability that $Y = 0$ is smaller than

$$e^{-\mu/4k^2}$$

which is much much smaller than 2^{-k^2}. As $Y = 0$ if and only if there is no copy of H in G, and as the total number of graphs in $\mathcal{H}(k)$ is smaller than $2^{\binom{k}{2}}$, we conclude that G is induced universal for $\mathcal{H}(k)$ with high probability. This completes the proof of Theorem 2.1. □

2.2 Symmetric graphs

Recall that we called a graph on k vertices asymmetric if no induced subgraph of it has more than k^{4m} automorphisms, where $m = 2\sqrt{k \log k}$. Call a graph on k vertices *symmetric* if it is not asymmetric. Let $\mathbf{T}(k) = \mathcal{F}(k) - \mathcal{H}(k)$ denote the set of all symmetric graphs on k vertices

In this subsection we construct a small induced universal graph for the family $\mathbf{T}(k)$. To this end it is desirable to obtain some useful structural properties of graphs with a large automorphism group by utilizing known results about large subgroups of the symmetric group. There is a rich literature about automorphism groups of graphs, see, for example [4] and its many references. It seems hopeless to try to characterize all graphs with at most k vertices and at least $k^{8\sqrt{k \log k}}$ automorphisms. Fortunately, for our purpose here it suffices to prove and apply some partial information about their structure, as described in what follows.

The *minimal degree* of a permutation group is the size of the minimum support of a nontrivial element in it. There are several results stating that large permutation groups have nontrivial elements with small supports (or equivalently, many fixed points), see, for example, [8] and the references therein. For our purpose here the following simple fact suffices.

Lemma 2.4 *For any $p > 1$ and t, any subgroup S of size at least p^{4t} of the symmetric group S_p contains a permutation with at least t and at most $p - 3t$ fixed points.*

Proof: Consider the subgroup as a group of permutations of $[p] = \{1, 2, \ldots, p\}$. By the pigeonhole principle there is a subset $A = \{a_1, a_2, \ldots, a_t\}$ of t elements of $[p]$ so that there are at least $\frac{|S|}{p(p-1)\cdots(p-t+1)} > p^{3t}$ permutations σ in S satisfying $\sigma(i) = a_i$ for all $i \in [t] = \{1, 2, \ldots, t\}$. For any two such permutations σ_1, σ_2, the product $\sigma_1\sigma_2^{-1}$ fixes all points of A . Let S' be the subgroup of S that fixes all points of A. Then $|S'| > p^{3t}$.

The number of permutations in S' that fixes all points but at most i is clearly at most $\binom{p-t}{i} i! < p^i$, and since $p^{3t-1} < p^{3t}$ there is an element of S' that fixes at most $p - 3t$ points. $\square$

Corollary 2.5 *Let $T = (V, E)$ be a graph in $\mathbf{T}(k)$. Then there are three pairwise disjoint sets of vertices A, B, C of T, each of cardinality m, so that the following holds. There is a numbering of the elements of A, B, C: $A = \{a_1, a_2, \ldots, a_m\}$, $B = \{b_1, b_2, \ldots, b_m\}$ and $C = \{c_1, c_2, \ldots, c_m\}$ such that for any $1 \leq i, j \leq m$, $a_i b_j$ is an edge of T if and only if $a_i c_j$ is an edge of T. That is, for each $1 \leq j \leq m$, b_j and c_j have exactly the same neighbors in the set A.*

Proof: By the definition of $\mathbf{T}(k)$ there is an induced subgraph T' of T on $p \leq k$ vertices whose group of automorphisms S is of size at least $k^{4m} \geq p^{4m}$. By Lemma 2.4 this group contains a permutation σ with at least m and at most $p - 3m$ fixed points. Let $A = \{a_1, a_2, \ldots a_m\}$ be m of these fixed points and consider the expression of σ as a product of (nontrivial) cycles. The total length of these cycles is at least $3m$. From each cycle $(w_1, w_2, \ldots, w_r)$ of length r, define $\lfloor r/2 \rfloor$ disjoint pairs

$$(w_1, w_2), (w_3, w_4), \ldots (w_{2\lfloor r/2 \rfloor - 1}, w_{2\lfloor r/2 \rfloor}).$$

Altogether we get at least m such pairs, let (b_i, c_i), $(1 \leq i \leq m)$ be m of them. Observe, now, that for every $1 \leq j \leq m$, σ maps b_j to c_j and fixes all elements a_i. As σ is an automorphism of T this means that for every i, $a_i b_j$ is an edge of T if and only if so is $a_i c_j$. $\square$

Lemma 2.6 *Let T' be the graph obtained from a complete graph on k vertices by removing all edges of a complete bipartite graph $K_{m,m}$ in it, where, as before, $m = 2\sqrt{k \log k}$. Let $[k] = \{1, 2, \ldots, k\}$ be the set of vertices of T' and suppose $A' = \{1, 2, \ldots, m\}$, $B' = \{m+1, \ldots, 2m\}$, $C' = \{2m+1, 2m+2, \ldots, 3m\}$ and $D' = \{3m+1, 3m+2, \ldots, k\}$, where there are no edges between A' and C' and all other pairs of vertices of T' are adjacent. Then there is an orientation T'' of the edges of T' in which all edges between A' and B' are oriented from A' to B' and the maximum outdegree of a vertex in T'' is at most $\frac{k}{2} - \frac{m^2}{k} + O(1) < \frac{k}{2} - 3 \log k$.*

Proof: We need two simple facts, as follows.

Fact 1: For any integer $g > 1$ the edges of the complete graph K_g on g vertices can be oriented so that every outdegree is at most $g/2$.

Indeed, if g is odd then K_g has an Eulerian orientation in which every outdegree is exactly $(g-1)/2$, and if g is even, omit a vertex from an Eulerian orientation of K_{g+1} to get an orientation as needed.

Fact 2: For any positive integers p, q and $r \leq q$ there is a bipartite graph with classes of vertices P and Q of sizes p and q, respectively, so that every vertex of P has degree exactly r and every vertex of Q has degree either $\lfloor pr/q \rfloor$ or $\lceil pr/q \rceil$.

One simple way to prove this fact is to number the vertices of Q: $u_1, u_2, \ldots, u_q$ and to connect, for each i, vertex number i of P to the vertices

$$u_{(i-1)r+1}, u_{(i-1)r+2}, \ldots, u_{ir}$$

where the indices are reduced modulo q.

Using these two facts construct an orientation T'' of T' as follows. Orient the edges of the complete graph on A', using Fact 1, so that each outdegree is at most $m/2$. Similarly, orient the edges of the complete graph on $B' \cup C'$ so that every outdegree is at most m, and orient the edges of the complete graph on D' so that every outdgree is at most $(k-3m)/2$. All edges between A' and B' are oriented from A' to B'. By Fact 2, for any real $x \in (0,1)$ the edges of the complete bipartite graph with vertex classes A' and D' can be oriented so that the outdgree of each vertex of A' is at most $x|D'|+1 = x(k-3m)+1$ and the outdegree of each vertex of D' is at most $(1-x)m + O(1)$. Similarly, for any real $y \in (0,1)$ the edges of the complete bipartite graph with vertex classes $B' \cup C'$ and D' can be oriented so that the outdegree of each vertex of $B' \cup C'$ is at most $y(k-3m)+1$ and the outdegree of each vertex of D' is at most $(1-y)2m + O(1)$. In the resulting orientation the outdegrees of the vertices of $A', B' \cup C'$ and D' are bounded, up to absolutely bounded additive terms, by

$$\frac{3}{2}m + x(k-3m), \quad m + y(k-3m) \quad \text{and} \quad \frac{k-3m}{2} + (1-x)m + (1-y)2m,$$

respectively. Since $m = o(k)$ it is not difficult to check that there are $x, y \in (0,1)$ (which are both $1/2 + o(1)$) so that these 3 quantities are equal. Although the precise expressions for x and y are not needed, we note here that these are:

$$x = \frac{1}{2} - \frac{m^2}{k(k-3m)}, \quad y = \frac{1}{2} - \frac{m^2}{k(k-3m)} + \frac{m}{2(k-3m)}.$$

With these x and y all three quantities above are exactly $\frac{k}{2} - \frac{m^2}{k}$.

It follows that there is an orientation T'' in which all outdegrees are equal, up to an $O(1)$ additive error, and as the total number of edges is $\binom{k}{2} - m^2$ this (as well as the explicit computation above) implies that every outdegree in T'' is at most $\frac{k}{2} - \frac{m^2}{k} + O(1) < \frac{k}{2} - 3\log k$, completing the proof of the lemma. □

Theorem 2.7 *There is an induced universal graph for* $\mathbf{T}(k)$ *whose number of vertices is at most* $2^{k/2-\log k}$.

Proof: An *adjacency labeling scheme* for a family of graphs is a way of assigning labels to the vertices of each graph in the family such that given the labels of two vertices in the graph it is possible to determine whether or not they are adjacent in the graph, without using any additional information besides the labels. It is easy and well known that a family of graphs has a labeling scheme in which every label contains L bits if and only if there is an induced universal graph for the family with at most 2^L bits. This is implicit in the work of Moon [11] and is mentioned explicitly in lots of subsequent papers starting with [7]. Indeed, for a given labeling scheme for a family, the graph whose vertices are all possible labels in which two vertices are adjacent if and only if their labels correspond to adjacent vertices is an induced universal graph for the family (and the converse is equally simple). It thus suffices to describe a labeling scheme for the family $\mathbf{T}(k)$ in which each label consists of at most $k/2 - \log k$ bits. Given a graph $T \in \mathbf{T}(k)$ we describe the labels of its vertices. By Corollary 2.5 T contains three disjoint subsets of vertices $A = \{a_1, a_2, \ldots, a_m\}$, $B = \{b_1, b_2, \ldots, b_m\}$ and $C = \{c_1, c_2, \ldots, c_m\}$ satisfying the assertion of the lemma. Number the vertices of T by the integers $1, 2, \ldots, k$ so that a_i gets the number i, b_i the number $m+i$ and c_i the number $2m+i$, where the rest of the numbering is arbitrary. Let T'' be the graph constructed in Lemma 2.6. The label of vertex number i of T is the number i followed by one bit for each outneighbor j of i in T'', in order. This bit is 0 if i and j are not adjacent in T, and is 1 if they are adjacent. Note that the length of each label is at most $\log k + k/2 - 2\log k \leq k/2 - \log k$, where the first $\log k$ bits are used to present the number of the vertex.

It is not difficult to check that this is a valid labeling scheme. Given the labels of two vertices, if it is not the case that one of them lies in A and the other in C, one of the labels contains the information about the adjacency between the two vertices, and the labels (together with the graph T'' which is known as part of the labeling scheme) determine this information. The only exceptional case is when one of the two vertices is a_i and the other is c_j. But in this case the label of a_i determines whether or not a_i is adjacent to b_j, and this determines the information about the adjacency between a_i and c_j as well, by Corollary 2.5. This completes the proof of the theorem. □

2.3 The proof of Theorem 1.1

The assertion of Theorem 1.1 clearly follows from that of Theorem 2.1 together with that of Theorem 2.7. The required induced universal graph is simply the vertex disjoint union of the graph in Theorem 2.1 and the one in Theorem 2.7. Note that the size of the second graph is negligible compared to that of the first one, and thus the proof gives that the minimum possible number of vertices of an induced universal graph for $\mathcal{F}(k)$, namely, the function $f(k)$, satisfies

$$f(k) \leq 2^{(k-1)/2}(1 + O(\frac{\log^{3/2} k}{\sqrt{k}})).$$

□

3 Random graphs

In this section we establish a sharp estimate for the minimum integer $n = n(k)$ so that the random graph $G(n, 0.5)$ is induced universal for $\mathcal{F}(k)$ with high probability. The argument is similar to the one used in the proof of Theorem 2.1, with one additional simple trick that simplifies the computation. The idea is to insist on getting each labelled k vertex graph $F \in \mathcal{F}(k)$ as an induced subgraph of the labelled random graph $G = G(n, 0.5)$ on the set of vertices $[n]$ by an order preserving mapping, namely, the embedding of H in G respects the order of the labels. Note that for such embeddings, for every fixed labelled F and every set K of k vertices of G, the probability that the labelled induced subgraph of G on K is (monotonely) isomorphic to F is exactly $2^{-\binom{k}{2}}$, independently of the number of automorphisms of F.

Let n be the smallest integer that satisfies

$$\binom{n}{k} 2^{-\binom{k}{2}} \geq 4k^4. \tag{3}$$

As in subsection 2.1, since the ratio between $\binom{n+1}{k}$ and $\binom{n}{k}$ is $\frac{n+1}{n-k+1}$ which is very close to 1 for the relevant parameters, the left-hand-side of (3) for this smallest n is $1 + o(1)$ and thus, by Stirling's Formula

$$n = \frac{k}{e} 2^{(k-1)/2}(1 + O(\frac{\log k}{k})),$$

and in particular, $n = (1 + o(1))\frac{k}{e}2^{(k-1)/2}$.

Theorem 3.1 *Let n be as above and let $G = (V, E) = G(n, 0.5)$ be the random graph on a set $V = \{1, 2, \ldots, n\}$ of n labelled vertices. Then, with high probability every labelled*

$F \in \mathcal{F}(k)$ is an induced subgraph of G with an order preserving embedding. In particular, G is an induced universal graph for $\mathcal{H}(k)$ with high probability.

Proof: Let F be a fixed member of $\mathcal{F}(k)$ considered as a labelled graph on $[k] = \{1, 2, \ldots, k\}$, and let G be the random graph $G(n, 0.5)$ on $[n] = \{1, 2, \ldots, n\}$, where n is as chosen in (3). For every subset $K \subset V$ of size $|K| = k$ let X_K denote the indicator random variable whose value is 1 if the induced subgraph of G on K is isomorphic to F with a monotone embedding and let $X = \sum_K X_K$, where the summation is over all subsets $K \subset V$ of cardinality k. Thus X is the number of copies of F in G. The expectation of each X_K is clearly

$$E(X_K) = 2^{-\binom{k}{2}},$$

and thus, by linearity of expectation,

$$E(X) = \binom{n}{k} 2^{-\binom{k}{2}} \geq 4k^4$$

where the last inequality follows from (3).

As before, say that two copies of F in G have a nontrivial intersection if they share at least two vertices. Put $\mu = E(X)$, then $\mu = (1 + o(1))4k^4$. Let Z denote the random variable $Z = \sum_{K,K'} X_K X_{K'}$ where the summation is over all (ordered) pairs of k-subsets K, K' of V that satisfy $2 \leq |K \cap K'| \leq k - 1$. Thus, Z is the number of pairs of copies of F in G that have a nontrivial intersection. We next compute the expectation of Z and show that it is much smaller than μ (and hence also much smaller than μ^2). Put $\Delta = E(Z)$ and note that $\Delta = \sum_{j=2}^{k-1} \Delta_j$ where Δ_j is the expected number of pairs K, K' with $X_K = X_{K'} = 1$ and $|K \cap K'| = j$.

Claim: For each $2 \leq j \leq k - 1$

$$\Delta_j \leq \mu \frac{1}{n^{0.48}} \tag{4}$$

Proof of Claim: Consider two possible cases, as follows.

Case 1: $2 \leq j \leq 3k/4$.

In this case

$$\Delta_j \leq \binom{n}{k} 2^{-\binom{k}{2}} \binom{k}{j} \binom{n-k}{k-j} 2^{-\binom{k}{2}+\binom{j}{2}}.$$

Indeed, there are $\binom{n}{k}$ ways to choose the set K, and $\binom{k}{j}\binom{n-k}{k-j}$ ways to choose K' with $|K \cap K'| = j$. There is a unique way to place a copy of F monotonely in K and a unique way to place a copy of F in K'. This is an overcount, as these two copies have to agree on the edges in their common part. This determines all the edges in the induced graph on K and on K', and the probability that G indeed has exactly these edges and nonedges is

$$2^{-\binom{k}{2}} \cdot 2^{-\binom{k}{2}+\binom{j}{2}}$$

Therefore,

$$\frac{\Delta_j}{\mu^2} \le \frac{\binom{k}{j}\binom{n-k}{k-j}2^{\binom{j}{2}}}{\binom{n}{k}} \le (\frac{k^2 2^{(j-1)/2}}{n})^j \le (\frac{1}{n^{1/4-0.005}})^j \le \frac{1}{n^{0.49}}.$$

Here we used the fact that $k = (2 + o(1)) \log_2 n$ and that since $j \le 3k/4$ it follows that $2^{(j-1)/2} \le n^{3/4+o(1)}$. Recall that $\mu = (1 + o(1))4k^4 \le n^{0.01}$ and thus

$$\frac{\Delta_j}{\mu} \le \mu \frac{\Delta_j}{\mu^2} \le \frac{1}{n^{0.48}}$$

as claimed.

Case 2: $j = k - i, \ i \le k/4$.

In this case we also have

$$\Delta_j \le \binom{n}{k}\binom{k}{j}\binom{n-k}{k-j} 2^{-2\binom{k}{2}+\binom{j}{2}}.$$

Indeed, there are

$$\binom{n}{k}\binom{k}{j}\binom{n-k}{k-j}$$

ways to choose the sets K, K' having intersection $j = k - i$, and for each such choice there are at most one way to place copies of F in each of them monotonely. The probability that this coincides with all edges and nonedges of the induced subgraph of G on K and on K' is $2^{-2\binom{k}{2}+\binom{j}{2}}$. Since $\mu \ge 4k^4 > 1$ this implies that

$$\frac{\Delta_j}{\mu^2} < \frac{\Delta_j}{\mu} \le \binom{k}{j}\binom{n-k}{k-j} 2^{-\binom{k}{2}+\binom{j}{2}}$$

$$\le \binom{k}{i}\binom{n-k}{i} 2^{-i(k-i)} \le (kn2^{-(k-i)})^i \le \frac{1}{n^{0.5-o(1)}} \le \frac{1}{n^{0.48}}.$$

This completes the proof of the claim.

Returning to the proof of the theorem, note that by the claim, the variance of the random variable X satisfies

$$Var(X) \le E(X) + \sum_{K,K'} Cov(X_K, X_{K'})$$

where the summation is over all ordered pairs K, K' where $2 \le |K \cap K'| \le k-1$ (since for all other pairs the covariance is zero). Since

$$Cov(X_K, X_{K'}) = E(X_K X_{K'}) - E(X_K)E(X_{K'}) \le E(X_K X_{K'})$$

it follows from the claim above that

$$Var(X) \le \mu + \Delta \le (1+o(1))\mu.$$

Therefore, by Chebyshev's Inequality, the probability that $X \ge 3\mu/4$ is (much) bigger than 3/4. By Markov's inequality, the probability that $\Delta \le \mu/4$ is also (much) bigger than 3/4 and hence with probability larger than 1/2, both events happen simultaneously, that is, the number of copies of F in G is at least $3\mu/4$ and the number of pairs of copies of F with nontrivial intersection is smaller than $\mu/4$. By removing one copy of F from each pair with a nontrivial intersection we conclude that if this is the case, then G contains a family of at least $\mu/2$ copies of F with no two having a nontrivial intersection.

Let $h(G)$ be the maximum cardinality of a family of copies of F in G in which no two members have a nontrivial intersection, and let Y be the random variable $Y = h(G)$. We next apply Talagrand's Inequality (Theorem 2.2) to deduce that the probability that Y is zero is tiny.

By the above discussion, the probability that Y is at least $\mu/2$ exceeds 1/2. It is also clear that the value of $Y = h(G)$ can change by at most 1 if we add or delete one edge to G, and that h is f-certifiable where $f(s) = s\binom{k}{2}$. Therefore, by Theorem 2.2 with $b = \mu/2$ and $t = \sqrt{\mu}/k$ we conclude that the probability that $Y = 0$ is smaller than

$$e^{-\mu/4k^2}$$

which is much smaller than 2^{-k^2}. As $Y = 0$ if and only if there is no (monotone) copy of F in G, and as the total number of labelled graphs F in $\mathcal{F}(k)$ is $2^{\binom{k}{2}}$ we conclude that G is induced universal for $\mathcal{F}(k)$ with high probability. This completes the proof of Theorem 3.1. □

Remark: The quantity $\binom{n}{k}2^{-\binom{k}{2}}$ is exactly the expected number of cliques of size k in $G = G(n, 0.5)$. Therefore, if this number is $o(1)$ then the probability that G contains such a clique is $o(1)$, by Markov's Inequality. It follows that the smallest n such that $G(n, 0.5)$ is induced universal for $\mathcal{F}(k)$ with high probability is at least $(1+o(1))\frac{k}{e}2^{(k-1)/2}$. By Theorem 3.1 this is tight up to the $o(1)$-term and the smallest n is in fact $(1+o(1))\frac{k}{e}2^{(k-1)/2}$. Note that the above estimate suffices to show that given n, the largest k so that the random graph $G = G(n, 0.5)$ is induced universal for $\mathcal{F}(k)$ with high probability is always one of two consecutive numbers and for "most" values of n it is a single number. It is well known that this is the case for the clique number of G, as proved in [10], [5]. The result here shows that for most values of n, with high probability, the value k which is the size of the largest clique in G is equal to the largest k so that G is induced universal for $\mathcal{F}(k)$ (and for every n these two numbers differ by at most 1 with high probability). The reason is that since the clique of size k has the largest number of automorphisms among all graphs of size k, once the random graph contains it, it typically contains also all the graphs on k vertices with a smaller number of automorphisms. Here "most" means that for any large N, if n is chosen randomly and uniformly among the numbers between N and $2N$, then with probability that tends to 1 as N tends to infinity there is a unique k so that the largest clique of $G = G(n, 0.5)$ is k with high probability and G is induced universal for $\mathcal{F}(k)$ (with the same k) with high probability. Indeed, this is the case whenever n satisfies the following: if k_0 is the largest k for which $\binom{n}{k}2^{-\binom{k}{2}} \geq 1$ then in fact $\binom{n}{k_0}2^{-\binom{k_0}{2}} \geq 4k_0^4$.

It is worth noting that there are values of n for which the probability that the size of the largest clique of $G = G(n, 0.5)$ is one more than the largest k for which G is induced universal for $\mathcal{F}(k)$, is bounded away from zero and one. Indeed, if $\lambda > 0$ is bounded away from 0 and 1 and $\binom{n}{k}2^{-\binom{k}{2}} = \lambda$, then one can use the Stein-Chen method to conclude that the probability that the size of the maximum clique in G is k, is $(1 + o(1))(1 - e^{-\lambda})$. This is also the probability that the size of the maximum independent set in G is k, and these two events are nearly independent. In this case with probability $(1 + o(1))e^{\lambda}(1 - e^{\lambda})$ the size of the maximum independent set is smaller by 1 than that of the maximum clique, and the largest k for which G is induced universal for $\mathcal{F}(k)$ is, with high probability, the smaller of these two numbers. We omit the detailed argument, but mention that it is very similar to the one appearing in [1], Section 2.

4 Extensions

The proof of Theorem 1.1 can be extended to supply similar tight estimates for several related questions. In this section we describe some examples. In most of them the adaptation of the proof is simple, the only case that requires several additional ideas is that of bipartite graphs, described in subsection 4.5.

4.1 Tournaments

A *tournament* on k vertices is an orientation of a complete graph on k vertices, that is, a directed graph on k vertices so that for every pair u, v of distinct vertices there is either a directed edge from u to v or a directed edge from v to u (but not both). It is clear that the number of tournaments on k labelled vertices is $2^{\binom{k}{2}}$, and therefore the number of pairwise non-isomorphic tournaments on k vertices is at least

$$\frac{2^{\binom{k}{2}}}{k!}.$$

Let $\mathbf{T}_k$ denote the set of all tournaments on k vertices. Call a tournament G *induced universal* for $\mathbf{T}_k$ if it contains every member of $\mathbf{T}_k$ as an induced sub-tournament, and let $t(k)$ denote the minimum possible number of vertices of such a universal tournament. The obvious counting argument shows that $t(k) \geq 2^{(k-1)/2}$ and Moon [12] showed that $t(k) \leq O(k2^{k/2})$. This has been improved in [3], where it is proved that $t(k) \leq 16 \cdot 2^{\lceil k/2 \rceil}$. Our method here suffices to determine $t(k)$ up to a lower order additive term.

Theorem 4.1 *The minimum possible number of vertices $t(k)$ of a tournament that contains a copy of every k-vertex tournament satisfies*

$$2^{(k-1)/2} \leq t(k) \leq 2^{(k-1)/2}(1 + O(\frac{\log^{3/2} k}{\sqrt{k}})).$$

Therefore $t(k) = (1 + o(1))2^{(k-1)/2}$.

The proof is similar to that of Theorem 2.1. We first show, using Talagrand's Inequality, that a random tournament on n vertices, where n is as in (1), contains, with high probability, a copy of every tournament on k vertices in which any induced subgraph has less than $k^{8m\sqrt{k \log k}}$ automorphisms. Next we prove that there is a much smaller tournament that contains a copy of all the k-vertex tournaments containing a subgraph with that many automorphisms. This is done by proceeding as in Section 2, using the large automorphism group to describe an appropriate labeling scheme for these tournaments. The desired universal tournament is the vertex disjoint union of the random tournament with the smaller one above. We omit the details.

4.2 Directed graphs

Let $\mathcal{D}(k)$ denote the set of all directed graphs on k vertices. Any two vertices u, v in a member $D \in \mathcal{D}(k)$ can be either nonadjacent, or connected by a directed edge from u to

v, or by one from v to u, or by both. Therefore the cardinality of $\mathcal{D}(k)$ is at least

$$\frac{4^{\binom{k}{2}}}{k!}.$$

A directed graph G is induced universal for $\mathcal{D}(k)$ if every member of $\mathcal{D}(k)$ is an induced subgraph of G. Let $d(k)$ denote the minimum possible number of vertices of such a graph. Simple counting shows that $d(k) \geq 2^{k-1}$. In [3] the authors prove that this is tight up to a constant factor, proving that $d(k) \leq 16 \cdot 2^{k-1}$ and mention the open problem of closing the gap between the upper and lower bound. Our method here gives the following tight estimate.

Theorem 4.2 *The minimum possible number of vertices $d(k)$ of a directed graph that contains a copy of every k-vertex directed graph satisfies*

$$2^{k-1} \leq d(k) \leq 2^{k-1}(1 + O(\frac{\log^{3/2} k}{\sqrt{k}})).$$

Therefore $d(k) = (1 + o(1))2^{k-1}$.

The proof is a straightforward modification of the ones of Theorem 1.1 and Theorem 4.1.

4.3 Oriented graphs

Let $\mathcal{O}(k)$ denote the set of all oriented graphs on k vertices. Any two vertices u, v in a member $O \in \mathcal{O}(k)$ can be either nonadjacent, or connected by a directed edge from u to v, or by one from v to u, but not by both. Therefore the cardinality of $\mathcal{D}(k)$ is at least

$$\frac{3^{\binom{k}{2}}}{k!}.$$

An oriented graph G is induced universal for $\mathcal{O}(k)$ if every member of $\mathcal{O}(k)$ is an induced subgraph of G. Let $r(k)$ denote the minimum possible number of vertices of such a graph. Simple counting shows that $r(k) \geq 3^{(k-1)/2}$. Our technique shows that this is tight up to a small additive error. The proof, which is a simple variant of the previous ones, is omitted.

Theorem 4.3 *The minimum possible number of vertices $r(k)$ of an oriented graph that contains an induced copy of every k-vertex oriented graph satisfies*

$$3^{(k-1)/2} \leq r(k) \leq 3^{(k-1)/2}(1 + O(\frac{\log^{3/2} k}{\sqrt{k}})).$$

Therefore $r(k) = (1 + o(1))3^{(k-1)/2}$.

4.4 Edge colored complete graphs

For a fixed positive integer r, let $\mathcal{K}(k,r)$ denote the set of all complete graphs on k vertices whose edges are colored by the r colors $\{1,2,\ldots,r\}$. The number of members of $\mathcal{K}(r)$ is thus at least

$$\frac{r^{\binom{k}{2}}}{k!}.$$

Note that the case $r=2$ is equivalent to that of all undirected graphs on k vertices. Let $kr(k)$ denote the minimum possible number of vertices of an induced universal graph G for $\mathcal{K}(r,k)$, that is, the minimum number of vertices of a complete r-edge colored graph that contains every member of $\mathcal{K}(r,k)$ as an induced subgraph. Counting shows that $kr(k) \geq r^{(k-1)/2}$. Our method here gives

Theorem 4.4 *For every fixed r and large k, the function $kr(k)$ satisfies*

$$r^{(k-1)/2} \leq kr(k) \leq r^{(k-1)/2}(1+O(\frac{\log^{3/2} k}{\sqrt{k}})).$$

Therefore $kr(k) = (1+o(1))r^{(k-1)/2}$.

4.5 Bipartite graphs

Let $\mathcal{B}(k)$ denote the set of all bipartite graphs $B=(U_1,U_2,E)$ on k vertices. Here the vertex classes of B are U_1,U_2. Call B a (k_1,k_2)-bipartite graph if $|B_1|=k_1$ and $|B_2|=k_2$ (where $k_1+k_2=k$). A bipartite graph G with vertex classes V_1,V_2, each of size n, is an induced universal graph for $\mathcal{B}(k)$ if for any member $B=(U_1,U_2,E)$ of $\mathcal{B}(k)$ there are sets $U_1' \subset V_1$ and $U_2' \subset V_2$ and bijections from U_1 to U_1' and from U_2 to U_2' which map B to an isomorphic induced subgraph of G. Note that we insist here that the vertices of U_1 are mapped to vertices in V_1, and the same holds for U_2,V_2. Allowing arbitrary mappings from the vertices of B to those of G is also possible, and does not cause any significant changes, hence we prefer to consider the definition given above. Let $b(k)$ denote the smallest possible n so that an induced universal bipartite graph as above for $\mathcal{B}(k)$ exists. Since this graph has to contain all $(\lfloor k/2 \rfloor, \lceil k/2 \rceil)$-bipartite graphs as induced subgraphs, a simple counting argument shows that for even k, $b(k) \geq 2^{k/4}$ whereas for odd k,

$$b(k) \geq 2^{(k^2-1)/4k} = 2^{k/4}(1-O(1/k)).$$

In [9] it is shown that $b(k) \leq O(k^2 2^{k/4})$ and in [3] it is proved that $b(k) \leq c2^{k/4}$ for some absolute constant c (which is not specified in the paper, but is certainly less than 100.) Here we show that the lower bound is tight, up to a lower order additive term.

Theorem 4.5 *The function $b(k)$ satisfies*

$$2^{k/4}(1 - O(1/k)) \leq b(k) \leq 2^{k/4}(1 + O(\frac{\log^{3/2} k}{\sqrt{k}}))$$

Therefore $b(k) = (1 + o(1))2^{k/4}$.

The proof is similar to the previous ones, but requires a few additional arguments. We first note that as shown in [3], it is easy to deal with all unbalanced bipartite graphs. This is stated in the following lemma.

Lemma 4.6 ([3], Theorem 8.1) *For every r there is a bipartite graph G_r with classes of vertices V_1, V_2, each of size at most $n = 2k \cdot 2^{k/4 - r^2/k}$, that contains every $(k/2-r, k/2+r)$-bipartite graph (U_1, U_2, E) as an induced subgraph (with U_1 embedded in V_1 and U_2 in V_2.)*

The proof is by orienting the edges of the complete bipartite graph with classes of vertices of sizes $k/2 - r$ and $k/2 + r$ so that every outdegree is at most $k/4 - r^2/k + 1$. Such an orientation exists by Fact 2 in subsection 2.2. We can then label each vertex by its number (which is a number not exceeding k) and by the bits describing its adjacency relations to its outneighbors in the above orientation, in order. This assigns to each vertex a label with at most $\log_2 k + k/4 - r^2/k + 1$ bits, and the labels of any two vertices suffice to determine whether or not they are adjacent. The existence of the desired graph G_r now follows from the standard simple connection between adjacency labeling schemes and induced universal graphs.

Note that by taking the vertex disjoint union of the graphs G_r for all r (positive or negative) satisfying, say, $2\sqrt{k \log k} \leq |r|$ ($< k/2$) the total number of vertices in each side is less than $2^{k/4}/k^2$ which is smaller than $b(k)/k^2$. It thus remains to deal with the (k_1, k_2)-bipartite graphs in which k_1 and k_2 are close to each other, hence we assume from now on that each of them is at least, say, $0.45k$.

As in the proof of Theorem 1.1, we deal separately with bipartite graphs in which every large induced subgraph has a relatively small number of automorphisms, which we call asymmetric, and with the others, called symmetric. The asymmetric graphs will be contained in a random bipartite graph, and the symmetric ones in a smaller graph, constructed using an appropriate adjacency labelling scheme. We proceed with the details. In the rest of this section an automorphism of a bipartite graph $B = (U_1, U_2, E)$ with vertex classes U_1, U_2 means an automorphism that maps each U_i to itself. Recall that in all (k_2, k_2) bipartite graphs considered from now on $k_1 + k_2 = k$ and each k_i is between $0.45k$ and $0.55k$.

Call a (k_1, k_2)-bipartite graph on $k_1 + k_2 = k$ vertices *asymmetric* if any induced subgraph of it on at least $0.9k$ vertices has at most k^{8m} automorphisms, where $m = 2\sqrt{k \log k}$. Note that, in particular, the number of automorphisms of any such graph is at most k^{8m}. Let $\mathcal{B}'(k)$ denote the family of all asymmetric bipartite graphs on k vertices.

Let n be the smallest integer that satisfies the following inequality

$$\binom{n}{\lfloor k/2 \rfloor}\binom{n}{\lceil k/2 \rceil}\frac{\lfloor k/2 \rfloor!\lceil k/2 \rceil!}{k^{16m}}2^{-\lfloor k/2 \rfloor \cdot \lceil k/2 \rceil} \geq 1. \tag{5}$$

When increasing n to $n+1$, the left-hand-side of the last inequality increases by $1 + o(1)$ for the relevant parameters, and thus it follows that the above smallest n satisfies

$$n = 2^{k/4}(1 + O(1/k)).$$

In particular, $n = (1 + o(1))2^{k/4}$.

Theorem 4.7 *Let n be as above and let $G = (V_1, V_2, E) = G(n, n, 0.5)$ be the random bipartite graph with vertex classes V_1, V_2, each of size n, where each pair of vertices $v_1 \in V_1$ and $v_2 \in V_2$ form an edge randomly and independently with probability $1/2$. Then, with high probability G is an induced universal bipartite graph for $\mathcal{B}'(k)$.*

As in the proof of Theorem 2.1, the proof here is based on Talagrand's Inequality (Theorem 2.2).

We need the following lemma.

Lemma 4.8 *Let $B \in \mathcal{B}'(k)$ have s automorphisms. Let the sizes of the vertex classes of B be k_1, k_2 (and recall that each k_i is at least $0.45k$.) Suppose j_1, j_2 are integers satisfying $1 \leq j_1 = k_1 - i_1 \leq k_1$, $1 \leq j_2 = k_2 - i_2 \leq k_2$, $j_1 + j_2 = j = k - i$ and assume $1 \leq i \leq 0.1k$. Let K_1, K_1' be two sets of vertices, each of size k_1, and let K_2, K_2' be two sets of vertices, each of size k_2, where $|K_1 \cap K_2| = j_1$, $|K_2' \cap K_2'| = j_2$ and K_1, K_1' do not intersect K_2, K_2'. Then the number of ways to choose the edges and nonedges among all $2k_1k_2 - j_1j_2$ pairs consisting of a vertex of K_1 and a vertex of K_2 or a vertex of K_1' and a vertex of K_2' so that the induced subgraph on $K_1 \cup K_2$ is isomorphic to B and the induced subgraph on $K_1' \cup K_2'$ is also isomorphic to B (and both isomorphisms map the first vertex class of B to K_1 and to K_1') is at most*

$$\frac{k_1!k_2!}{s}k^{i_1+i_2}k^{8m}.$$

Proof: There are exactly $\frac{k_1!k_2!}{s}$ ways to place a copy of B on $K_1 \cup K_2$. For each such fixed copy, we bound the number of embeddings of B in $K_1' \cup K_2'$. There are at most

$$k_1(k_1 - 1) \cdots (k_1 - i_1 + 1)k_2(k_2 - 1) \ldots (k_2 - i_2 + 1) < k^{i_1+i_2}$$

ways to choose the vertices of B mapped to the vertices of $K_1' - K_1$ and to the vertices of $K_2 - K_2'$. Fix a set T of these i vertices and their embedding. In order to complete the embedding, the induced subgraph of the copy of B placed in $K_1 \cup K_2$ on the set of vertices $(K_1 \cap K_1') \cup (K_2 \cap K_2')$ has to be isomorphic to the induced subgraph of B on $V(B) - T$. If so, then the number of ways to embed these $k-i$ vertices is the number of automorphisms of this induced subgraph of B, which is, by the definition of $\mathcal{B}'(k)$, at most k^{8m}. □

Proof of Theorem 4.7: Let B be a fixed member of $\mathcal{B}'(k)$, let k_1, k_2 denote the number of vertices in its vertex classes, and let s be the size of its automorphsim group. Then

$$s \leq k^{8m} = k^{16\sqrt{k \log k}}.$$

Let $G = (V_1, V_2, E)$ be the random bipartite graph $G(n, n, 0.5)$, where n is as chosen in (5). For every pair of subsets $K_1 \subset V_1,\ K_2 \subset V_2$ of sizes $|K_1| = k_1,\ |K_2| = k_2$, let X_{K_1,K_2} denote the indicator random variable whose value is 1 if the induced subgraph of G on $K_1 \cup K_2$ is isomorphic to B and let $X = \sum_{K_1,K_2} X_{K_1,K_2}$, where the summation is over all pairs of subsets $K_1 \subset V_1$ of cardinality k_1 and $K_2 \subset V_2$ of cardinality k_2. Thus X is the number of copies of B in G. The expectation of each X_{K_1,K_2} is clearly

$$E(X_{K_1,K_2}) = \frac{k_1! k_2!}{s} 2^{-k_1 k_2}.$$

Thus, by linearity of expectation,

$$E(X) = \binom{n}{k_1}\binom{n}{k_2} \frac{k_1! k_2!}{s} 2^{-k_1 k_2} \geq k^{8m}$$

where in the last inequality we used (5) and the fact that $s \leq k^{8m}$. Note also that since

$$k_1 k_2 \geq \frac{k^2}{4} - \frac{k^2}{400}$$

it follows that the expectation of X is at most $(1+o(1))k^{16m} 2^{k^2/400} < n^{0.02}$ (even if $s = 1$).

We say that two copies of B in G have a *nontrivial intersection* if they share at least one vertex in each vertex class. Put $\mu = E(X)$. Let Z denote the random variable $Z = \sum_{K,K'} X_{K_1,K_2} X_{K_1',K_2'}$ where the summation is over all (ordered) pairs $K = (K_1, K_2)$ and $K' = (K_1', K_2')$ that satisfy $K_1 \cap K_1' \neq \emptyset$ and $K_2 \cap K_2' \neq \emptyset$. Thus, Z is the number of pairs of copies of B in G that have a nontrivial intersection. We next compute the expectation of Z and show that it is much smaller than μ (and hence also much smaller than μ^2). Put $\Delta = E(Z)$. Then $\Delta = \sum_{j_1,j_2} \Delta_{j_1,j_2}$, where the summation is over all (j_1, j_2) satisfying $1 \leq j_1 \leq k_1$, $1 \leq j_2 \leq k_2$ and $j_1 + j_2 < k_1 + k_2 = k$. Here Δ_{j_1,j_2} is the expected number of pairs $K = (K_1, K_2), K' = (K'_1, K'_2)$ with $X_{K_1,K_2} = X_{K'_1,K'_2} = 1$ and $|K_1 \cap K'_1| = j_1$, $|K_2 \cap K'_2| = j_2$.

Claim: For all admissible j_1, j_2

$$\Delta_{j_1,j_2} \leq \mu \frac{1}{n^{0.1}} \tag{6}$$

Proof of Claim: Consider two possible cases, as follows.

Case 1: $j_1 + j_2 \leq 0.9k$.

In this case

$$\Delta_{j_1,j_2} \leq \binom{n}{k_1}\binom{n}{k_2}\binom{k_1}{j_1}\binom{k_2}{j_2}\binom{n-k_1}{k_1-j_1}\binom{n-k_2}{j_2-k_2}\frac{k_1!k_2!}{s}\frac{k_1!k_2!}{s}2^{-2k_1k_2+j_1j_2}.$$

Indeed, there are $\binom{n}{k_1}\binom{n}{k_2}$ ways to choose the sets K_1, K_2. For each such choice there are

$$\binom{k_1}{j_1}\binom{k_2}{j_2}\binom{n-k_1}{k_1-j_1}\binom{n-k_2}{j_2-k_2}$$

ways to choose K'_1, K'_2 with $|K_1 \cap K'_1| = j_1$, $|K_2 \cap K'_2| = j_2$. There are $\frac{k_1!k_2!}{s}$ ways to place a copy of B in $K_1 \cup K_2$ and $\frac{k_1!k_2!}{s}$ ways to place a copy of B in $K'_1 \cup K'_2$ (this is an overcount, as these two copies have to agree on the edges in their common part). This determines all the edges and nonedges in the induced subgraph on $K_1 \cup K_2$ and on $K'_1 \cup K'_2$. The probability that the bipartite graph G indeed has exactly these edges is

$$2^{-2k_1k_2+j_1j_2}.$$

Therefore,

$$\frac{\Delta_{j_1,j_2}}{\mu^2} \leq k_1^{j_1}k_2^{j_2}(\frac{k_1}{n})^{j_1}(\frac{k_2}{n})^{j_2}2^{j_1j_2} < (\frac{k^2}{n})^{j_1+j_2}2^{((j_1+j_2)/2)^2} = [\frac{k^2 2^{(j_1+j_2)/4}}{n}]^{j_1+j_2}$$

$$< (\frac{1}{n^{0.1-o(1)}})^2 < \frac{1}{n^{0.15}}.$$

Here we used the fact that $k = (4 + o(1)) \log_2 n$ and that $2^{(j_1+j_2)/4} \leq 2^{0.9k/4}$. Recall that $\mu \leq n^{0.02}$ and thus

$$\frac{\Delta_{j_1,j_2}}{\mu} \leq \mu \frac{\Delta_{j_1,j_2}}{\mu^2} < \frac{1}{n^{0.1}}$$

as claimed.

Case 2: $j_1 = k_1 - i_1, j_2 = k_2 - i_2 \;\; 1 \leq i = i_1 + i_2 \leq 0.1k$.

In this case we have, by Lemma 4.8

$$\Delta_{j_1,j_2} \le \binom{n}{k_1}\binom{n}{k_2}\binom{k_1}{j_1}\binom{n-k_1}{k_1-j_1}\binom{k_2}{j_2}\binom{n-k_2}{k_2-j_2}\frac{k_1!k_2!}{s}k^{i_1+i_2}k^{8m}2^{-2k_1k_2+j_1j_2}.$$

Indeed, there are

$$\binom{n}{k_1}\binom{n}{k_2}\binom{k_1}{j_1}\binom{n-k_1}{k_1-j_1}\binom{k_2}{j_2}\binom{n-k_2}{k_2-j_2}$$

ways to choose the sets K_1, K_2, K_1', K_2' with the required intersections, and by Lemma 4.8 for each such choice there are at most $\frac{k_1!k_2}{s}k^{i_1+i_2}k^{8m}$ ways to place copies of B in each of them. The probability that this coincides with all edges and nonedges of the induced subgraph of G on $K_1 \cup K_2$ and on $K_1' \cup K_2'$ is $2^{-2k_1k_2+j_1j_2}$.

Since $\mu \ge k^8 > 1$ this implies that

$$\frac{\Delta_{j_1,j_2}}{\mu^2} < \frac{\Delta_{j_1,j_2}}{\mu} \le \binom{k_1}{i_1}\binom{n-k_1}{i_1}\binom{k_2}{i_2}\binom{n-k_2}{i_2}k^{i_1+i_2}k^{8m}2^{-k_1k_2+j_1j_2}$$

$$< (k^2n)^{i_1+i_2}n^{o(1)}2^{-i_2(k_1-0.5i_1)-i_1(k_2-0.5i_2)}$$

$$< (k^2n)^{i_1+i_2}2^{-0.4k(i_1+i_2)}n^{o(1)} < (n^{-0.6+o(1)})^i < n^{-0.1}.$$

This completes the proof of the claim.

Returning to the proof of the theorem, note that by the claim, the variance of the random variable X satisfies

$$Var(X) \le E(X) + \sum_{K,K'} Cov(X_{K_1,K_2}, X_{K_1',K_2'})$$

where the summation is over all ordered pairs $K = (K_1, K_2), K' = (K_1', K_2')$ with nontrivial intersections (since for all other pairs the covariance is zero). Since

$$Cov(X_{K_1,K_2}, X_{K_1',K_2'}) \le E(X_{K_1,K_2}X_{K_2',K_2'})$$

it follows from the claim above that

$$Var(X) \le \mu + \Delta \le (1 + o(1))\mu.$$

Therefore, by Chebyshev's Inequality, the probability that $X \ge 3\mu/4$ is (much) bigger than $3/4$. By Markov's inequality, the probability that $\Delta \le \mu/4$ is also (much) bigger than $3/4$ and hence with probability larger than $1/2$, both events happen simultaneously, that is, the number of copies of B in G is at least $3\mu/4$ and the number of pairs of copies of

B with nontrivial intersection is smaller than $\mu/4$. By removing one copy of B from each pair with a nontrivial intersection we conclude that if this is the case, then G contains a family of at least $\mu/2$ copies of B with no two having a nontrivial intersection.

Let $h(G)$ be the maximum cardinality of a family of copies of B in G in which no two members have a nontrivial intersection, and let Y be the random variable $Y = h(G)$. We next apply Talagrand's Inequality (Theorem 2.2) to deduce that the probability that Y is zero is tiny.

By the above discussion, the probability that Y is at least $\mu/2$ exceeds $1/2$. It is also clear that the value of $Y = h(G)$ can change by at most 1 if we add or delete one edge to G, and that h is f-certifiable where $f(s) = s \cdot k_1 k_2 \leq sk^2/4$. Therefore, by Theorem 2.2 with $b = \mu/2$ and $t = \sqrt{2\mu}/k$ we conclude that the probability that $Y = 0$ is smaller than

$$e^{-\mu/2k^2}.$$

This is much much smaller than one over the cardinality of $\mathcal{B}'(k)$, which is less than $k2^{k^2/4}$. As $Y = 0$ if and only if there is no copy of B in G, we conclude that G is induced universal for $\mathcal{B}'(k)$ with high probability. This completes the proof of Theorem 4.7. $\square$

In order to complete the proof of Theorem 4.5 it suffices to show that there is bipartite graph with at most, say, $2^{k/4}/k$ vertices which is induced universal for the symmetric bipartite graphs, that is, for all (k_1, k_2)-bipartite graphs with each k_i being at least $0.45k$ that have induced subgraphs on at least $0.9k$ vertices with at least k^{8m} automorphisms, where $m = 2\sqrt{k \log k}$. This is similar to the proof of Theorem 2.7. Here is a sketch of the argument. By Lemma 2.4 it follows that any such symmetric (k_1, k_2)-bipartite graph $F = (U_1, U_2, E)$ has an automorphism with at least $2m$ fixed points and at least $6m$ non-fixed points. Such an automorphism must contain at least m fixed points in one vertex class and at least $3m$ non-fixed points in the other. (Indeed, it has at least m fixed points in some vertex class and at least $3m$ non-fixed points in some class. If these happen to be the same class, then the other class contains either many fixed points or many non-fixed points, implying what's needed.) Without loss of generality assume there are m fixed points in U_1 and $3m$ non-fixed points in U_2. As in Corollary 2.5 this implies that there is a set $A = \{a_1, a_2, \ldots, a_m\}$ of m vertices in U_1 and two disjoint sets $B = \{b_1, b_2, \ldots, b_m\}$ and $C = \{c_1, c_2, \ldots, c_m\}$ in U_2 such that for any $1 \leq i, j \leq m$, $a_i b_j$ is an edge of F if and only if $a_i c_j$ is an edge of F.

The next ingredient needed is an appropriate bipartite analogue of Lemma 2.6, which follows.

Lemma 4.9 *Let H be the bipartite (k_1, k_2)-graph obtained from the complete bipartite graph with vertex classes U_1, U_2, ($|U_i| = k_i$), by omitting all edges connecting A and C, where A is a subset of cardinality m of U_1, and B, C are disjoint subsets of cardinality m of U_2, and $m = 2\sqrt{k \log k}$. Then there is an orientation H' of H in which all edges between A and B are oriented from A to B and the maximum outdegree of a vertex in H' is at most*

$$\frac{k_1 k_2}{k} - \frac{m^2}{k} + O(1) \ (\ < \frac{k}{4} - 3 \log k).$$

The proof is by applying Fact 2 used in the proof of Lemma 2.6. Let $D_1 = U_1 - A$ and $D_2 = U_2 - (B \cup C)$. By Fact 2 for any real $x \in (0, 1)$ there is an orientation of the edges of the complete bipartite graph with vertex classes A and D_2 so that the outdegree of each vertex of A is at most $x|D_2| + 1 = x(k_2 - 2m) + 1$ and the outdegree of each vertex of D_2 is at most $(1 - x)m + O(1)$. Similarly, for any real $y \in (0, 1)$ there is an orientation of the edges of the complete bipartite graph between $B \cup C$ and D_1 so that the outdegree of every vertex of $B \cup C$ is at most $y|D_1| + 1 = y(k_1 - m) + 1$ and the outdgree of each vertex of D_1 is at most $(1 - y)2m + O(1)$. Finally, there is an orientation of all edges connecting D_1 and D_2 so that each vertex of D_1 has outdegree at most $z(k_2 - 2m) + 1$ and each vertex of D_2 has outdgree at most $(1 - z)(k_1 - m) + O(1)$.

Recalling that each k_i is between $0.45k$ and $0.55k$, and that $m = o(k)$, it follows that there are $x, y, z \in (0, 1)$, all rather close to $1/2$, so that all 4 quantities

$$m + x(k_2 - 2m), \quad y(k_1 - m), \quad (1 - y)2m + z(k_2 - 2m), \quad \text{and} \quad (1 - x)m + (1 - z)(k_1 - m)$$

which represent (up to an $O(1)$ additive error) the outdegrees of the vertices in A, $B \cup C$, D_1 and D_2 respectively, are equal. This means that in our orientation, in which all edges between A and B are oriented from A to B, the outdegrees of all vertices are essentially the same, and hence each of them is the average outdegree, which is

$$\frac{k_1 k_2}{k} - \frac{m^2}{k},$$

up to an additive constant error. This establishes the lemma. □

The last lemma can be used, as in the proof of theorem 2.7, to show that for any fixed k_1, k_2 there is an adjacency labelling scheme for the above symmetric bipartite graphs in which each label has at most $\frac{k}{4} - 2 \log k$ bits. This supplies an induced universal bipartite graph for the required symmetric bipartite graphs, whose total number of vertices is at most $2^{k/4}/k$. Together with Theorem 4.7, this completes the proof of Theorem 4.5.

5 Concluding remarks

Our upper and lower bounds for the various induced-universal graphs considered here differ by lower order additive terms. It may be interesting to try and get even tighter estimates. How close are the counting lower bounds to the correct values ?

To be specific, consider the function $f(k)$ discussed in Theorem 1.1, namely, the minimum possible number of vertices in an induced-universal graph for the set $\mathcal{F}(k)$ of all k-vertex undirected graphs. The trivial counting lower bound shows that $n = f(k)$ must satisfy

$$\binom{n}{k} \geq |\mathcal{F}(k)|.$$

Equality would hold here if there would have been a graph G on $n = (1 + o(1))2^{(k-1)/2}$ vertices containing every graph $F \in \mathcal{F}(k)$ as an induced subgraph exactly once. It is not difficult to show that this is impossible, and in fact any graph G of this size must contain at least $2^{\Omega(k)}$ induced copies of some k-vertex graphs. This is proved in the following proposition.

Proposition 5.1 *For any positive constant c and all sufficiently large k the following holds. For any graph $G = (V, E)$ on at least 2^{ck} vertices there is a graph H on k vertices so that G contains at least $2^{(c/2+o(1))k}$ induced copies of H.*

Proof: By the standard proof of Ramsey's Theorem there are vertices $v_1, v_2, \ldots, v_m$, where $m = (ck - \log k)/2$ and a set U of at least k vertices containing none of the vertices v_i so that one of the following holds. Either

(i) There are no edges among the vertices v_i and no edges connecting any v_i to any vertex of U, or

(ii) The vertices v_i form a clique and each v_i is adjacent to all vertices of U.

Indeed, starting with $U_0 = V$ choose arbitrarily a vertex $x_1 \in U_0$ and let U_1 be either the set of all its neighbors in U_0 or the set of all its non-neighbors, whichever is bigger. Next choose $x_2 \in U_1$ and let U_2 be the set of all its neighbors in U_1 or all its non-neighbors, whichever is bigger. Proceeding in the same way, after $2m - 1$ steps we get a set $\{x_1, x_2, \ldots, x_{2m-1}\}$ of vertices and a set $U = U_{2m-1}$ of more than k vertices of G. If in at least m of the steps we have chosen non-neighbors, we get (i), else we get (ii).

To complete the proof fix a set U' of $k - m/2$ vertices of U and observe that the induced graphs on the union of U' with any set of $m/2$ of the vertices v_i are all isomorphic. □

In particular, if G is an induced universal graph for $\mathcal{F}(k)$ with $(1+o(1))2^{(k-1)/2}$ vertices, whose existence is proved in Theorem 1.1, then by the above proposition G contains at least $2^{(1+o(1))k/4}$ induced copies of some k-vertex graphs. (The proof in fact shows that this is the case for any subgraph consisting of $m/2$ vertices v_i and any $k-m/2$ vertices of U.) Note that counting shows that most k-vertex graphs appear much less, that is, only $2^{o(k)}$ times, as induced subgraphs of G.

It is not difficult to show that the maximum number of automorphisms of a tournament on k vertices is only exponential in k - much smaller than the maximum number of automorphisms of a k-vertex undirected graph (which is $k!$) Indeed, let $g(k)$ denote the maximum number of automorphisms of a tournament on k vertices, and let T be a tournament attaining this maximum. Like any tournament, T contains a vertex v whose outdegree d satisfies $k/4+O(1) \leq d \leq 3k/4+O(1)$. If an automorphism of T maps v to u, then its outneighbors are mapped to the outneighbors of u and its inneighbors are mapped to the inneighbors of u. This shows that $g(k) \leq kg(d)g(k-1-d)$ where d is as above, and it is easy to verify that this gives an exponential upper bound for $g(k)$. This, together with the arguments in the proof of Theorem 2.1, imply that there is an absolute constant $c>1$ so that a random tournament with $c2^{(k-1)/2}$ vertices is induced universal for the set $\mathbf{T}_k$ of all tournaments on k vertices with high probability. Although this does not imply the sharp statement of Theorem 4.1, it shows that for tournaments, unlike for undirected graphs, the random construction is larger than the best one only by a (small) constant factor.

All the induced universal graphs appearing in the theorems in the paper contain a large random part, namely, the constructions here are not explicit, unlike the ones in [11], [12], [3]. It would be interesting to find explicit constructions of the same sizes.

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