# Divisible subdivisions

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**Abstract**

We prove that for every graph $H$ of maximum degree at most 3 and for every positive integer $q$ there is a finite $f = f(H, q)$ such that every $K_f$-minor contains a subdivision of $H$ in which every edge is replaced by a path whose length is divisible by $q$. For the case of cycles we show that for $f = O(q \log q)$ every $K_f$-minor contains a cycle of length divisible by $q$, and observe that this settles a recent problem of Friedman and the second author about cycles in (weakly) expanding graphs.

## 1 Introduction

There are several known results asserting that any graph with a sufficiently large minimum (or average) degree contains a cycle of prescribed length modulo a given parameter. An early result of this form appears in [3]: for every odd $k$ there exists a $c(k)$ so that every graph with minimum degree at least $c(k)$ contains a cycle of length $\ell$ modulo $k$ for every integer $\ell$. A similar result holds for every even $k$ and every even $\ell$ (but of course not for even $k$ and odd $\ell$ as shown by dense bipartite graphs.) Thomassen proved in [11] that for non-bipartite 2-connected graphs a result as above exists also for even $k$ and odd $\ell$.

For graphs with large chromatic number stronger conclusions hold. Another result established in [11] addresses this case: For any two positive integers $m$ and $k$ there exists a number $c(m, k)$ such that the following holds. For every assignment of two natural numbers $k(e) \leq k$ and $d(e)$ for each edge $e$ of $K_m$, any graph of chromatic number at least

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$c(m, k)$ contains a subdivision of $K_m$ in which each edge $e$ corresponds to a path of length $d(e)$ modulo $k(e)$.

A more recent result with a similar flavor is proved in [10]: for every $k$ there is a $c(k)$ so that every graph with average degree at least $c(k)$ contains a subdivision of $K_k$ in which every edge is subdivided the same number of times.

A common feature of these results and related ones is that they apply only to graphs with a rather large average degree. In particular, if the average degree is just a bit above 2, then none of these results holds, and indeed there are simple examples showing no such result can hold if we do not make any additional assumptions. In this note we prove a result applicable to very sparse graphs. The sufficient condition we give is based on complete minors; since there exist minors of arbitrarily large complete graphs with average degree arbitrarily close to 2 in every subgraph, this imposes essentially the weakest possible condition in terms of graph density.

**Theorem 1.** *For every graph $H$ of maximum degree at most 3, and for every positive integer $q$ there is a finite $f = f(H, q)$ such that every $K_f$-minor $G$ contains a subdivision of $H$ in which every edge is replaced by a path whose length is divisible by $q$.*

**Remark:** For every $f$ and $q$ there is a $K_f$-minor $G$ with maximum degree 3, in which any path between two vertices of degree 3 is of length divisible by $q$; obviously such $G$ does not contain a subdivision of any graph $H$ with maximum degree $\Delta(H) > 3$. Hence, the assumption that $H$ has maximum degree at most 3 is needed. Similarly the conclusion about paths of length 0 modulo $q$ cannot be replaced by any other residue modulo $q$, unlike the results in [3] or [11] mentioned above.

We make essentially no attempt to optimize the value of $f = f(H, q)$, and the problem of determining its asymptotic behavior remains open.

It is well known that graphs without small separators contain large complete minors (the line of research establishing these results started in [2] and culminated with [7]). Our main result is thus applicable to such graphs. Also, non-existence of sublinear separators is essentially equivalent to weak expansion (see [9] for a discussion), hence Theorem 1 can be applied to the class of (weakly) expanding graphs.

In the very special case of obtaining a cycle of length divisible by $q$ we get a nearly tight bound, proving the following.

**Theorem 2.** *For every positive integer $q$ there is a $g = g(q) = O(q \log q)$ such that every $K_g$-minor $G$ contains a cycle of length divisible by $q$. Moreover, if $q$ is a prime then $g(q) < 4q$.*

This, together with the fact that expanders contain large clique minors (see [9] for a background), settles a problem raised recently in [6].

The proofs are described in the next two sections. The final section contains some concluding remarks and open problems.

## 2 Subdivisions

In this section we prove Theorem 1. We start by proving a key lemma.

**Lemma 2.1.** *There is a function $f_1(k,q)$ satisfying the following condition. Let $k, q \geq 2$ be integers. Let $T$ be a tree with edges labeled by the elements of $Z_q$, and let $L$ be a set of specified leaves of $T$ of cardinality $|L| = f_1(k,q)$. Then there are a subset $L_0 \subset L$ of $k$ leaves and a residue $a \in Z_q$ such that for every three leaves $x_1, x_2, x_3 \in L_0$ there is a vertex $v \in V(T)$ with all paths from $v$ to $x_i$ in $T$ being disjoint outside of $v$ and having all weight $a$ modulo $q$.*

*Proof.* We set with foresight

$$f_1 = f_1(k,q) = ((k-1)q+1)^{(k-1)q^2+1}.$$

Assume that $T, L$ are as given in the lemma. Observe that if the lemma's conclusion holds for a subtree $T' \subseteq T$ containing $L$ then it holds for $T$. Hence we can assume that $T$ is a minimal by inclusion tree containing $L$.

For convenience root $T$ at an arbitrary vertex $r \in V(T)$ with $d_T(r) \geq 3$. For a vertex $v \in V(T)$ denote by $T_v$ the subtree of $T$ rooted at $v$ (with respect to $r$).

Consider first the case where there is a vertex $v \in V(T)$ with $d_T(v) \geq (k-1)q+2$. By the minimality of $T$, for each child $u$ of $v$ the subtree $T_u$ contains a leaf $\ell(u) \in L$, the paths from $v$ to all such $\ell(u)$ are disjoint outside of $v$. By the pigeonhole principle there is a subset $U_0$ of the children of $v$ in $T$ of cardinality $|U_0| = \lceil \frac{d_T(v)-1}{q} \rceil \geq k$ such that all paths from $v$ to $\ell(u)$, $u \in U_0$, are of the same weight $a$ modulo $q$. The set $L_0 = \{\ell(u) : u \in U_0\}$ fulfills then the requirement of the lemma.

Now we treat the complementary case $\Delta(T) \leq (k-1)q+1$. Let $t$ be the maximal number of vertices of degree at least three on a path from $r$ to a leaf $x \in L$. Since every $x \in L$ is uniquely determined by the sequence of edges leaving the vertices of degree at least three on the unique path from $r$ to $x$ in $T$, and the number of such sequences is obviously at most $(\Delta(T))^t$, we obtain: $|L| \leq (\Delta(T))^t$, implying $t \geq (k-1)q^2+1$. Let $P$ be a path from $r$ to a leaf of $T$ with $t$ vertices of degree at least three in $T$ along it, and let $U_1 \subset V(P)$ be these vertices, $|U_1| = t \geq (k-1)q^2 + 1$. By the pigeonhole principle, there is a subset $U_2 \subset U_1$ of cardinality $|U_2| = \lceil \frac{|U_1|}{q} \rceil \geq (k-1)q + 1$ such that all subpaths of $P$ from $r$ to $u \in U_2$ have the same weight modulo $q$. This implies that for every $u_1 \neq u_2 \in U_2$ the subpath of $P$ between $u_1$ and $u_2$ has weight 0 modulo $q$. By the minimality of $T$, every vertex $u \in U_2$ contains a leaf $\ell(u) \in L$ in its subtree $T_u$, where all these leaves are distinct. Applying the pigeonhole principle again, we derive the existence of a subset $U_3 \subset U_2$ of cardinality $|U_3| = k$ such that every path between $u \in U_3$ and the corresponding leaf $\ell(u) \in T_u$ has the same weight $a$ modulo $q$. Set $L_0 = \{\ell(u) : u \in U_3\}$. We claim that $L_0$ meets the requirement of the lemma. Indeed, let $u_1, u_2, u_3$ be distinct vertices in $U_3$ ordered in the order of their appearance along $P$. Then the paths from $u_2$ to $\ell(u_i)$, $i = 1, 2, 3$, are all disjoint outside of $u_2$ and have total weight $a$ modulo $q$ (the part along $P$ has weight 0 modulo $q$, and the appended part from $u_i$ to $\ell(u_i)$ has weight $a(\text{mod } q)$). □

We can now prove Theorem 1, making no attempt to optimize the estimate for $f$. Let $\Gamma$ be a Ramsey graph in $q$ colors for the $(q-1)$-subdivision of $H$. Assume that the vertex set of $\Gamma$ is $[N]$, denote $k = 2|E(\Gamma)|$. Let $M = f_1(k, q)$, with $f_1(k, q)$ from Lemma 2.1. Finally, set:

$$f = M + (N-1)q + 1\,.$$

Assume $G$ is a minor of $K_f$ with supernodes $X_1, \ldots, X_{(N-1)q+1}, Y_1, \ldots, Y_M$. For a pair of supernodes $X_i, Y_j$, if $e = (x, y)$ is an edge connecting $X_i$ to $Y_j$ then split $e$ by a vertex $z$, assign weights $w((x, z)) = 0$, $w((z, y)) = 1$, and append $(x, z)$ to $X_i$. We assign weight 1 to all remaining edges of $G$.

Let $T_i$ be a spanning tree of $X_i$ and let $L_i$ be a set of $M = f_1(q, k)$ distinct leaves of $T_i$, each connected to a different supernode $Y_j$. Apply Lemma 2.1 to $(T_i, L_i)$ to get a subset $L'_i$ of cardinality $|L'_i| = k$ and a residue $a_i \in Z_q$ with the properties guaranteed by the lemma. Invoking the pigeonhole principle with respect to the multiset of residues $\{a_i\}$ we conclude that there exists a subset $I \subset [(N-1)q + 1]$ of cardinality $|I| = N$ with all residues $a_i$, $i \in I$, taking the same value $a$. By renumbering if necessary we can assume $I = [N]$.

Now we go sequentially over all edges $e = (i_1, i_2) \in E(\Gamma)$ and connect the corresponding supernodes $X_{i_1}, X_{i_2}$ as follows: choose a previously unused supernode $Y_{j_1}$ having a neighbor in $L'_{i_1}$, choose a distinct and previously unused supernode $Y_{j_2}$ having a neighbor in $L'_{i_2}$, and then connect $Y_{j_1}$ and $Y_{j_2}$. Concatenating, we obtain a path $P_e$ in $G$ from a leaf $x_1 \in L'_{i_1}$ to a leaf $x_2 \in L'_{i_2}$, where all these paths are vertex disjoint for different edges

$e = (i_1, i_2) \in E(\Gamma)$.

Define a coloring $c : E(\Gamma) \to Z_q$ as follows: for $e = (i_1, i_2) \in E(\Gamma)$, its color $c(e)$ is equal to the weight modulo $q$ of the path $P_e$ between $X_{i_1}$ and $X_{i_2}$. By the choice of $\Gamma$, the so obtained coloring $c$ induces a monochromatic copy $H^*$ of the $(q-1)$-subdivision of $H$, say in color $b \in Z_q$. Let $I_0 \subset [N]$ be the subset of supernodes corresponding to the vertices of $H$ in this subdivision. By construction, for each edge $f = (i_1, i_2) \in E(H)$, $i_1, i_2 \in I_0$, the graph $G$ contains a path $Q_f$ from $X_{i_1}$ to $X_{i_2}$ passing through a sequence of $q-1$ intermediate supernodes $X_i$, with the intermediate supernodes being distinct for distinct edges. Each such path enters and leaves $X_i$ through vertices of $L'_i$; by the definition of $L'_i$ and the choice of $a$, the entrance and departure point in $X_i$ can be connected by a path of length $2a$ modulo $q$. The weight of $Q_f$ between two consecutive intermediate supernodes $X_i$, and also between $X_{i_1}$ and the first intermediate supernode, and between the last intermediate supernode and $X_{i_2}$ is $b$ modulo $q$. Finally, for each supernode $X_i$, $i \in I_0$, the (at most three) paths $Q_f$ leaving $X_i$ all depart from the vertices of $L'_i$, hence we can choose a vertex $v_i \in X_i$ connected to the departure points by disjoint paths of weight $a$ each. Collecting all weights, we conclude that for each edge $f = (i_1, i_2) \in E(H)$, the path between $v_{i_1}$ and $v_{i_2}$ in $G$ has total weight:

$$2a + (q-1) \cdot 2a + q \cdot b = (2a + b)q \equiv 0 (\mathrm{mod}\ q)\,.$$

We have thus obtained the required subdivision of $H$, completing the proof. □

## 3 Cycles

In this section we prove Theorem 2. In the lemma below and later, a complete digraph is a digraph in which every pair of vertices is connected by one edge in each of the two directions.

**Lemma 3.1.** *Let $q \geq 2$ be an integer, and let $\Gamma = (V, E)$ be a complete digraph on $\lceil 2q \ln q \rceil$ vertices with weights $w(e) \in Z_q$ on its edges. Then $\Gamma$ contains a directed cycle $C$ of total weight divisible by $q$.*

*Proof.* The proof borrows its main idea from the argument in [1]. Here though we are in a more favorable situation dealing with the complete digraph and can thus allow ourselves to employ a simpler probabilistic tool for the proof – the union bound (instead of the Local Lemma used in [1]).

Let $c : V \to Z_q$ be a random labeling of $V$ by the elements of $Z_q$. For a vertex $u \in V$ denote by $A_u$ the event "$u$ has an outneighbor $v$ satisfying $c(v) = c(u) + w(uv)$". In order to estimate $Pr[\overline{A_u}]$ observe that by conditioning on the label $c(u)$, the probability that none of the outneighbors $v$ of $u$ satisfies $c(v) = c(u) + w(uv)$ is $(1 - 1/q)^{|V|-1} < 1/|V|$. Hence by the union bound there is a choice of $c$ for which all of the events $A_u$ hold. Fix such a choice, and for every vertex $u \in V$ choose an outgoing edge $(u, v)$ so that $c(v) = c(u) + w(uv)$. In the subgraph of $\Gamma$ obtained this way every outdegree is 1 and hence it has a directed cycle $C = (u_1, \ldots, u_\ell, u_1)$. Summing all weights along the edges of $C$ and denoting $u_{\ell+1} = u_1$ we obtain:

$$\sum_{i=1}^{\ell} w(u_i u_{i+1}) = \sum_{i=1}^{\ell} (c(u_{i+1}) - c(u_i)) \equiv 0 (\text{mod } q),$$

as required. □

If $q$ is a prime number then the logarithmic term in the above lemma can be omitted, thus giving an asymptotically optimal order of magnitude.

**Lemma 3.2.** *Let $q \geq 2$ be a prime, and let $\Gamma = (V, E)$ be a complete digraph on $2q - 1$ vertices with weights $w(e) \in Z_q$ on its edges. Then $\Gamma$ contains a directed cycle $C$ of total weight divisible by $q$.*

*Proof.* If there are two vertices $u, v$ so that $w(u, v) = -w(v, u)$ (modulo $q$) then there is a directed cycle consisting of two edges satisfying the requirement, hence we may and will assume that there is no such pair of vertices. In this case we proceed to prove that for every $k < q$ there are distinct vertices $x_0, x_1, y_1, x_2, y_2, \ldots x_k, y_k$ in $\Gamma$ and a set $S$ of $k + 1$ distinct residues modulo $q$ so that for every $s \in S$ there is a directed path $P_s$ from $x_0$ to $x_k$ of total weight $s$ modulo $q$. Each path $P_s$ consists of $k$ subpaths $p_1, p_2, \ldots, p_k$, in this order, where each $p_i$ is either the single edge $x_{i-1}x_i$ or the two-edge path $x_{i-1}y_i x_i$. This is proved by induction on $k$. For $k = 0$ the required path is the trivial path with no edges and $S = \{0\}$. Assuming the result holds for $k < q - 1$ we prove it for $k + 1$. Let $u, v$ be two vertices of $\Gamma$ that are not in the set $\{x_0, x_1, x_2, \ldots, x_k, y_k\}$. If $w(x_k, u) = w(x_k, v) + w(v, u)$ and $w(x_k, v) = w(x_k, u) + w(u, v)$ then $w(u, v) = -w(v, u)$ contradicting the assumption. Thus at least one of these equalities does not hold; by renaming $u$ and $v$ if needed we may assume that $w(x_k, u) \neq w(x_k, v) + w(v, u)$. Define $x_{k+1} = u$ and $y_{k+1} = v$. Let $T$ be the set of the two distinct residues $w(x_k, u) = w(x_k, x_{k+1})$ and $w(x_k, v) + w(v, u) = w(x_k, y_{k+1}) + w(y_{k+1}, x_{k+1})$. Clearly, for every residue $s \in S + T$, there is a path from $x_0$ to $x_{k+1}$ of total weight $s$ in $\Gamma$. By the Cauchy-Davenport theorem (for the easy special case in which one of the sets is of size 2), $|S + T| \geq |S| + |T| - 1 = k + 1$ establishing the induction step. Taking $k = q - 1$, the result shows that for every residue class $s$ modulo

$q$ there is a directed simple path from $x_0$ to $x_{q-1}$ of weight $s$. Choosing $s = -w(x_{q-1}, x_0)$ and adding the edge $x_{q-1}x_0$ gives the required cycle. $\square$

We can now prove Theorem 2. If $q$ is not a prime define $N = \lceil 2q \log q \rceil$ and $g = 2N$, if it is a prime define $N = 2q - 1$ and $g = 2N$. Given a $K_g$-minor on the $2N$ supernodes $X_i^+$ and $X_i^-$ for $1 \leq i \leq N$, assume without loss of generality that the induced subgraph on each supernode is a tree and that there is exactly one edge connecting each pair of supernodes. For each $i$ let $b_i$ be the weight of the unique edge, $x_i^- x_i^+$ connecting $X_i^-$ and $X_i^+$ (1 if there are no weights). For each $i \neq j$ let $w'(ij)$ be the total weight modulo $q$ of the unique path in the induced tree on the vertices $X_i^+ \cup X_j^-$ from the vertex $x_i^+$ to the vertex $x_j^-$. By the two lemmas above applied to the auxiliary complete directed graph on the vertices $\{1, 2, \ldots, N\}$ with the weights $w(ij) = b_i + w'(ij)$ there is a directed cycle of total weight 0 modulo $q$ in this auxiliary digraph. This gives the required cycle in the original graph. $\square$

# 4 Concluding remarks

- There are several ways to improve the bound for $f(H, q)$ in the proof of Theorem 1. In particular, one can use the constructions of Ramsey graphs for subdivisions given in [8, 5]. In addition, it is possible to take $k = \Delta(\Gamma)$, where $\Gamma$ is the Ramsey graph used, and $M = 2|E(\Gamma)| + f_1(q, k)$. Then we first choose $N$ supernodes $X_i$ with the same value of $a_i$ as in the present proof, fix a bijection between these $N$ nodes and the vertex set of $\Gamma$, and then for each $i = 1, ..., N$, look at the leaves of $X_i$ connected to previously unused supernodes $Y_j$ (altogether we use at most $2|E(\Gamma)|$ such supernodes $Y_j$, two per each edge of $\Gamma$ throughout the proof). There are at least $f_1(q, k)$ of them, this would be our set $L_i$. Next apply to it Lemma 2.1 to get a set of cardinality $k$ (in fact here the degree of vertex $i$ in $\Gamma$ suffices), and then put aside the supernodes $Y_j$ to which the edges from this $k$-set of leaves in $L_i$ lead. This gives some improvement, but as is frequently the case with Ramsey-type results, the bound obtained is still huge, surely far from being optimal. It may be interesting to try to determine or to estimate the asymptotic behavior of the best possible bound for $f(H, q)$.

- An $\alpha$-expander is a graph on $n$ vertices in which every set $X$ of at most $n/2$ vertices has at least $\alpha|X|$ neighbors outside $X$, see [9] for a general discussion. It is easy to see that any such graph has no sublinear separators and thus contains a $K_f$-minor for $f \geq c(\alpha)\sqrt{n}$ by [7]. Our results thus apply to such graphs and in particular imply that any such graph contains cycles of length divisible by $q$ for any $q \leq \tilde{O}(n^{1/2})$. This settles a question posed explicitly in [6]. See [6] for further results about cycle lengths in $\alpha$-expanders.

- There is a substantial amount of research on Ramsey-type problems for structures labelled by elements of an abelian group. Questions of this type are called zero-sum problems, see [4] for a survey of the subject (until the mid 90s). A typical problem in the subject is to determine or estimate the smallest number $f$ so that any complete graph with edges labelled by the elements of $Z_q$ contains a subgraph of a prescribed type in which the total weight of the edges is 0 modulo $q$. This problem for complete directed graphs, where the desired subgraph is a directed cycle, is addressed in Lemma 3.1 and Lemma 3.2. It seems plausible to believe that the first lemma is not tight and that the function $g(q)$ in Theorem 2 is linear in $q$ for any integer $q$.

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