Size and degree anti-Ramsey numbers

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Abstract

A copy of a graph H in an edge colored graph G is called **rainbow** if all edges of H have distinct colors. The **size anti-Ramsey number** of H, denoted by $AR_s(H)$, is the smallest number of edges in a graph G such that any of its proper edge-colorings contains a rainbow copy of H. We show that $AR_s(K_k) = \Theta(k^6/\log^2 k)$. This settles a problem of Axenovich, Knauer, Stumpp and Ueckerdt. The proof is probabilistic and suggests the investigation of a related notion, which we call the degree anti-Ramsey number of a graph.

Keywords: anti-Ramsey, proper edge coloring, rainbow subgraph.

1 The main result

A copy of a graph H in an edge colored graph G is called **rainbow** if all edges of H have distinct colors. Following [3] define the **size anti-Ramsey number** of a graph H, denoted by $AR_s(H)$, to be the smallest number of edges in a graph G such that any proper edge-coloring of G contains a rainbow copy of H.

This notion is related to the **anti-Ramsey number** of H, denoted $AR(H)$, which is the smallest n such that any proper edge coloring of the complete graph K_n on n vertices contains a rainbow copy of K_k. This was introduced by Erdős, Simonovits and Sós [5], and has been studied by several researchers. In particular, Babai [4] and Alon, Lefmann, and Rödl [1] determined the order of magnitude of $AR(K_k)$, showing that

$$AR(K_k) = \Theta(k^3/\log k). \tag{1}$$

This clearly implies that

$$AR_s(K_k) = O(k^6/\log^2 k). \tag{2}$$

In [3] Axenovich, Knauer, Stumpp and Ueckerdt proved that $AR_s(K_k) = \Omega(k^5/\log k)$ and raised the problem of closing the gap between the upper and lower bounds. This is done in the following theorem.

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Theorem 1.1. *There exists an absolute constant c so that $AR_s(K_k) \geq ck^6/\log^2 k$. Therefore, $AR_s(K_k) = \Theta(k^6/\log^2 k)$.*

To prove the lower bound one has to show that any graph G with fewer than $ck^6/\log^2 k$ edges admits a proper coloring with no rainbow copy of K_k. The proof proceeds, like the proof in [3], by splitting the graph into two induced parts, one on the vertices of high degree and the other on the vertices of low degree. The induced subgraph on the high degree vertices can be colored by applying (1). The induced subgraph on the low degree vertices is colored by a probabilistic argument whose analysis relies on several interesting ideas. This is the main novelty in the proof here, improving the argument in [3] in which this induced subgraph is colored deterministically using Vizing's Theorem. The details are presented in the next section. The final section contains a brief discussion of the notion of degree anti-Ramsey numbers, which is motivated by the main argument.

2 The proof

Throughout the proofs we make no attempt to optimize the absolute constants. To simplify the presentation we omit all floor and ceiling signs whenever these are not crucial. We also assume, whenever this is needed, that k is sufficiently large. The main part of the proof is the following result, showing that any graph in which the maximum degree is not too large has a proper edge coloring with no rainbow copy of K_k. Note that the total number of edges of the graph can be arbitrarily large.

Theorem 2.1. *There exists an absolute constant c_1 so that any graph G with maximum degree at most $c_1 k^3/\log k$ admits a proper edge coloring with no rainbow copy of K_k.*

This easily implies the assertion of Theorem 1.1, as shown next.

Proof of Theorem 1.1 assuming Theorem 2.1: In view of (2) it suffices to prove the lower bound. To do so, we need to show that the edges of any graph G with fewer than $ck^6/\log^2 k$ edges can be properly colored avoiding a rainbow K_k.

Consider such a graph G, let V_0 be the set of all vertices of G of degree at least $d = c_2 k^3/\log k$, and let $V_1 = V(G) - V_0$ be the set of all remaining vertices. Here $c_2 = c_1/8$ where c_1 is the constant from Theorem 2.1. Clearly,

$$|V_0| \leq \frac{2|E(G)|}{d} \leq \frac{2ck^3}{c_2 \log k},$$

and thus, for an appropriate choice of c, $2c/c_2$ is sufficiently small and it follows from (1) that the induced subgraph $G[V_0]$ of G on V_0 can be properly colored without creating a rainbow copy of $K_{k/2}$.

Put $G_1 = G[V_1]$. The maximum degree of G_1 satisfies $\Delta(G_1) \leq d \leq c_1(k/2)^3/\log(k/2)$. Therefore, by Theorem 2.1 there exists a proper edge coloring of the induced subgraph $G[V_1]$ of G on V_1 that avoids rainbow copies of $K_{k/2}$ as well. Coloring the edges between V_0 and V_1 arbitrarily, ensuring that the coloring is proper, gives a proper coloring of G without a rainbow K_k. □

We proceed with the proof of Theorem 2.1, which is the main technical part of this note.

Proof of Theorem 2.1: Let $G = (V, E)$ be a graph with maximum degree at most $d = c_1 k^3/\log k$, where c_1 is an absolute constant to be chosen later. Define a (random) edge-coloring of G in two steps as follows. First, let $C = [10d] = \{1, 2, \ldots, 10d\}$ be a palette of $10d$ colors and let $f : E \to C$ be a random coloring of the edges of G obtained by picking, for each edge e of G, randomly and independently, a uniformly chosen color in C. Call f the initial coloring of G. This coloring will be used for most of the argument.

Let $C' = \mathbb{N} - C$ be another (infinite) palette of colors. In the second step, any edge e for which $f(e) = f(e')$ for some edge e' incident with e is recolored with a new color from C' which is used only once in our coloring. This modified coloring is the final coloring of G.

The final coloring is clearly a proper edge coloring of G. Our objective is to show that with positive probability this coloring contains no rainbow copy of K_k.

Fix a copy K of K_k in G. We next show that the probability that this copy is rainbow in our final coloring is very small. This, together with the local lemma will suffice to show that with positive probability no copy of K_k is rainbow. To bound the probability that K is rainbow we prove the following lemma.

Lemma 2.2. *The probability that K is rainbow is smaller than*

$$e^{-10^{-10}k^4/d} = e^{-(10^{-10}/c_1)k\log k}.$$

Proof: Split the vertices of K into two disjoint sets V_1 and V_2, where $|V_1| = 0.9k$ and $|V_2| = 0.1k$. Let $K_1 = (V_1, E_1)$ be the clique on V_1 and let $K_2 = (V_2, E_2)$ be the clique on V_2. Expose, now, the value of $f(e)$ for all edges of G besides those in $E_1 \cup E_2$. Given these values define, for each edge $e \in E_1 \cup E_2$, the set $S(e)$ of all colors in C that differ from all values $f(e')$ for $e' \in E - (E_1 \cup E_2)$ that is incident with e. That is:

$$S(e) = \{c \in C : \text{ for all } \ e' \in E - (E_1 \cup E_2) \text{ satisfying } e \cap e' \neq \emptyset,\ f(e') \neq c\}.$$

We now expose the random values of $f(e_1)$ for each $e_1 \in E_1$. For each edge $e_2 \in E_2$, let $F(e_2) \subset S(e_2)$ denote the set of all colors c in $S(e_2)$ so that

(i) there is exactly one edge $e_1 \in E_1$ for which $f(e_1) = c$, and

(ii) $f(e_1) = c \in S(e_1)$.

Note that if $c \in F(e_2)$ and $e_1 \in E_1$ is the unique edge of E_1 satisfying (i) and (ii), then in the final coloring the color of e_1 stays c. Therefore, if in the final coloring the color of e_2 belongs to $F(e_2)$, then the copy of K is not rainbow.

Let $B(e_2)$ denote the event that $|F(e_2)| \leq k^2/10$.

Claim 2.3. *For each $e_2 \in E_2$, the probability of $B(e_2)$ is at most $e^{-\Theta(k^2)}$.*

Proof of claim: Given the fixed values of $f(e)$ for all $e \in E - (E_1 \cup E_2)$, let g denote the restriction of the random function f to the edges of E_1. For this random function $g : E_1 \to C = [10d]$, let $L(g)$ be the random variable given by $L(g) = |F(e_2)|$. We first claim that the expectation of $L(g)$ satisfies

$$E[L(g)] \geq |E_1| 0.6 \left(1 - \frac{1}{10d}\right)^{|E_1|-1} > 0.5\binom{0.9k}{2} > 0.2k^2. \tag{3}$$

Indeed, for each fixed edge $e_1 \in E_1$, the probability that $f(e_1) \in S(e_1) \cap S(e_2)$ is at least 0.6, as each $S(e_i)$ is of size at least $8d$ and hence the cardinality of $S(e_1) \cap S(e_2)$ is at least $6d$. Given the value of $f(e_1)$, the conditional probability that $f(e') \neq f(e_1)$ for each $e' \in E_1,\ e' \neq e_1$, is

$$\left(1 - \frac{1}{10d}\right)^{|E_1|-1} > 1 - \frac{|E_1|}{10d} > 0.9,$$

where here we used that $d = c_1 k^3/\log k$, $|E_1| = \binom{0.9k}{2}$ and k is sufficiently large. It thus follows that the probability that $f(e_1)$ contributes 1 to the expectation of $|F(e_2)|$ is bigger than $0.6 \cdot 0.9 > 0.5$, and hence, by linearity of expectation, (3) follows.

Put $m = |E_1| = \binom{0.9k}{2}$ and let $X_0, X_1, \ldots, X_m$ be the Doob martingale for the random variable $L(g)$ defined as follows. Fix an arbitrary ordering $h_1, h_2, \ldots, h_m$ of the edges of E_1 and define X_i to be the conditional expectation of $L(g)$ given the values of $g(h_1), g(h_2), \ldots, g(h_i)$. Therefore, X_0 is a constant, namely, the expectation of $L(g)$, whereas X_m is the variable $L(g)$ itself. It is not difficult to check that if two functions g_1 and g_2 differ on a single edge of E_1, then $|L(g_1) - L(g_2)| \leq 2$. Indeed, by changing the value $f(e)$ of a single edge e in E_1 from color c_1 to c_2, we may add c_2 to the set $F(e_2)$, and may also add c_1 to $F(e_2)$ (in case the color c_1 appeared twice among the colors of edges of E_1 before the change). Obviously we cannot add any other color to $F(e_2)$. It thus follows, by a simple consequence of Azuma's Inequality (see [2], Theorem 7.4.2 and the discussion preceding it), that the probability that $L(g)$ deviates from its expectation by at least s is at most $e^{-s^2/(8m)}$. In particular, the probability that $L(g) = |F(e_2)|$ is smaller than $k^2/10$ is at most

$$e^{-\Omega(k^4/(8m))} = e^{-\Omega(k^2)},$$

completing the proof of the claim. It is worth noting that instead of considering the Doob martingale as above, one can apply the bounded differences inequality of McDiarmid ([6], Lemma 1.2). □

Returning to the proof of the lemma, as explained in the beginning of its proof, we now expose all initial colors $f(e)$ for $e \in E_1$ (in addition to the already exposed initial colors of the edges $e \in E - (E_1 \cup E_2)$). We further assume that none of the events $B(e_2)$ occurs, that is, assume that $|F(e_2)| \geq k^2/10$ for all $e_2 \in E_2$. By Claim 2.3 this happens with probability $1 - e^{\Theta(k^2)}$. Conditioning on this, we now expose the random values of $f(e_2)$ for all $e_2 \in E_2$ one by one. Recall that each of them is a uniform random number in $C = [10d]$.

For each edge $e_2 \in E_2$ independently of all other edges, the probability that $f(e_2) \in F(e_2)$ is $\frac{|F(e_2)|}{10d} \geq \frac{k^2}{100d}$. Let B_1 denote the event that less than $s_1 = 0.5|E_2|\frac{k^2}{100d}$ edges $e_2 \in E_2$ satisfy $f(e_2) \in F(e_2)$. The probability that B_1 occurs is clearly at most the probability that a binomial random variable with parameters $|E_2|$ and $p = \frac{k^2}{100d}$ is at most half its expectation. It follows, by Chernoff's Inequality (see, e.g., [2], Theorem A.1.13), that

$$Prob[B_1] \leq e^{-|E_2|p/8} \leq e^{-k^4/(300000d)}.$$

While revealing the values of $f(e_2)$ for each $e_2 \in E_2$ one by one according to some fixed order, let B_2 denote the event that there are at least $2|E_2|\frac{k^2}{2000d}$ edges $e_2 \in E_2$ so that $f(e_2) = f(e_2')$ for some edge $e_2' \in E_2$ that appears before e_2 according to this order. Note that for each e_2, given any history of the f values of all earlier edges, the conditional probability that $f(e_2)$ equals one of them is smaller than $|E_2|/(10d) < \frac{k^2}{2000d} = p$. Therefore, the probability that the event B_2 occurs is at most the probability that a binomial random variable with parameters $|E_2|$ and $p = \frac{k^2}{2000d}$ gets a value which is at least twice its expectation. Applying, again, the known estimates for binomial distributions (see, e.g., [2], Theorem A.1.11) we conclude that

$$Prob[B_2] \leq e^{-|E_2|p/27} \leq e^{-k^4/(10^9 d)}.$$

Note, finally, that if both B_1 and B_2 fail, then there are (many) edges $e_2 \in E_2$ so that $f(e_2) \in F(e_2)$ and $f(e_2)$ is different than $f(e_2')$ for any other edge $e_2' \in E_2$. But in this case $f(e_2)$ is also the final color of e_2 and hence K is not rainbow. It thus follows that if none of the events B_1, B_2 and $B(e_2)$ for $e_2 \in E_2$, occurs then K is not rainbow, implying, by our estimates above, the assertion of the lemma. □

We can now complete the proof of the theorem using the Lovász Local Lemma (see, e.g., [2], Chapter 5). For each copy K of K_k in G, let A_K denote the event that K is rainbow in our final coloring. Construct a dependency graph for the events A_K, where A_K and $A_{K'}$ are adjacent if and only if the distance between K and K' in G is at most 1. Since the final coloring of the edges of a clique is determined completely by the colorings of the edges of the clique and the edges incident to it, it follows that indeed each event A_K is mutually independent of all events $A_{K'}$ besides those adjacent to it in the dependency graph. As the maximum degree of G is d, the maximum degree in the dependency graph is most $kd\binom{d}{k} < k^{4k}/4$, as for any fixed k-clique K in G there are at most kd vertices within distance 1 from K, and each such vertex lies in at most $\binom{d}{k}$ cliques. Therefore, by Lemma 2.2 and by the Local Lemma, if $10^{-10}/c_1 \geq 4$, that is, if we choose, say, $c_1 = 10^{-10}/4$, then with positive probability our random coloring contains no rainbow copy of K_k, completing the proof of the theorem. □

3 Degree anti-Ramsey numbers

Theorem 2.1 motivates the introduction of a new notion, the *degree anti-Ramsey number* $AR_d(H)$ of a graph H, as follows.

Definition 3.1. *For a graph H, the* **degree anti-Ramsey number** *$AR_d(H)$ of H is the minimum value d so that there is a graph G with maximum degree at most d such that any proper edge coloring of G contains a rainbow copy of H.*

It is clear that $AR_d(H) \leq AR(H) - 1$ for all H. It is also clear that $AR_d(H) \geq |E(H)| - 1$ for any graph H, since if the maximum degree of a graph G is at most $|E(H)| - 2$ it admits, by Vizing's Theorem [7], a proper edge coloring with at most $|E(H)| - 1$ colors. This coloring cannot contain a rainbow copy of H.

For any tree H this is nearly tight, namely, $AR_d(H) \leq |E(H)|$. Indeed, if $m = |E(H)|$ and G is a rooted tree with all internal vertices of degree m and height at least $|V(H)|$, then in any proper edge-coloring of G we can find a rainbow copy of H greedily. To see this, let $v_0, v_1, v_2, \ldots, v_m$ be an ordering of the vertices of H so that each v_i has a unique neighbor among the previous vertices $v_0, \ldots, v_{i-1}$. Define a bijective homomorphism of $V(H) = \{v_0, \ldots, v_m\}$ to the vertices of G as follows. First map v_0 to the root of G. Assuming $v_0, \ldots v_{i-1}$ have already been mapped so that their image spans a rainbow tree isomorphic to the induced subtree of H on $\{v_0, \ldots v_{i-1}\}$, map v_i maintaining this property. To do so, let v_j, $j < i$, be the unique neighbor of v_i among the previous vertices, and suppose v_j has been mapped to the vertex u_j of G. Then v_i will be mapped to a child w of u_j so that the color of the edge $u_j w$ is different from the colors of all other edges in the image of the partial tree we have so far. This is always possible, as there are m distinct colors of edges incident with the vertex u_j, and only $i - 1 < m$ of these have already been used. This shows that indeed for every tree H, $|E(H)| - 1 \leq AR_d(H) \leq |E(H)|$ and in fact the same bounds hold for any forest H, by a similar reasoning.

For any matching of $m > 2$ edges, it is easy to see that $AR_d(H) = m - 1$. This is shown by taking as the graph G any vertex disjoint union of m graphs, each being an $m - 1$ regular graph of class 2 (namely, of chromatic index m).

Our main technical result here (Theorem 2.1) shows that for the complete graph K_k, $AR_d(K_k) = \Theta(k^3/\log k)$.

It seems interesting to study the function $AR_d(H)$ for general graphs H.

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References

[1] N. Alon, H. Lefmann and V. Rödl, On an anti-Ramsey type result. In: Sets, graphs and numbers (Budapest, 1991), volume 60 of Colloq. Math. Soc. János Bolyai, pages 9–22, North-Holland, Amsterdam, 1992.

[2] N. Alon and J. H. Spencer, *The Probabilistic Method, Third Edition*, Wiley, 2008, xv+352 pp.

[3] M. Axenovich, K. Knauer, J. Stumpp and T. Ueckerdt, Online and size anti-Ramsey numbers, Journal of Combinatorics 5 (2014), 87–114.

[4] L. Babai, An anti-Ramsey theorem, Graphs Combin. 1(1985), 23–28.

[5] P. Erdős, M. Simonovits and V. T. Sós, Anti-Ramsey theorems. In: Infinite and finite sets (Colloq., Keszthely, 1973), Vol. II, pp. 633–643. Colloq. Math. Soc. Janos Bolyai, Vol. 10, North-Holland, Amsterdam, 1975.

[6] C. McDiarmid, On the method of bounded differences. In: Surveys in Combinatorics, 1989 (Norwich 1989), London Math. Soc. Lecture Note Ser., Vol. 141, Cambridgae Univ. Press, Cambridge, 1989, pp. 148–188.

[7] V. G. Vizing, On an estimate on the chromatic class of a p-graph (in Russian), Diskret. Analiz. 3 (1964), 25-30.