Subgraphs with a Large Cochromatic Number

Noga Alon * Michael Krivelevich † Benny Sudakov ‡

Abstract

The cochromatic number of a graph $G = (V, E)$ is the smallest number of parts in a partition of V in which each part is either an independent set or induces a complete subgraph. We show that if the chromatic number of G is n, then G contains a subgraph with cochromatic number at least $\Omega(\frac{n}{\ln n})$. This is tight, up to the constant factor, and settles a problem of Erdős and Gimbel.

1 Introduction

All graphs considered here are finite and simple. For a graph G, let $\chi(G)$ denote the chromatic number of G. The *cochromatic number* of $G = (V, E)$ is the smallest number of sets into which the vertex set V can be partitioned so that each set is either independent or induces a complete graph. We denote by $z(G)$ the cochromatic number of G.

The cochromatic number was originally introduced by L. Lesniak and H. Straight [6] and is related to coloring problems and to Ramsey theory. The subject has been studied by various researches (see [8] for several references). A natural question is to find a connection between the chromatic and the cochromatic numbers of a graph. A complete graph on n vertices shows that a graph G with a high chromatic number may have a low cochromatic number. Thus to get a nontrivial result one should consider subgraphs of G. P. Erdős and J. Gimbel [3] studied this question and proved that if $\chi(G) = n$, then G contains a subgraph whose cochromatic number is at least $\Omega(\sqrt{n/\ln n})$. They conjectured (see also [7] and [8], pp. 262–263) that the square root can be omitted. In this note we prove the following theorem which settles this conjecture.

Theorem 1.1 *Let G be a graph with chromatic number n, then G contains a subgraph with cochromatic number at least $(\frac{1}{4} + o(1))\frac{n}{\log_2 n}$.*

*Department of Mathematics, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, Israel. Email: noga@math.tau.ac.il. Research supported in part by a USA Israeli BSF grant and by the Fund for Basic Research administered by the Israel Academy of Sciences.

†Department of Mathematics, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, Israel. Email: krivelev@math.tau.ac.il. Research supported in part by a Charles Clore Fellowship.

‡Department of Mathematics, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, Israel. Email: sudakov@math.tau.ac.il.

Mathematics Subject Classification (1991): 05C15, 05C35.

Note that the result of the above theorem is best possible up to a constant factor, as shown by a clique on n vertices together with the simple result of [2], [4] that the cochromatic number of any graph on n vertices is at most $(2+o(1))\frac{n}{\log_2 n}$.

2 The Proof

In this section we prove the main result. This is done using a probabilistic argument. Throughout, we assume that n is sufficiently large. To simplify the presentation, we omit all floor and ceiling signs whenever these are not crucial. Let $G = (V, E)$ be a graph with chromatic number n. We can assume that G does not contain a clique of size n. Otherwise, by the known results about Ramsey numbers (see, e.g., [5], [1]), G contains an n-vertex subgraph with neither a clique nor an independent set of size at least $2\log_2 n$, whose cochromatic number is at least $\frac{n}{2\log_2 n}$, as needed.

As the first step we reduce the size of the problem. More precisely, we prove that it is enough to consider graphs with at most n^2 vertices. This can be done by the following lemma.

Lemma 2.1 *Let $G = (V, E)$ be a graph with chromatic number n. Then either $z(G) \geq n/\ln n$ or G contains a subgraph $G_1 = (V_1, E_1)$, such that $\chi(G_1) = (1 + o(1))n$ and $|V_1| \leq n^2$.*

Proof: Suppose that $z(G) < n/\ln n$. Let $V = \bigcup_{i=1}^{k} U_i \cup \bigcup_{j=1}^{l} W_j$ be a partition of the set of vertices of G into independent sets U_i and cliques W_j, such that $k + l < n/\ln n$. Define $V_1 = \cup_{j=1}^{l} W_j$. Since G has no clique of size n, $|V_1| \leq n^2/\ln n < n^2$. Let G_1 be the subgraph of G induced on the set V_1. Then any coloring of G_1 together with the sets U_i forms a coloring of G. Thus

$$n = \chi(G) \leq \chi(G_1) + k \leq \chi(G_1) + n/\ln n.$$

Therefore $\chi(G_1) \geq n - n/\ln n = (1 + o(1))n$. $\square$

Theorem 1.1 is now a straightforward consequence of the following lemma.

Lemma 2.2 *Let $G_1 = (V_1, E_1)$ be a graph on at most n^2 vertices with $\chi(G_1) = (1+o(1))n$. Let H be a subgraph of G_1, obtained by choosing each edge of G_1 randomly and independently with probability $1/2$. Then almost surely*

$$z(H) \geq (\frac{1}{4} + o(1))\frac{n}{\log_2 n}.$$

Proof: The probability that H contains a clique of size $4\log_2 n$ is clearly at most

$$\begin{aligned}\binom{n^2}{4\log_2 n}\left(\frac{1}{2}\right)^{\binom{4\log_2 n}{2}} &\leq \left(\frac{en^2}{4\log_2 n}\right)^{4\log_2 n}\left(\frac{1}{2}\right)^{8(\log_2 n)^2 - 2\log_2 n} \\ &\leq \left[\frac{n^2}{\log_2 n}\frac{\sqrt{2}}{n^2}\right]^{4\log_2 n} = o(1),\end{aligned}$$

where here we used the estimate $\binom{m}{k} \leq (\frac{em}{k})^k$ which is valid for all m and k.

The probability that there exists a subset $V_0 \subseteq V_1$ such that the induced subgraph $G_1[V_0]$ of G_1 on V_0 has minimum degree at least $4\log_2 n$ and V_0 becomes an independent set in H, is at most

$$\sum_{k=4\log_2 n}^{n^2} \binom{n^2}{k} \left(\frac{1}{2}\right)^{\frac{4k\log_2 n}{2}} \leq \sum_{k=4\log_2 n}^{n^2} \left[\frac{en^2}{k}\frac{1}{n^2}\right]^k$$
$$= \sum_{k=4\log_2 n}^{n^2} \left(\frac{e}{k}\right)^k \leq n^2 \left(\frac{1}{\log_2 n}\right)^{\log_2 n} = o(1).$$

This implies that almost surely (that is, with probability tending to 1 as n tends to infinity) any independent set V_0 in H induces a subgraph of G_1 with chromatic number at most $4\log_2 n$. Indeed, if $\chi(G_1[V_0]) > 4\log_2 n$, then $G_1[V_0]$ contains a color-critical subgraph $G_2 = (V_2, E_2)$ with $\chi(G_2) = 4\log_2 n + 1$ which must have minimum degree at least $4\log_2 n$. Since V_2 is independent in H this almost surely does not happen, by the above argument.

Now, let $V_1 = \bigcup_{i=1}^k U_i \cup \bigcup_{j=1}^l W_j$ be a partition of the vertex set of H into independent sets U_i and cliques W_j, satisfying $k + l = z(H)$. Then almost surely

$$\begin{aligned}(1+o(1))n = \chi(G_1) &\leq \sum_{i=1}^k \chi(G_1[U_i]) + \sum_{j=1}^l |W_j| \leq k \cdot 4\log_2 n + l \cdot 4\log_2 n \\ &= (k+l)4\log_2 n = z(H)4\log_2 n,\end{aligned}$$

implying $z(H) \geq \frac{1+o(1)}{4}\frac{n}{\log_2 n}$. $\square$

References

[1] N. Alon and J. Spencer, *The Probabilistic Method*, Wiley, New York, 1992.

[2] Y. Caro, *On the covering number of combinatorial structures*, Ars Combinatoria 29B (1990), 111-124.

[3] P. Erdős and J. Gimbel, *Some problems and results in cochromatic theory*, in: "Quo Vadis, Graph Theory?", Annals of Discrete Mathematics 55 (1993), 261–264.

[4] P. Erdős, J. Gimbel and D. Kratsch, *Some extremal results in cochromatic and dichromatic theory*, J. Graph Theory 15 (1992), 579–585.

[5] R. L. Graham, B. L. Rothschild and J. H. Spencer, *Ramsey Theory*, Wiley, New York, 1980.

[6] L. Lesniak and H. J. Straight, *The cochromatic number of a graph*, Ars Combinatoria 3 (1977), 39–46.

[7] J. Gimbel, *Problem*, Discrete Mathematics 101 (1992), 353–354.

[8] T. Jensen and B. Toft, *Graph coloring problems*, Wiley, New York, 1995.