

Perfect matchings in ϵ -regular graphs

Noga Alon

*School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540
and Department of Mathematics,
Raymond and Beverly Sackler Faculty of Exact Sciences,
Tel Aviv University, Tel Aviv, Israel; Email: noga@math.tau.ac.il.

Vojtech Rödl

†Department of Mathematics and Computer Science,
Emory University, Atlanta, USA; Email: rodl@mathcs.emory.edu.

Andrzej Ruciński

‡Department of Discrete Mathematics,
Faculty of Mathematics and Computer Science,
Adam Mickiewicz University, Poznań, Poland;
Email: rucinski@math.amu.edu.pl.

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Abstract

A super (d, ϵ) -regular graph on $2n$ vertices is a bipartite graph on the classes of vertices V_1 and V_2 , where $|V_1| = |V_2| = n$, in which the minimum degree and the maximum degree are between $(d - \epsilon)n$ and $(d + \epsilon)n$, and for every $U \subset V_1, W \subset V_2$ with $|U| \geq \epsilon n, |W| \geq \epsilon n$, $\left| \frac{e(U, W)}{|U||W|} - \frac{e(V_1, V_2)}{|V_1||V_2|} \right| < \epsilon$. We prove that for every $1 > d > 2\epsilon > 0$ and $n > n_0(\epsilon)$, the number of perfect matchings in any such graph is at least $(d - 2\epsilon)^n n!$ and at most $(d + 2\epsilon)^n n!$. The proof relies on the validity of two well known conjectures for permanents; the Minc conjecture, proved by Brégman, and the van der Waerden conjecture, proved by Falikman and Egorichev.

An ϵ -regular graph on $2n$ vertices is a bipartite graph on the classes of vertices V_1 and V_2 , where $|V_1| = |V_2| = n$, in which for every $U \subset V_1, W \subset V_2$ with $|U| \geq \epsilon n, |W| \geq \epsilon n$,

$$\left| \frac{e(U, W)}{|U||W|} - \frac{e(V_1, V_2)}{|V_1||V_2|} \right| < \epsilon, \quad (1)$$

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 where here $e(X, Y)$ denotes the number of edges between X and Y . The quantity $\frac{e(V_1, V_2)}{|V_1||V_2|}$ is called the *density* of the graph.

Such a graph is a *super (d, ϵ) -regular graph* if, in addition, its minimum degree δ and its maximum degree Δ satisfy

$$(d - \epsilon)n \leq \delta \leq \Delta \leq (d + \epsilon)n.$$

In this note we prove the following result

Theorem 1 *Let G be a super (d, ϵ) -regular graph on $2n$ vertices, where $d > 2\epsilon$ and $n > n_0(\epsilon)$. Then the number $M(G)$ of perfect matchings of G satisfies*

$$(d - 2\epsilon)^n n! \leq M(G) \leq (d + 2\epsilon)^n n!.$$

Thus, the number of perfect matchings in any super (d, ϵ) -regular graph on $2n$ vertices is close to the expected number of such matchings in a random bipartite graph with edge probability d (which is clearly $d^n n!$). This result is combined with some additional ideas in [7] to derive a new proof of the Blow-Up Lemma of Komlós, Sárközy and Szemerédi.

The upper bound in Theorem 1 is true for all bipartite graphs with maximum degree at most $(d + \epsilon)n$ on at least one side, and is an easy consequence of the Minc conjecture [6] for permanents, proved by Brégman [2] (c.f. also [1] for a probabilistic description of a proof of Schrijver). Indeed, the Minc conjecture states that the permanent of an n by n matrix A with $(0, 1)$ entries satisfies

$$\text{per}(A) \leq \prod_{i=1}^n r_i!^{1/r_i},$$

where r_i is the sum of the entries of the i -th row of A . To derive the upper bound in Theorem 1 apply this estimate to the matrix $A = (a_{u,v})_{u \in V_1, v \in V_2}$ in which $a_{u,v} = 1$ if u, v are adjacent and $a_{u,v} = 0$ otherwise. Here $M(G) = \text{per}(A)$. Since the function $x!^{1/x}$ is increasing, $M(G) \leq (k!)^{n/k}$, where $k = \lfloor (d + \epsilon)n \rfloor$, and the upper bound follows by applying the Stirling approximation formula for factorials.

It is worth noting that since every ϵ -regular graph with density d and $2n$ vertices contains, in each color class, at most ϵn vertices of degree higher than $(d + \epsilon)n$, some version of the above upper bound is also true for any ϵ -regular graph of density d . Namely, one can show that for every $d > 0$, if ϵ is sufficiently small as a function of d , then for every ϵ -regular graph G on $2n$ vertices with density d we have

$$M(G) < (d + 3\epsilon)^n n!$$

provided $n > n_0(\epsilon)$.

To prove the lower bound observe that by the van der Waerden conjecture, proved by Falikman [4] and Egorichev [3], the number of perfect matchings in a bipartite k -regular graph with n vertices in each color class is at least $(k/n)^n n!$. Thus it suffices to show that our graph contains a spanning k -regular subgraph (a k -factor), where $k = \lceil (d - 2\epsilon)n \rceil$. This is proved in the next lemma.

Lemma 2 *Let G be a super (d, ϵ) -regular graph on $2n$ vertices, $d > 2\epsilon$. Then G contains a spanning k -factor, where $k = \lceil (d - 2\epsilon)n \rceil$.*

In the proof of this lemma we will apply the following criterion for containing a k -factor, which can be found e.g. in [5], page 70, Thm. 2.4.2.

Theorem 3 *Let G be a bipartite graph on $2n$ vertices in the classes V_1 and V_2 , where $|V_1| = |V_2| = n$. Then G has a k -factor if and only if for all $X \subseteq V_1$ and $Y \subseteq V_2$*

$$k|X| + k|Y| + e(V_1 - X, V_2 - Y) \geq kn \quad \square \quad (2)$$

Proof of Lemma 2. We first assume, to simplify the notation and avoid using floor and ceiling signs when these are not crucial, that $(d - 2\epsilon)n$ is an integer.

By Theorem 3, all we need is to prove inequality (2). If $|X| + |Y| \geq n$ then the left-hand side of (2) is at least nk , and we are done. Assume, thus, that $|X| + |Y| < n$. Without loss of generality we may and will assume that $|V_1 - X| \geq |V_2 - Y|$. If $|V_2 - Y| < \epsilon n$, then, since $|X| + |Y| < n$, it follows that $|X| < |V_2 - Y| < \epsilon n$ and thus every vertex of $V_2 - Y$ has at least $\delta - |X| > (d - 2\epsilon)n = k$ neighbors in $V_1 - X$, implying that $e(V_1 - X, V_2 - Y) \geq (n - |Y|)k$, and showing that the left-hand side of (2) is at least $k|X| + k|Y| + k(n - |Y|) \geq kn$, as needed. Otherwise, $|V_1 - X| \geq |V_2 - Y| \geq \epsilon n$, and thus, by the ϵ -regularity assumption and the obvious fact that $e(V_1, V_2)/(|V_1||V_2|) \geq d - \epsilon$, it follows that $e(V_1 - X, V_2 - Y) > (d - 2\epsilon)(n - |X|)(n - |Y|)$. Therefore, the left-hand side of (2) is at least

$$\begin{aligned} k|X| + k|Y| + e(V_1 - X, V_2 - Y) &\geq (d - 2\epsilon)(n|X| + n|Y| + (n - |X|)(n - |Y|)) \\ &= (d - 2\epsilon)(n^2 + |X||Y|) \geq (d - 2\epsilon)n^2 = kn. \end{aligned}$$

This completes the proof. \square

Remark: Note that in the last proof the assumption (1) may be relaxed, as we only used the fact that for every $U \subset V_1, W \subset V_2$, of cardinality at least ϵn each, $\frac{e(W,U)}{|W||U|} \geq \frac{e(V_1,V_2)}{|V_1||V_2|} - \epsilon$. For the lower bound in Theorem 1 the assumption about the maximum degree of G as well as the assumption that n is sufficiently large as a function of ϵ can also be omitted.

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