Asymptotically tight bounds for some multicolored Ramsey numbers

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Abstract

Let $H_1, H_2, \ldots, H_{k+1}$ be a sequence of $k+1$ finite, undirected, simple graphs. The (multicolored) Ramsey number $r(H_1, H_2, \ldots, H_{k+1})$ is the minimum integer r such that in every edge-coloring of the complete graph on r vertices by $k+1$ colors, there is a monochromatic copy of H_i in color i for some $1 \leq i \leq k+1$. We describe a general technique that supplies tight lower bounds for several numbers $r(H_1, H_2, \ldots, H_{k+1})$ when $k \geq 2$, and the last graph H_{k+1} is the complete graph K_m on m vertices. This technique enables us to determine the asymptotic behaviour of these numbers, up to a polylogarithmic factor, in various cases. In particular we show that $r(K_3, K_3, K_m) = \Theta(m^3 \text{poly} \log m)$, thus solving (in a strong form) a conjecture of Erdős and Sós raised in 1979. Another special case of our result implies that $r(C_4, C_4, K_m) = \Theta(m^2 \text{poly} \log m)$ and that $r(C_4, C_4, C_4, K_m) = \Theta(m^2 / \log^2 m)$. The proofs combine combinatorial and probabilistic arguments with spectral techniques and certain estimates of character sums.

1 Introduction

All graphs considered here are finite, undirected and simple, unless otherwise specified. Let $H_1, H_2, \ldots, H_{k+1}$ be a sequence of $k+1$ graphs. The multicolored Ramsey number $r(H_1, H_2, \ldots, H_{k+1})$ is the minimum integer r such that in every edge-coloring of the complete graph on r vertices by $k+1$ colors, there is a monochromatic copy of H_i in color i for some $1 \leq i \leq k+1$.

The determination or estimation of these numbers is usually a very difficult problem. When all graphs H_i are complete graphs with more than two vertices, the only values that are known precisely are those of $r(K_3, K_m)$ for $m \leq 9$, $r(K_4, K_4)$, $r(K_4, K_5)$ and $r(K_3, K_3, K_3)$. Even the determination of the asymptotic behaviour of Ramsey numbers up to a constant factor is a hard problem, and despite a lot of efforts by various researchers (see, e.g., [16], [10] and their references), there are only a few infinite families of graphs for which this behaviour is known. A particularly

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interesting example is the result of Kim [17] together with that of Ajtai, Komlós and Szemerédi [2] that show that $r(K_3, K_m) = \Theta(m^2/\log m)$.

The situation is even worse for multicolored Ramsey numbers, that is, for the case of at least 3 colors. Even the asymptotic behaviour of $r(K_3, K_3, K_m)$ has been very poorly understood, and Erdős and Sós raised the following conjecture in [15] (see also [23], [10], p. 23).

Conjecture 1.1 (Erdős and Sós, [15])

$$\lim_{m \mapsto \infty} \frac{r(K_3, K_3, K_m)}{r(K_3, K_m)} = \infty.$$

Here we describe a general technique that supplies tight lower bounds for several numbers $r(H_1, H_2, \ldots, H_{k+1})$ when $k \geq 2$, and the last graph H_{k+1} is the complete graph K_m on m vertices. In particular we show that $r(K_3, K_3, K_m) = \Theta(m^3 \text{poly} \log m)$, thus solving, in a strong form, the above mentioned conjecture. The technique can be used to deal with more than 3 colors as well. For two graphs H, K and for an integer k, let $r_k(H; K)$ denote the Ramsey number $r(H_1, H_2, \ldots, H_k, K)$, where $H_i = H$ for all $i \leq k$. Our method shows that for every fixed integer $k \geq 1$,

$$r_k(K_3; K_m) = \Theta(m^{k+1} \text{poly} \log m). \tag{1}$$

The method is particularly effective for determining the asymptotic behaviour of the numbers $r_k(H; K_m)$, when H is bipartite and $k \geq 2$ (and even more effectively, when $k \geq 3$.) Surprisingly, we can often get tight estimates for these numbers even in cases where the asymptotic behaviour of $r_1(H; K_m) = r(H, K_m)$ is far from being understood.

In particular, it is not known if $r(C_4, K_m) = O(m^{2-\epsilon})$ for some absolute constant $\epsilon > 0$, and Erdős conjectured in [12], (see also [10]. p. 19), that this is the case. Using our technique here we show that $r(C_4, C_4, K_m) = \Theta(m^2 \text{poly} \log m)$ and that for every fixed $k \geq 3$

$$r_k(C_4; K_m) = \Theta(m^2/\log^2 m), \tag{2}$$

thus determining these numbers up to a constant factor for every fixed number of colors exceeding 3.

More generally, we get similar estimates for other complete bipartite graphs H. We show that for every fixed t and for every fixed $s \geq (t-1)! + 1$, $r(K_{t,s}, K_{t,s}, K_m) = \Theta(m^t \text{poly} \log m)$, and for every $k \geq 2$,

$$r_k(K_{t,s}; K_m) = \Theta(m^t/\log^t m). \tag{3}$$

Similar tight results are obtained when H is a cycle of length 6 or 10. The proofs combine combinatorial and probabilistic arguments with spectral techniques and certain estimates of character sums.

Our notation is rather standard. As usual, for two functions $f(n)$ and $g(n)$ we write that $f(n) = O(g(n))$ if there exists a positive constant c so that $f(n) \leq cg(n)$ for all sufficiently large n, and write that $f(n) = \Omega(g(n))$ if $g(n) = O(f(n))$. We write $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $g(n) = O(f(n))$. We write $f(n) = \tilde{O}(g(n))$ if there is an absolute constant c such that

$f(n) \leq g(n)(\log n)^c$ for all sufficiently large n. Similarly, $f(n) = \tilde{\Omega}(g(n))$ if there is a constant b such that $f(n) \geq g(n)(\log n)^b$ for all sufficiently large n. Therefore, $f(n) = \tilde{\Omega}(g(n))$ if and only if $g(n) = \tilde{O}(f(n))$. Finally, $f(n) = \tilde{\Theta}(g(n))$ if $f(n) = \tilde{O}(g(n))$ and $f(n) = \tilde{\Omega}(g(n))$, that is, f and g are equal up to polylogarithmic factors.

Throughout the paper we assume, whenever this is needed, that n is sufficiently large. We make no attempt to optimize the various absolute constants in our estimates. To simplify the presentation, we omit all floor and ceiling signs whenever these are not crucial. All logarithms are in the natural base e.

The rest of the paper is organized as follows. In Section 2 we bound the maximum possible number of independent sets of a given size in regular graphs with small nontrivial eigenvalues. In Section 3 we describe our basic technique for obtaining lower bounds for multicolored Ramsey numbers; the bounds are obtained by considering random shifts of appropriate pseudo random graphs, or of blow-ups of such graphs. We proceed with the proofs of the specific results mentioned above. The proof of (1) is described in subsection 3.1, and that of (2) in subsection 3.2. In subsection 3.3 we present the proof of (3), and in subsection 3.4 we consider additional even cycles. The final section, Section 4, contains some concluding remarks.

2 The number of independent sets in graphs with small nontrivial eigenvalues

An $(n, d, \lambda)-$*graph* is a d-regular graph $G = (V, E)$ on n vertices, such that the absolute value of every eigenvalue of the adjacency matrix of G, besides the largest one, is at most λ. It is well known that if λ is much smaller than d, then any (n, d, λ)-graph has some strong pseudo-random properties; see, e.g., [7], Chapter 9.2. Here we prove a new property of such graphs: they do not contain many large independent sets.

We will be interested in graphs on n vertices in which d behaves like n^α for some fixed α between 0 and 1. Some of our graphs will have loops (at most one loop per vertex), with a loop contributing 1 to the degree of the corresponding vertex. We call a set of vertices independent if it contains no edges besides, possibly, loops (that is, if the corresponding set in the simple graph obtained from our graph by omitting all loops is independent.)

It is easy to see that the number of independent sets of size, say, $m = \frac{n}{2(d+1)}$, in any d-regular graph on n vertices, (without any assumption on its eigenvalues) is at least

$$\frac{n(n-d-1)(n-2d-2)\dots(n-(m-1)(d+1))}{m!} > (\frac{n}{2m})^m = (d+1)^m.$$

Somewhat surprisingly, if the graph is an (n, d, λ)-graph and we consider slightly larger independent sets, for example sets of size $m = \frac{n \log^2 n}{d}$, then their number cannot exceed

$$(\frac{\lambda}{\log^{2-o(1)} n})^m.$$

This is proved in the following theorem.

Theorem 2.1 *Let $G = (V, E)$ be an (n, d, λ)-graph. Then for any $m \geq \frac{2n \log n}{d}$, the number of independent sets of size m in G is at most*

$$[\frac{emd^2}{4\lambda n \log n}]^{\frac{2n \log n}{d}} [\frac{e2\lambda n}{md}]^m. \quad (4)$$

In particular, for any $\epsilon > 0$ and any $n > n_0(\epsilon)$, if $m = \frac{n}{d} \log^2 n$, then the number of independent sets of size m is at most

$$(\frac{\lambda}{(\log n)^{2-\epsilon}})^m. \quad (5)$$

To prove the theorem, we need the following simple lemma. Some versions of this lemma appear in various places, see, e.g., [7], Chapter 9.

Lemma 2.2 *Let $G = (V, E)$ be an (n, d, λ)-graph, and let $B \subset V$ be a subset of bn vertices of G. Define*

$$C = \{u \in V : \ |N(u) \cap B| \leq \frac{db}{2} \},$$

where here and in what follows $N(u)$ denotes the set of all neighbors of u (including u itself, if there is a loop at u). Then

$$|B||C| < \frac{4\lambda^2}{d^2} n^2.$$

In particular, if $|B| \geq \frac{2\lambda}{d} n$ then $|C| < \frac{2\lambda}{d} n$, and consequently for every $B \subset V$, $|B \cap C| < \frac{2\lambda}{d} n$.

Proof . Let A denote the adjacency matrix of G, and define a vector $x = (x_v : v \in V)$ by $x_v = -b$ if $v \notin B$ and $x_v = 1 - b$ if $v \in B$. As the sum of coordinates of x is zero, it is orthogonal to the all 1 vector which is the eigenvector of the largest eigenvalue of A, and hence

$$||Ax||_2^2 = x^t A^t A x \leq \lambda^2 x^t x.$$

However, $x^t x = (n - |B|)b^2 + |B|(1-b)^2 = b(1-b)n$, and

$$||Ax||_2^2 = \sum_{v \in V} (|N(v) \cap B|(1-b) - (d - |(N(v) \cap B|)b)^2 = \sum_{v \in V} (|N(v) \cap B| - db)^2.$$

Therefore

$$\sum_{v \in V} (|N(v) \cap B| - db)^2 \leq \lambda^2 x^t x.$$

Each $v \in C$ contributes to the left hand side more than $d^2 b^2 / 4$, and hence

$$|C| d^2 b^2 / 4 < \lambda^2 b(1-b)n < \lambda^2 bn,$$

implying that $\frac{d^2 |B||C|}{4} < \lambda^2 n^2$, as needed. □

Proof of Theorem 2.1. Consider the number of ways to choose an ordered set $v_1, v_2, \ldots, v_m$ of m vertices of G which form an independent set. Starting with $B_0 = V$, let B_i denote the set of all vertices that are not adjacent to any vertex among the first i chosen vertices $v_1, \ldots, v_i$. Obviously, all the vertices v_j for $j > i$ have to lie in B_i. Define, also,

$$C_i = \{u \in V : |N(u) \cap B_i| \le \frac{d|B_i|}{2n}\}.$$

Note that if the next chosen vertex, v_{i+1}, is not a member of C_i, then $|B_{i+1}| < (1 - \frac{d}{2n})|B_i|$, and hence throughout the process there cannot be more than $\frac{2n}{d} \log n$ choices like that, since otherwise the corresponding set of non-neighbors will be empty before the process terminates, not allowing us to choose the next vertex.

It follows that with at most $s = \frac{2n}{d} \log n$ possible exceptions, each vertex v_{i+1} has to lie in $B_i \cap C_i$. However, by Lemma 2.2, this intersection is always of size at most $2\lambda n/d$. Therefore, the total number of choices for the ordered set $v_1, v_2, \ldots, v_m$ is at most

$$\binom{m}{s} n^s (\frac{2\lambda}{d} n)^{m-s}.$$

Indeed, there are $\binom{m}{s}$ possibilities to choose a set of s indices covering all indices i in which the vertex v_i has not been chosen in $C_i \cap B_i$. Then, there are at most n ways to choose each such vertex v_i, and at most $\frac{2\lambda}{d} n$ ways to choose each vertex v_j for each other index j.

Dividing by $m!$ in order to get an upper bound for the number of unordered independent sets of size m, and plugging in the value of $s = \frac{2n}{d} \log n$, we conclude that this number is at most

$$\frac{1}{m!} (\frac{em}{s})^s n^s (\frac{2\lambda}{d} n)^{m-s} \le [\frac{emn}{s 2\lambda n/d}]^s \, [\frac{e 2\lambda n}{md}]^m$$

$$= [\frac{emd^2}{4\lambda n \log n}]^{\frac{2n \log n}{d}} \, [\frac{e 2\lambda n}{md}]^m,$$

as claimed.

In particular, if $m = \frac{n}{d} \log^2 n$ then the last quantity is equal to

$$[\frac{ed \log n}{4\lambda}]^{\frac{2n \log n}{d}} \, [\frac{e 2\lambda}{\log^2 n}]^{\frac{n \log^2 n}{d}} \le (\frac{\lambda}{(\log n)^{2-\epsilon}})^{\frac{n \log^2 n}{d}},$$

provided $n > n_0(\epsilon)$. □

3 Tight bounds for Ramsey numbers

Recall that for a positive integer k and for two graphs H and K, $r_k(H; K)$ denotes the Ramsey number $r(H_1, \ldots, H_k, K)$, where $H_i = H$ for all $1 \le i \le k$.

We need the following simple lemma.

Lemma 3.1 *Let G be a graph on n vertices, and let M denote the number of independent sets of size m in G. If, for a positive integer $k \ge 2$, $M^k < (\binom{n}{m})^{k-1}$, then there is a collection $G_1, G_2 \ldots, G_k$ of k graphs on the same set V of n vertices, where each G_i is isomorphic to G, and where the graph whose edges are all pairs of vertices of V that do not lie in any G_i contains no clique of size m.*

Therefore, if G contains no copy of H for some fixed graph H, then the Ramsey number $r_k(H; K_m)$ satisfies $r_k(H; K_m) > n$.

Proof For each i, $1 \le i \le k$, let G_i be a random copy of G on V, that is, a graph obtained from G by mapping its vertices to those of V according to a random one to one mapping. The probability that a fixed set of m vertices of V will be an independent set in each G_i is precisely

$$(\frac{M}{\binom{n}{m}})^k,$$

implying, by our assumption that $M^k < (\binom{n}{m})^{k-1}$, that with positive probability there is no such independent set. This gives the existence of the graphs G_i as required.

By coloring each edge of the complete graph on V by the minimum i such that it belongs to G_i, if there is such an i, and by $k+1$ otherwise, we conclude that if G contains no copy of H then $r_k(H; K_m) > n$. □

3.1 Triangles

The *r-blow-up* G' of a graph G is the graph obtained by replacing each vertex v of G by an independent set S_v of size r, and each edge uv of G by the set of all edges xy with $x \in S_u$, $y \in S_v$. It is easy to see that the adjacency matrix of G' is the tensor product of the adjacency matrix of G with an all-one r by r matrix, and hence all the nonzero eigenvalues of G' are simply those of G multiplied by r. It follows that if G is an (n, d, λ)-graph, then G' is an $(nr, dr, \lambda r)$-graph.

The following theorem determines the asymptotic behaviour of $r_k(K_3; K_m)$ for every fixed k, up to poly-logarithmic factors.

Theorem 3.2 *For every fixed $k \ge 1$, the Ramsey number $r_k(K_3; K_m)$ satisfies $r_k(K_3; K_m) = \tilde{\Theta}(m^{k+1})$.*

Proof For $k = 1$, $r_k(K_3; K_m) = r(K_3, K_m) = \Theta(m^2/\log m)$ as proved by Ajtai, Komlós and Szemerédi [2] and by Kim [17]. We next prove, by induction on k, that for every fixed $k \ge 1$,

$$r_k(K_3; K_m) \le c_k \frac{m^{k+1}(\log\log m)^{k-1}}{(\log m)^k}.$$

This holds for $k = 1$, by the above mentioned result. Assuming the result holds for $k-1$, we prove it for k. Given an edge-coloring of K_N by $k+1$ colors with no monochromatic triangle in any of the first k colors, and no monochromatic K_m in the last color, consider the graph T consisting of all edges of the first k colors. We claim that the maximum degree of T is at most $D = k(r_{k-1}(K_3; K_m) - 1) < kr_{k-1}(K_3; K_m)$. Indeed, otherwise there is a vertex v incident with at least $r_{k-1}(K_3; K_m)$ edges of color i for some $i \le k$. The induced subgraph of T on the set of all vertices connected to v by edges of color i cannot contain edges of color i, and thus must contain either a monochromatic triangle of color j for some $j \le k$, $j \ne i$, or an independent set of size m, leading, in each of these cases, to a contradiction. Therefore, the maximum degree of T is at most D. Let s be the Ramsey number $r(H_1, H_2, \ldots, H_k)$, with $H_i = K_3$ for all i. It is known that $s \le O(k!)$ but here we only need the fact that it is a finite function of k. Obviously T contains no copy of K_s. By a result of Shearer [22] this implies that T contains an independent set of size at least $\Omega(\frac{N \log D}{D \log \log D})$. As this set must be of size smaller than m we conclude that for some $c = c(k) > 0$,

$$c \frac{N \log [kr_{k-1}(K_3; K_m)]}{kr_{k-1}(K_3; K_m) \log \log [kr_{k-1}(K_3; K_m)]} < m,$$

which, together with the induction hypothesis, implies the desired upper bound.

To get the lower bound, we apply Theorem 2.1 and Lemma 3.1 to appropriate blow-ups of an explicit family of graphs constructed in [3]. In that paper it is shown that for every $n = 2^{3f}$, with f not divisible by 3, there is a triangle-free (n, d, λ)-graph with $d = 2^{f-1}(2^{f-1} - 1) = (\frac{1}{4} + o(1))n^{2/3}$ and $\lambda = 9 \cdot 2^f + 3 \cdot 2^{f/2} + 1/4 = (9 + o(1))n^{1/3}$. Let G be an r blow-up of such a graph, where $r = n^{k/3-2/3}(\log n)^{2-\delta}$ for some $\delta > 0$. Then G is triangle-free, and is an (N, D, Λ)- graph with $N = nr$, $D = dr$, $\Lambda = (9 + o(1))n^{1/3}r$. By Theorem 2.1 it follows that for $m = \frac{N \log^2 N}{D} = c(k)n^{1/3}(\log n)^2$, the number M of independent sets of size m in G satisfies

$$M \leq [\frac{n^{1/3}r}{\log^{2-\epsilon}(nr)}]^m,$$

provided n is sufficiently large as a function of ϵ. If ϵ is sufficiently small as a function of δ, then it is not difficult to check that

$$M^k < [\binom{N}{m}]^{k-1},$$

implying, by Lemma 3.1, that

$$r_k(K_3; K_m) > N.$$

Since $m = c(k)n^{1/3}(\log n)^2$ and $N = nr = n^{(k+1)/3}(\log n)^{2-\delta}$ we conclude that for all $\delta > 0$ and all sufficiently large m,

$$r_k(K_3; K_m) \geq \Omega(\frac{m^{k+1}}{(\log m)^{2k+\delta}}).$$

This completes the proof. □

3.2 Bipartite graphs and 4-cycles

Our technique is particularly effective for bounding $r_k(H; K_m)$ when H is a bipartite graph. In this case we can sometimes determine the asymptotic behaviour of $r_k(H; K_m)$ up to a constant factor for every fixed $k > 2$. In this subsection we illustrate this fact by considering the Ramsey numbers $r_k(C_4; K_m)$. We start, however, with a simple upper bound for the numbers $r_k(H; K_m)$ when H is a fixed bipartite graph. Recall that the *Turán number* $ex(n, H)$ of a graph H is the maximum possible number of edges of a simple graph on n vertices which contains no copy of H. It is well known that these numbers are sub-quadratic for every fixed bipartite H (see [20]).

Lemma 3.3 *Let H be a fixed bipartite graph, and suppose that the Turán number of H satisfies $ex(n, H) \leq O(n^{2-1/t})$, where $t > 1$ is a real. Then, for every fixed k there exists a constant $c = c(k, H)$ such that*

$$r_k(H; K_m) \leq c \frac{m^t}{(\log m)^t}.$$

Proof Put $n = r_k(H; K_m) - 1$. Given an edge-coloring of K_n by $k+1$ colors with no monochromatic copy of H in each of the first k colors, and no monochromatic K_m in the last color, let T be the graph whose edges are all edges of K_n colored by one of the first k colors. The total number of edges of T is clearly at most $k \cdot ex(n, H) \le b(k, H)n^{2-1/t}$. Moreover, the neighborhood of any vertex of degree d in T contains at most $k \cdot ex(d, H) \le b(k, H)d^{2-1/t}$ edges of T. It thus follows from the results in [1] that if D is the average degree of T then it contains an independent set of size at least $\Omega(n \log D/D) \ge \Omega(n^{1/t} \log n)$. (In fact, as shown in [5], the chromatic number of T is at most $O(D/\log D)$). Since the independence number of T is smaller than m it follows that $\Omega(n^{1/t} \log n) < m$, implying the desired result. □

The Erdős-Rényi graph G, constructed in [14], is the polarity graph of a finite projective plane of order p. This graph is an (n, d, λ)-graph, where $n = p^2+p+1$, $d = p+1$ and $\lambda = \sqrt{p}$, and it exists for every prime power p. It has $p+1$ vertices incident with loops. By Theorem 2.1, the number of independent sets of size $m = \frac{n}{d} \log^2 n$ in this graph is at most $(\frac{\lambda}{\log^{2-\epsilon} n})^m$, and this, together with Lemma 3.1 and a simple computation implies that

$$r(C_4, C_4, K_m) > n = \Theta(m^2/\log^4 m).$$

Note that by Lemma 3.3 above this implies that $r(C_4, C_4, K_m) = \tilde{\Theta}(m^2)$.

For more colors our method suffices to determine the asymptotic behaviour of $r_k(C_4; K_m)$ up to a constant factor. Indeed, for any fixed $c > 6$ and all sufficiently large $n = p^2 + p + 1$, a simple computation, using Theorem 2.1 and Lemma 3.1, implies that

$$r_3(C_4; K_{c\sqrt{n}\log n}) > n.$$

This, together with the fact that $r_k(C_4; K_m) \ge r_3(C_4, K_m)$ for all $k \ge 3$, and together with Lemma 3.3 implies the second part of the following theorem, whose first part has been established in the previous paragraph.

Theorem 3.4 *The Ramsey numbers $r_k(C_4; K_m)$ satisfy the following:*
(i) $r_2(C_4; K_m) = \tilde{\Theta}(m^2)$.
(ii) For every fixed $k \ge 3$ there are two positive constants c_1, c_2 such that

$$c_1 \frac{m^2}{\log^2 m} \le r_k(C_4; K_m) \le c_2 \frac{m^2}{\log^2 m}.$$

□

The results above are surprising in view of the fact that the asymptotic behaviour of the Ramsey number $r_1(C_4; K_m) = r(C_4, K_m)$ is much less understood. In [12] Erdős conjectured that this number is at most $O(n^{2-\epsilon})$ for some fixed $\epsilon > 0$, but the best known bounds are only (see [24], [13]):

$$\Omega(\frac{m^{3/2}}{\log^{3/2} m}) \le r(C_4, K_m) \le O(\frac{m^2}{\log^2 m}).$$

Theorem 3.4 shows that the situation becomes clearer as the number of colors increases. In the next two subsections we show several additional examples exhibiting this phenomenon.

3.3 Complete bipartite graphs

The *projective norm graphs* $G(p,t)$ have been constructed in [6], modifying an earlier construction given in [19]. The construction is the following. Let $t > 2$ be an integer, let p be a prime, let $GF(p)^*$ denote the multiplicative group of the finite field with p elements, and let $GF(p^{t-1})$ denote the field with p^{t-1} elements. The set of vertices of the graph $G = G(p,t)$ is the set $V = GF(p^{t-1}) \times GF(p)^*$. Two distinct (X,a) and $(Y,b) \in V$ are adjacent if and only if $N(X+Y) = ab$, where the norm N is understood over $GF(p)$, that is, $N(X) = X^{1+p+\cdots+p^{t-2}}$. Note that $|V| = p^t - p^{t-1}$. If (X,a) and (Y,b) are adjacent, then (X,a) and $Y \neq -X$ determine b. Thus G is regular of degree $p^{t-1}-1$.

These graphs can be defined in the same manner starting with a prime power q instead of the prime p, but for our purpose here the prime case suffices. The main property of the graphs $G(p,t)$, proved in [6] by applying some tools from algebraic geometry developed in [19], is the following.

Lemma 3.5 ([6]) *The graph $G(p,t)$ contains no subgraph isomorphic to $K_{t,(t-1)!+1}$.*

We need to bound the eigenvalues of $G(p,t)$. It turns out that we can, in fact, compute these eigenvalues precisely. These have been computed independently by T. Szabó [25].

Lemma 3.6 *Let $G = G(p,t)$ be as above. Then every eigenvalue of $G(p,t)$, besides the trivial one, is either $p^{(t-1)/2}$ or $-p^{(t-1)/2}$ or 0 or 1 or -1. Therefore, G is an (n,d,λ)-graph with $n = p^t - p^{t-1}$, $d = p^{t-1}-1$ and $\lambda = p^{(t-1)/2}$.*

Proof Put $q = p^{t-1}$ and let A be the adjacency matrix of $G = G(p,t)$. The rows and columns of this matrix are indexed by the ordered pairs of the set $GF(q) \times GF(p)^*$. Let ψ be a character of the additive group of $GF(q)$, and let χ be a character of the multiplicative group of $GF(p)$. Consider the vector $v : GF(q) \times GF(p)^* \mapsto C$ defined by $v(X,a) = \psi(X)\chi(a)$. For each non-zero element $c \in GF(p)^*$, define $S_c = \{Z \in GF(q) : N(Z) = c\}$. Since the norm of each nonzero member of $GF(q)$ lies in $GF(p)^*$, the sets S_c form a partition of all nonzero elements of $GF(q)$. We now compute the vector Av:

$$[Av](X,a) = \sum_{b \in GF(p)^*} \sum_{Y : N(X+Y)=ab} \psi(Y)\chi(b) = \sum_{b \in GF(p)^*} \chi(b) \sum_{Y \in S_{ab}-X} \psi(Y)$$

$$= \sum_{b \in GF(p)^*} \chi(b) \sum_{Z \in S_{ab}} \psi(Z)\overline{\psi(X)} = \sum_{b \in GF(p)^*} \sum_{Z \in S_{ab}} \chi(ab)\psi(Z)\overline{\psi(X)\chi(a)}$$

$$= \sum_{b \in GF(p)^*} \sum_{Z \in S_{ab}} \chi(N(Z))\psi(Z)\overline{\psi(X)\chi(a)} = [\sum_{Z \in GF(q), Z \neq 0} \psi(Z)\chi(N(Z))]\overline{\psi(X)\chi(a)}$$

$$= [\sum_{Z \in GF(q), Z \neq 0} \psi(Z)\chi(N(Z))]\overline{v(X,a)}.$$

Since $\overline{v(X,a)}$ is also a product of an additive character by a multiplicative one, another application of A shows that

$$A^2 v = |\sum_{Z \in GF(q), Z \neq 0} \psi(Z)\chi(N(Z))|^2 v.$$

Since the vectors $\psi(X)\chi(a)$, as ψ ranges over all additive characters of the large field, and χ ranges over all multiplicative characters of the small field, are pairwise orthogonal, we conclude that all the eigenvalues of the matrix A^2 are given by the expressions

$$|\sum_{Z \in GF(q), Z \neq 0} \psi(Z)\chi(N(Z))|^2.$$

Set $\chi'(Z) = \chi(N(Z))$ for all nonzero Z in $GF(q)$. Note that as the norm is multiplicative, χ' is a multiplicative character of the large field, and hence all the last expressions are squares of absolute values of Gauss sums. It is well known (c.f., e.g., [11], page 66), that the value of each such square, besides the trivial ones (that is, when either ψ or χ' are principal), is q. For the sake of completeness, we include a short proof of this fact. Put

$$S = |\sum_{Z \in GF(q), Z \neq 0} \psi(Z)\chi'(Z)|^2,$$

where ψ is a non-principal additive character and χ' is a non-principal multiplicative character. Then

$$S = q - 1 + \sum_{Z_1 \neq 0} \sum_{Z_2 \neq 0, Z_1} \psi(Z_1)\overline{\psi(Z_2)}\chi'(Z_1)\overline{\chi'(Z_2)}$$

$$= q - 1 + \sum_{Z_1 \neq 0} \sum_{Z_2 \neq 0, Z_1} \psi(Z_1 - Z_2)\chi'(Z_1/Z_2) = q - 1 + \sum_{Y \neq 0} \psi(Y) \sum_{Z_2 \neq 0, -Y} \chi'(\frac{Z_2 + Y}{Z_2})$$

$$= q - 1 + \sum_{Y \neq 0} \psi(Y) \sum_{Z_2 \neq 0, -Y} \chi'(1 + \frac{Y}{Z_2}).$$

When Z_2 ranges over all field elements besides $0, -Y$, the quantity $1 + \frac{Y}{Z_2}$ ranges over all nonzero field elements besides 1, and as the sum of $\chi'(X)$ over all elements X of the multiplicative group of the field is 0 and $\chi'(1) = 1$ it follows that $\sum_{Z_2 \neq 0, -Y} \chi'(1 + \frac{Y}{Z_2}) = -1$. Therefore, the above sum is equal to

$$q - 1 - \sum_{Y \neq 0} \psi(Y) = q - 1 - (-1) = q,$$

where here we used the fact that $\sum_Y \psi(Y) = 0$ and that $\psi(0) = 1$.

This gives the values in the nontrivial cases. If χ' is principal and ψ is not, then the sum $\sum_{Z \neq 0} \psi(Z) = -1$ and hence its square is 1. If ψ is principal and χ' is not, then $\sum_{Z \neq 0} \chi'(Z) = 0$. This completes the proof. □

Combining the last two lemmas with Theorem 2.1, Lemma 3.1 and Lemma 3.3 together with the known fact proved in [20] that for every fixed $s \geq t \geq 2$, $ex(n, K_{t,s}) = O(n^{2-1/t})$, we get the following theorem. We omit the detailed computation, which is analogous to that described in the previous subsection.

Theorem 3.7 *The Ramsey number $r_k(K_{t,s}; K_m)$ satisfy the following:*
(i) For every fixed $t > 1$ and every fixed $s \geq (t-1)! + 1$, $r_2(K_{t,s}, K_m) = \tilde{\Theta}(m^t)$.

(ii) For every fixed $k \geq 3$, $t > 1$ and $s \geq (t-1)!+1$ there are two positive constants c_1, c_2 such that

$$c_1 \frac{m^t}{\log^t m} \leq r_k(K_{t,s}; K_m) \leq c_2 \frac{m^t}{\log^t m}.$$

□

3.4 Additional even cycles

For every q which is an odd power of 2, the incidence graph of the generalized 4-gon has a polarity. The corresponding polarity graph is a $q+1$-regular graph with q^3+q^2+q+1 vertices. See [9], [21] for more details. This graph contains no cycle of length 6 and it is not difficult to compute its eigenvalues (they can be derived, for example, from the the eigenvalues of the corresponding incidence graph, given in [26]). Indeed, all the eigenvalues, besides the trivial one (which is $q+1$) are either 0 or $\sqrt{2q}$ or $-\sqrt{2q}$. Combining this with the known fact that $ex(n, C_6) = O(n^{4/3})$ (c.f., e.g., [8]) we conclude from Theorem 2.1, Lemma 3.1 and Lemma 3.3 that the following theorem holds. We omit the detailed computation.

Theorem 3.8 *The Ramsey numbers $r_k(C_6; K_m)$ satisfy the following:*
(i) $r_2(C_6; K_m) = \tilde{\Theta}(m^{3/2})$.
(ii) For every fixed $k \geq 3$ there are two positive constants c_1, c_2 such that

$$c_1 \frac{m^{3/2}}{\log^{3/2} m} \leq r_k(C_6; K_m) \leq c_2 \frac{m^{3/2}}{\log^{3/2} m}.$$

□

For every q which is an odd power of 3, the incidence graph of the generalized 6-gon has a polarity. The corresponding polarity graph is a $q+1$-regular graph with $q^5+q^4+\cdots+q+1$ vertices. See [9], [21] for more details. This graph contains no cycle of length 10 and its eigenvalues can be easily derived, for example, from the the eigenvalues of the corresponding incidence graph, given in [26]. All the eigenvalues, besides the trivial one are either $\sqrt{3q}$ or $-\sqrt{3q}$ or $\sqrt{q}$ or $-\sqrt{q}$. Combining this with the known fact that $ex(n, C_{10}) = O(n^{6/5})$ (c.f., e.g., [8]) we conclude from Theorem 2.1, Lemma 3.1 and Lemma 3.3 that the following theorem holds. Here, too, we omit the detailed computation.

Theorem 3.9 *The Ramsey numbers $r_k(C_{10}; K_m)$ satisfy the following:*
(i) $r_2(C_{10}; K_m) = \tilde{\Theta}(m^{5/4})$.
(ii) For every fixed $k \geq 3$ there are two positive constants c_1, c_2 such that

$$c_1 \frac{m^{5/4}}{\log^{5/4} m} \leq r_k(C_{10}; K_m) \leq c_2 \frac{m^{5/4}}{\log^{5/4} m}.$$

□

4 Concluding remarks

- In [4] the authors describe, for every fixed $t \geq 2$, infinite families of (n, d, λ)-graphs that contain no copy of K_{t+2}, where $d = (1 + o(1))n^{1-1/t}$ and $\lambda = (1 + o(1))d^{1/2}$. By taking the r-blow ups of these graphs, with $r = n^{(1-1/t)(k/2-1)}$, we can follow the arguments described in subsection 3.1 and conclude that for every fixed $k \geq 2$ and for every fixed $t \geq 2$
$$r_k(K_{t+2}; K_m) \geq \tilde{\Omega}(m^{k(t-1)/2+1}). \qquad (6)$$

- Our lower bound for the Ramsey numbers $r_k(K_3; K_m)$ or $r_k(K_{t+2}; K_m)$ are obtained by taking random shifts of blow-ups of appropriate Ramsey type graphs with well behaved eigenvalues. Kim and Mubayi [18] noticed that in these cases the proof can be simplified, and the spectral approach is not needed. We can simply take random shifts of blow ups of Ramsey graphs. Indeed, if we know that $r(K_{t+2}, K_f) \geq n$, then the r-blow up of the appropriate graph contains at most
$$\frac{\binom{n}{f}(fr)^m}{m!}$$
independent sets of size m. This is because there are at most $\binom{n}{f}$ ways to choose a set of f blown vertices containing our independent set, and then each vertex of the independent set is one of the fr vertices in these blocks. Starting, for example, with Kim's lower bound $r(K_3, K_f) \geq \Omega(f^2/\log f)$, this enables us to prove, using our random shifts approach and Lemma 3.1, that indeed $r_k(K_3; K_m) \geq \tilde{\Omega}(m^{k+1})$ for all $k \geq 1$, as proved in Theorem 3.2. In fact, the logarithmic factor here is somewhat better than what follows from the spectral technique. This simplification does not work, of course, for bounding $r_k(H; K_m)$ for bipartite graphs H, and the spectral approach seems essential in these cases. On the other hand, since it is known that $r(K_{t+2}, K_m) \geq \tilde{\Omega}(m^{(t+3)/2})$ we can take appropriate shifts of blow-ups and conclude that
$$r_k(K_{t+2}; K_m) \geq \tilde{\Omega}(m^{k(t+1)/2+1})$$
improving the estimate in (6).

- Our technique can be used to provide lower bounds for additional multicolored Ramsey numbers. If G_i is a graph that contains no homomorphic image of H_i, then no blow-up of G_i will contain H_i, and hence our techniques will enable us to obtain lower bounds for $r(H_1, H_2, \ldots, H_k, K_m)$. By taking appropriate random graphs we can get this way lower bounds for various Ramsey numbers. A specific example are the numbers $r_k(C_{2t+1}; K_m)$. Here, using the method of [24] it is not difficult to show that there is a graph on $n = c(f/\log f)^{1+1/(2t-1)}$ vertices with girth $\geq 2t + 2$ and no independent set of size f. Using the technique described above with $k \geq 2$, $r = c'(n/f)^{k-1} \log f$, and $m = f \log f$ yields $r_k(C_{2t+1}; K_m) \geq \Omega(m^{1+k/(2t-1)}/(\log m)^{k+2k/(2t-1)})$.

- The method can obviously be used to provide bounds for multicolored Ramsey numbers $r(H_1, H_2, \ldots H_k, K_m)$, even when not all the graphs H_i are necessarily isomorphic. Thus, for example, we can use the graphs constructed in subsection 3.1 and subsection 3.3 to conclude that

$$r(K_3, K_{3,3}, K_m) \geq \tilde{\Omega}(m^3).$$

Acknowledgment: Part of this research was done during an Oberwolfach workshop on Combinatorics in January, 2002. We would like to thank the organizers of the workshop, L. Lovász and H. J. Prömel, as well as the participants, for helpful discussions.

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