Decreasing the Diameter of Bounded Degree Graphs *

Noga Alon †   András Gyárfás ‡   Miklós Ruszinkó §

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To the memory of Paul Erdős

Abstract

Let $f_d(G)$ denote the minimum number of edges that have to be added to a graph G to transform it into a graph of diameter at most d. We prove that for any graph G with maximum degree D and $n > n_0(D)$ vertices, $f_2(G) = n - D - 1$ and $f_3(G) \geq n - O(D^3)$. For $d \geq 4$, $f_d(G)$ depends strongly on the actual structure of G, not only on the maximum degree of G. We prove that the maximum of $f_d(G)$ over all connected graphs on n vertices is $n/\lfloor d/2 \rfloor - O(1)$. As a byproduct, we show that for the n-cycle C_n, $f_d(C_n) = n/(2\lfloor d/2 \rfloor - 1) - O(1)$ for every d and n, improving earlier estimates of Chung and Garey in certain ranges.

1 Preliminaries and results

Extremal problems concerning the diameter of graphs have been initiated by Erdős, Rényi and Sós in [4] and [5]. Problems concerning the change of diameter if edges are added or deleted have been initiated by Chung and Garey in [2], followed by a survey of Chung [1] which contains further references, e.g. the paper by Schoone, Bodlaender and Leeuwen [6]. A related problem, decreasing the diameter of a triangle-free graph by adding a small number of edges while preserving the triangle-free property, has

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†Department of Mathematics, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv, Israel. Email: noga@math.tau.ac.il. Research Supported in part by a USA Israeli BSF grant and by the Hermann Minkowski Minerva Center for Geometry at Tel Aviv University.

‡Computer and Automation Research Institute of the Hungarian Academy of Sciences, Budapest P.O.Box 63, Hungary-1518. Email: gyarfas@luna.aszi.sztaki.hu.

§Computer and Automation Research Institute of the Hungarian Academy of Sciences, Budapest, P.O.Box 63, Hungary-1518. Email: ruszinko@lutra.sztaki.hu. Research supported in part by OTKA Grants T 016414 and F 014919.

been considered by Erdős, Gyárfás and Ruszinkó [3]. In this paper we continue the direction initiated in [2] and investigate the minimum number of edges one has to add to a graph G to transform G into a graph of diameter at most d. Let $f_d(G)$ denote this minimum.

In general, by [6], it is NP-complete to determine the minimum number of edges to be added to a graph to make it of diameter d. On the other hand, in some cases it is trivial to find $f_d(G)$. For example, $f_d(G) = n-1$ for every $d \geq 2$ if G has n vertices and no edges, $f_1(G)$ is equal to the number of edges in the complement of G since only the complete graph has diameter one. The function $f_2(G)$ is already interesting. If G has n vertices and maximum degree $D(G)$ then $f_2(G) \leq n - D(G) - 1$, because G can be trivially extended into a graph of diameter at most two by adding all missing edges from an arbitrary vertex of degree D. We shall prove that this bound is tight for fixed D and large n (Theorem 2.1). For the case $d = 3$ we show (Theorem 2.3) that for any n-vertex graph G with maximum degree D, $f_3(G) \geq n - O(D^3)$ (and there are examples for which $f_3(G) \leq n - O(D^2)$). If these results are applied to the cycle C_n we get that $f_2(C_n) = n - 3$ for sufficiently large n and $n - 100 \leq f_3(C_n) \leq n - 6$ (Corollaries 2.2 and 2.4).

For general d we prove that $f_d(G) \leq n/\lfloor d/2 \rfloor$ for any connected graph G on n vertices (Theorem 3.1) and we also show that this is tight for every n and $d \geq 2$, up to a constant additive term (Theorems 3.2, 3.4). This is proved by considering the value of $f_d(G)$ where G is a path of length $n/\lfloor d/2 \rfloor$ with pending paths of length $\lfloor d/2 \rfloor - 1$ at each of its vertices.

These results (and their proof techniques) can be applied to get good lower bounds of $f_d(G)$ for other graphs G as well. We demonstrate this with the case when $G = C_n$, the cycle with n vertices: for an arbitrary positive integer h, $\lfloor n/(2h-1) \rfloor - 7 \leq f_{2h}(C_n) \leq \lfloor n/(2h-1) \rfloor$ and $\lfloor n/(2h-1) \rfloor - 155 \leq f_{2h+1}(C_n) \leq \lfloor n/(2h-1) \rfloor$ (Corollary 3.5). Thus $f_d(C_n)$ is determined up to an additive constant error term.

It is worth comparing our estimates to the ones of Chung and Garey [2]. They proved that for t even, the minimum diameter $C(n,t)$ which can achieved by adding t edges to C_n satisfies $n/(t+2) - 1 \leq C(n,t) \leq n/(t+2) + 3$. It is easy to check that for $t \geq c\sqrt{n}$ our lower bound is stronger, otherwise it is weaker. The importance of the magnitude of t is even more visible from a conjecture stated in [1]: One can decrease the diameter of a path P_n to $(n+t-1)/(t+1)$ by adding t edges (for t even). If t may depend on n, this conjecture says that it is enough to add $(n-4)/2$ edges to P_n to get a graph of diameter three. However, by Corollary 2.4, $n - 100$ edges

2 Diameter two and three

Theorem 2.1 *Let G be a graph of order n with maximum degree D. Then at least $n-D-1$ edges are needed to extend G into a graph of diameter at most 2, provided n is sufficiently large (as a function of D).*

Observe, that this result is tight, since – as already mentioned – adding all missing edges to a vertex of degree D we obtain a graph of diameter two.

Proof. Assume that G is extended by H. If H has at most $D+1$ tree components then $e(H) \geq n-D-1$ and the proof is finished. Otherwise, select $D+2$ tree components of H, C_i, and a vertex x_i of degree at most one (in H) from each C_i, $i = 1, 2, \dots, D+2$. Notice that from each x_i at most $D^2 + D$ additional components of H are reachable by paths of length at most 2 in $G \cup H$, because at most $D(D-1)+D$ components are reachable by such a path of G and at most D components are reachable by a 2 path which starts with an edge of H followed by an edge of G. Since $G \cup H$ is of diameter 2, this means that H has at most $D^2 + D + 1$ components.

Call a component of H *small* if it has no more than $h = 2D^3 + 5D^2 + 2D$ vertices, otherwise it is called *large*. We claim that a large component C of H has at least $|C| + D^2$ edges.

To see this, fix a large component C and select a point x_i of degree at most one in each of the other $D+1$ tree components of H. (Without loss of generality, we may assume that $i = 1, 2, \dots D+1$.) Let $A_i \subseteq C$ denote the vertices which are reachable from x_i by a 2-path of $G \cup H$ whose second edge is from G or by a 1-path of G. By an argument similar to the one above it follows that $|A_i| \leq D^2 + D$, so $|A| \leq (D+1)(D^2 + D)$, where $A = A_1 \cup A_2 \cup \dots A_{D+1}$. Each vertex $y \in (C \setminus A)$ is at distance at most two from x_i, $i = 1, 2, \dots, D+1$. By the definition of A, the shortest path from y to x_i in $G \cup H$ must be of length two and it must start with an edge yz_i of H with some $z_i \in A$. Since the degree of z_i in G is at most D, there are at least two distinct z_i-s. This implies that y is adjacent to at least two vertices of A in H. Thus C has at least $2(|C| - |A|)$ edges (in H) which, using that C is large, gives

$$2(|C| - |A|) \geq 2|C| - 2(D+1)(D^2 + D) \geq |C| + D^2$$

proving the claim.

Set $n_0 = (D^2 + D + 1)(2D^3 + 5D^2 + 2D - 1) + 1$. Then, if G has at least n_0 vertices, H has a large component C. Assume H has t other components. Since those t components span at least $n - |C| - t$ edges, we have

$$e(H) \geq n - |C| - t + e(C) \geq n - (D^2 + D + 1) + D^2 = n - D - 1,$$

completing the proof. □

Corollary 2.2 *For $n > n_0$, at least $n-3$ edges must be added to C_n to get a graph of diameter two.*

Obviously, Corollary 2.2 is tight, and perhaps the best possible n_0 can be determined with some additional effort. We have the following example, showing that the best possible value of n_0 is at least 11. Consider the Petersen graph with vertex set $\{1, \dots 10\}$ and with edges $(i, i+5)$ $(i \in \{1, \dots 5\})$, $(i, i+1 \ (mod\ 5))$ $(i \in \{1, \dots 5\})$, $(i, i+2)$ $(mod\ 5)$ $(i, i+2 \in \{6, \dots 10\})$. Join a new vertex 11 to vertices 1, 8 and 9. The resulting graph is a Hamiltonian graph of diameter two with eleven vertices and eighteen edges, showing that for $n = 11$, C_n can be extended by less than $n-3$ edges to a diameter two graph.

The following theorem shows that extending into diameter three does not require significantly less edges than extending into diameter two.

Theorem 2.3 *Suppose that G is a graph of order n with maximum degree D $(\ \geq 2)$. Then at least $n - 3(D+1)^3 - 2(D+1)^2 - 1$ edges are needed to extend G into a graph of diameter three.*

Proof. Assume that G is extended by H so that the diameter of $G \cup H$ is at most 3. Consider the components of H and denote by t the number of components which are trees. We shall fix $n - t$ edges of H by selecting all edges of the tree components and selecting a unicyclic spanning subgraph in each other component. We shall refer to these edges as the *fixed* edges. Observe that the average degree of any subgraph formed by fixed edges is at most 2. Select a vertex x_i of degree at most one in each tree component $(i = 1, 2, \dots, t)$. The edge of H incident with x_i is called the *root edge.* Since $G \cup H$ is of diameter at most three, there is at least one path of length at most three with endpoints x_i x_j for all pairs $1 \leq i < j \leq t$. Call a path *essential* , if it is of length three and its middle edge is an edge of H.

We claim that there are at least $\binom{t}{2} - ct$ essential paths connecting distinct pairs x_i, x_j, where $c = (D+1)^3/2$ depends on D only.

To see this observe, that for a fixed vertex x_i

(o) there is one x_j which can be reached from x_i by a (nonessential) path of length zero (x_i itself);

(i) there are at most D x_j-s which can be reached from x_i by (nonessential) paths of length one, since to get to a new component only the edges of G can be used (and it is of maximum degree D);

(ii) there are at most $D^2 + D$ x_j-s which can be reached from x_i by (nonessential) paths of length two, since x_i has at most $D+1$ neighbors in $G \cup H$ and from each of them at most D x_j-s can be reached;

(iii) there are at most $D(D^2 + D + 1)$ x_j-s which can be reached from x_i by nonessential paths of length three, since from x_i there are less than $D^2 + D + 1$ paths of length two having the second edge from G and from each of them at most D x_j-s can be reached. Therefore, from a fixed vertex x_i at most as many x_j-s can be reached by nonessential paths of length at most three as the sum of the above estimations, which is less than $(D+1)^3$ for $D \geq 2$. From this the claim with $c = (D+1)^3/2$ follows.

On the other hand, any edge of H can be the middle edge of at most $(D+1)^2$ essential paths, since to both endpoints of such an edge at most D x_j-s can be adjacent in G and at most one x_j can be adjacent in H. Moreover, the middle edges of essential paths are spanned by a set X of at most $t(D+1)$ vertices, since those ones have to be adjacent to some x_i. Using that the fixed edges form a graph of average degree at most two on any subset of vertices, at most $t(D+1)$ edges are fixed edges among the middle edges of essential paths. These considerations give the

$$\frac{\binom{t}{2} - \frac{t}{2}(D+1)^3}{(D+1)^2} - t(D+1) \tag{1}$$

lower bound for the non-fixed edges of H. Set $t_0 = 3(D+1)^3 + 2(D+1)^2 + 1$. For $0 \le t \le t_0$ H has at least $n - t \ge n - t_0 = n - 3(D+1)^3 - 2(D+1)^2 - 1$ (fixed) edges and for $t \ge t_0$ the number of fixed edges and the estimate (1) show that

$$\begin{aligned} |E(H)| &\ge n - t + \frac{\binom{t}{2} - \frac{t}{2}(D+1)^3}{(D+1)^2} - t(D+1) \\ &= n + t\left(\frac{(t-1)}{2(D+1)^2}\right) - t\left(\frac{3(D+1)}{2} + 1\right) \\ &\ge n + t\left(\frac{3(D+1)}{2} + 1\right) - t\left(\frac{3(D+1)}{2} + 1\right) \\ &= n \ge n - 3(D+1)^3 - 2(D+1)^2 - 1, \end{aligned}$$

from which the desired result follows. □

Unlike in Theorem 2.1, the bound in Theorem 2.3 is probably not tight and can be improved to $n - O(D^2)$. One can not expect better than that since if we take a graph G of maximum degree D which contains a diameter 2 subgraph G^* with approximately D^2 vertices (such G^* does exist, see [5]), then adding all missing edges from a vertex of G^* to $V(G) \setminus V(G^*)$ we get an extension which is of diameter at most three.

Corollary 2.4 *At least $n - 100$ edges must be added to C_n to get a graph of diameter three.*

We suspect that in fact for all $n > n_0$ at least $n - 6$ edges have to be added to C_n to get a graph of diameter three. If true, this is best possible as shown by the following example. Add the edge $(4, n-2)$ and the edges $(1, i)$ for $5 \le i \le n-3$. A similar solution is to replace $(4, n-2)$ by $(3, n-1)$.

3 Larger Diameter

We start this section with an upper bound on $f_d(G)$ for arbitrary d provided G is connected.

Theorem 3.1 *For any connected graph G of order n, $f_d(G) \le n/\lfloor d/2 \rfloor$.*

Proof. It is enough to prove the theorem for even d, say $d = 2h$ and one can also assume (by monotonicity) that G is a tree. Select a longest path $P = x_1 x_2 \ldots$ of G, we may assume that its length is at least h, otherwise G has diameter at most $h-1$. Remove the edge $x_h x_{h+1}$ from G. Each vertex of the subtree T_1 containing $x_h = y_1$ is at distance at most $h - 1 = d/2 - 1$ from x_h since P was maximal. The procedure is iterated on the subtree of G containing x_{h+1}. Clearly, this partitions G into subtrees T_i, each but the last one with at least h vertices. Moreover, each subtree T_i has a vertex y_i at distance at most $h-1$ from all vertices of T_i. This shows that there are at most $t = \lceil n/h \rceil$ subtrees. The required extension of G is obtained by adding the edges $y_t y_i$ for all $1 \le i < t$. □

We next show that for every $d \ge 2$ and n, the following tree, which we denote by $T(n,d)$, provides an example where Theorem 3.1 is tight up to a constant additive term. The tree $T(n,d)$ is defined as follows. Put $h = \lfloor d/2 \rfloor$ and take a path of $\lceil n/h \rceil$ vertices. This will be called the *horizontal path*, and its vertices $\overline{x}$ are called the *top* vertices. From each top vertex $\overline{x}$ of the horizontal path grow a path P_x with h vertices (including the top one). These paths are called the *vertical paths* and their endpoints $\underline{x}$ (not belonging to the horizontal path) are called *bottom* vertices. Finally, delete, if needed, $h\lceil n/h \rceil - n$ vertices from the last vertical path to make sure the total number of vertices is n. Thus, the tree $T(n,2) = T(n,3)$ is simply a path with n vertices, and, for even n, the tree $T(n,4)$ is called an *n-comb*. The general case (with h dividing n) appears in Figure 1.

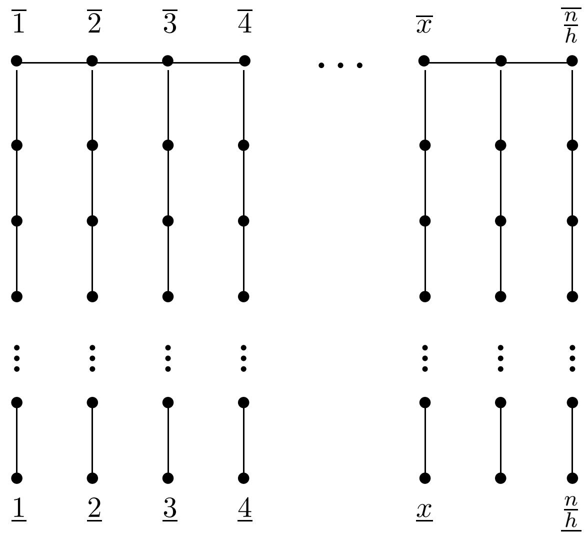

Figure 1.

For $d \ge 4$ the minimum number of edges one has to add to a graph to transform it to one of diameter at most d – in contrast to the cases $d \le 3$ – depends on the structure of the graph in an essential way. For example, one can transform $T(n,6)$ to a graph of diameter 4 by adding to it about

$n/3$ edges simply by connecting one given vertex to all vertices which are neither top nor bottom ones. On the other hand, significantly more ($\approx n/2$) edges have to be added to transform the n-comb $T(n,4)$ into a graph of diameter four. In the next theorem it is shown that indeed, for arbitrary even d, Theorem 3.1 is tight up to a constant additive term. (This remains true for odd d with a worse constant and with a slightly more complicated proof and will be stated in Theorem 3.4).

Theorem 3.2 *For every positive integer h and every n, $f_{2h}(T(n,2h)) \geq \lfloor n/h \rfloor - 6$.*

Proof. We may assume that h is a divisor of n. Take an arbitrary extension of $T(n,2h)$ into a graph G of diameter $2h$. Call the original edges *black* and the added ones *red*.

To capture the structure of the red edges, an auxiliary (multi)graph R is defined as follows. To each vertical path P_i of $T(n,2h)$ a vertex i is associated, i.e., R has n/h vertices denoted by $1, 2, \ldots, (n/h)$. For every red edge with endpoints on the vertical paths P_i, P_j ($1 \leq i \leq j \leq n/h$), let ij be an edge of R. Notice that R can have multiple edges and (since $i = j$ is possible), multiple loops as well. Clearly, the number of edges of R is equal to the number of red edges of G.

To any set $A \subseteq V(R)$ a subgraph $G(A)$ of G is defined as follows. The vertex set X of $G(A)$ is the union of vertices on the vertical paths corresponding to A i.e. $X = \{\cup V(P_a) | a \in A\}$. The edges of $G(A)$ are the edges of the subgraph induced by X in G *except* the (black) edges of the horizontal path. (All red edges of G and the black edges on the vertical paths remain.) Observe that if $A \subseteq V(R)$ induces a subtree in R then $G(A)$ is a tree as well (union of $|A|$ vertical paths with $|A| - 1$ red connecting edges). Let $l_A(u,v)$ denote the distance of vertices u, v in $G(A)$. We shall use the following lemma.

Lemma 3.3 *Assume that $A \subseteq V(R)$ induces a subtree in R. Then there exists $x \in A$ such that*
(i) $l_A(\underline{x}, \overline{x}) = h - 1$.
(ii) There exists at most one top vertex $\overline{y} \in V(G(A))$ for which $l_A(\underline{x}, \overline{y}) = h$.
(iii) For every top vertex $\overline{z} \in V(G(A)) \setminus \{\overline{x}, \overline{y}\}$, $l_A(\underline{x}, \overline{z}) \geq h + 1$.

Proof. We apply induction on $|A|$, the case $|A| = 1$ is trivial ($G(A)$ is a path of length $h - 1$). For the inductive step, delete $y \in A$ such that y is of degree one in the subtree $R[A]$. By the inductive hypothesis there exists $x \in A^* = A - y$ satisfying the lemma (with A^*).

Claim: either x or y satisfies the lemma (with A). To prove the claim, consider the (unique) red edge ab where $b \in P_y$. If x does not satisfy the lemma (with A) then $l_A(\underline{x}, \overline{y}) \leq h$ and (since the shortest path from $\underline{x}$ to $\overline{y}$ must go through ab it follows that

$$l_A(\underline{x}, a) + 1 + l_A(b, \overline{y}) \leq h.$$

Now we prove that y satisfies the lemma (with A). Since (i): $l_A(\underline{y}, \overline{y}) = h - 1$ is obvious and $l_A(\underline{y}, \overline{x}) \geq h$ is immediate from the assumption $l_A(\underline{x}, \overline{y}) \leq h$, it is enough to show that $l_A(\underline{y}, \overline{z}) \geq h + 1$ holds for every $\overline{z} \notin \{\overline{x}, \overline{y}\}$. If this fails then (since the shortest path from $\underline{y}$ to $\overline{z}$ must traverse the red edge ab from b to a) we get

$$l_A(\underline{y}, b) + 1 + l_A(a, \overline{z}) \leq h.$$

By adding the two displayed inequalities we conclude that

$$l_A(\underline{x}, a) + l_A(a, \overline{z}) + l_A(\underline{y}, b) + l_A(b, \overline{y}) \leq 2h - 2.$$

Since $l_A(\underline{y}, b) + l_A(b, \overline{y}) = h - 1$ this implies that

$$l_{A^*}(\underline{x}, \overline{z}) = l_A(\underline{x}, \overline{z}) \leq l_A(\underline{x}, a) + l_A(a, \overline{z}) \leq h - 1$$

which contradicts the choice of x (in A^*) and hence completes the proof of the claim and the lemma. □

Now we return to the proof of Theorem 3.2. Assume that one can transform $T(n, 2h)$ to a graph of diameter at most $2h$ by adding at most $n/h - 7$ (red) edges. This implies that the graph R has at least seven tree components $C_1, C_2, \dots, C_t$ where $t \geq 7$. Take the bottom vertex $\underline{x_{C_i}}$ in each tree $G(V(C_i))$ according to Lemma 3.3.

Define another auxiliary (simple) graph R_1 with directed and undirected edges as follows. The vertices of R_1 will be the tree components C_i of R. Connect C_i to C_j by an undirected edge in R_1 if the distance of $\overline{x_{C_i}}$ and $\overline{x_{C_j}}$ along the horizontal path is two. Define a directed edge in R_1 from C_i to C_j if the distance of $\overline{x_{C_i}}$ and some top vertex of $G(V(C_j))$ along the horizontal path is one.

Assume that C_i and C_j are not adjacent in R_1. Consider a shortest path P from $\underline{x_{C_i}}$ to $\underline{x_{C_j}}$ in G. Let $\overline{u}$ and $\overline{v}$ be the first and last vertex encountered on the horizontal path when traversing P from $\underline{x_{C_i}}$ to $\underline{x_{C_j}}$. Using Lemma 3.3 and the assumption that C_i and C_j are not adjacent we can estimate $|P|$, the length of P as follows.

Case 1.: $\overline{u} = \overline{x_{C_i}}$ and $\overline{v} = \overline{x_{C_j}}$. Then $|P| \geq (h-1) + (h-1) + 3 = 2h + 1$.

Case 2.: $\overline{u} = \overline{x_{C_i}}$ and $\overline{v} \neq \overline{x_{C_j}}$ (or by symmetry $\overline{v} = \overline{x_{C_j}}$ and $\overline{u} \neq \overline{x_{C_i}}$.) Then $|P| \geq (h-1) + h + 2 = 2h + 1$.

Case 3.: $\overline{u} \neq \overline{x_{C_i}}$ and $\overline{v} \neq \overline{x_{C_j}}$. Then $|P| \geq h + h + 1 = 2h + 1$.

We get a contradiction to the assumption that G is of diameter at most $2h$ (we did not use the full strength of Lemma 3.3). Thus the graph R_1 must be complete so it has $\binom{t}{2}$ (directed or undirected) edges. However, from the definition, R_1 has at most $2t$ directed and at most $t - 1$ undirected edges therefore $3t - 1 \geq \binom{t}{2}$ must hold. This is a contradiction, since $t \geq 7$. □

The lower bound proof for $f_d(T(n,d))$ for odd values of d is similar to the even case and use the full strength of Lemma 3.3. It is stated in the following theorem (in which we make no attempt to optimize the additive constant term).

Theorem 3.4 *For every positive integer h and every n, $f_{2h+1}(T(n,2h+1)) \geq \lfloor n/h \rfloor - 154$.*

Proof. As in the proof of Theorem 3.2, we may and will assume that h is a divisor of n. Take an arbitrary extension of $T(n,2h+1)$ ($=T(n,2h)$) into a graph G of diameter at most $2h+1$ and like in the proof of Theorem 3.2, the added edges are called *red*. The auxiliary graph R is defined also as in the proof of Theorem 3.2 and $C_1, C_2, \ldots, C_t$ denote the tree components of R. Since the number of red edges is at least $\frac{n}{h} - t$, we may assume $t \geq 155$ throughout the proof.

Select vertices x_{C_i} in each tree component C_i according to Lemma 3.3. Let $\underline{x_{C_i}}$ and $\overline{x_{C_i}}$ denote the corresponding bottom and top vertices of $G(V(C_i))$. Moreover let $\underline{y_{C_i}}$ and $\overline{y_{C_i}}$ denote the (exceptional) bottom and top vertices defined in part (ii) of Lemma 3.3. (If they do not exist let $\underline{y_{C_i}} = \underline{x_{C_i}}$ and $\overline{y_{C_i}} = \overline{x_{C_i}}$.)

We need a refined definition of the second auxiliary graph R_1 (with undirected and directed edges) as follows. The t vertices of R_1 are the components C_i and C_i, C_j is defined as an undirected edge of R_1 in two cases: a. the distance of $\overline{x_{C_i}}$ and $\overline{x_{C_j}}$ along the horizontal path is two or three; b. the distance of $\overline{y_{C_i}}$ and $\overline{y_{C_j}}$ along the horizontal path is one. A directed edge of R_1 from C_i to C_j is defined in two cases: c. the distance of $\overline{x_{C_i}}$ and $\overline{y_{C_j}}$ along the horizontal path is two; d. the distance of $\overline{x_{C_i}}$ and a top vertex of $G(V(C_j))$ along the horizontal path is one.

Notice that at most 6 undirected edges are incident to any vertex of R_1 and at most 4 directed edges go out from any vertex of R_1. Thus R_1 has at most $7t$ edges. Therefore, by Turán's theorem, R_1 has an independent set of size at least $t/15$. The foregoing computations are not affected by assuming that $\frac{t}{15}$ is an integer. We restrict our attention to these $t/15$ tree components C_i, $1 \leq i \leq \frac{t}{15}$, and consider them as vertices in another (undirected) auxiliary graph R_2. Two vertices C_i and C_j in R_2 are considered adjacent if there is a red edge connecting two top vertices p, q where $p, \overline{x_{C_i}}$ and $q, \overline{x_{C_j}}$ are both edges of the horizontal path.

Assume that R_2 is a complete graph. Let S denote the set of top vertices which are at distance one along the horizontal path from some vertex $\overline{x_{C_i}}$, $1 \leq i \leq \frac{t}{15}$. Using that $\{C_i | 1 \leq i \leq \frac{t}{15}\}$ is independent in R_1 and complete in R_2 it follows that $\frac{2t}{15} - 2 \leq |S| \leq \frac{2t}{15}$ and the subgraph $G[S]$ induced by S in G has at least $\binom{\frac{t}{15}}{2}$ (distinct) red edges. On the other hand, one can define $\frac{n}{h} - t$ red edges of G by selecting the red edges corresponding to the edges of the t tree components of R plus selecting the red edges corresponding to edges of fixed unicursal spanning subgraphs of the other (non-tree) components of R. Notice that the selected $\frac{n}{h} - t$ red edges form a subgraph of G whose components are trees or unicursal graphs, in particular at most $|S| \le \frac{2t}{15}$ of them are in $G[S]$. Therefore the number of red edges in $G[S]$ is at least

$$\frac{n}{h} - t + \binom{t/15}{2} - 2t/15 = \frac{n}{h} + \frac{t(t-525)}{450} > \frac{n}{h} - 154$$

(because $\frac{t(t-525)}{450}$ has minimum value -153.125 at $t = 262.5$). Therefore, if R_2 is complete, the proof is finished.

Assume that there exist C_i and C_j which are adjacent neiher in R_1 nor in R_2. Using this and Lemma 3.3 (in full strength) we are going to show that $|P|$, the length of a shortest path P in G from $\underline{x_{C_i}}$ to $\underline{x_{C_j}}$ is at least $2h+2$ and this contradiction will finish the proof. Like in the proof of Theorem 3.2, let $\overline{u}$ and $\overline{v}$ denote the first and last vertex of P encountered in the horizontal path when P is traversed from $\underline{x_{C_i}}$ to $\underline{x_{C_j}}$.

Case 1.: $\overline{u} = \overline{x_{C_i}}$ and $\overline{v} = \overline{x_{C_j}}$; Then $|P| \ge (h-1) + (h-1) + 4 = 2h+2$.

Case 2.: $\overline{u} = \overline{x_{C_i}}$ and $\overline{v} = \overline{y_{C_j}}$; (or, by symmetry $\overline{u} = \overline{y_{C_i}}$ and $\overline{v} = \overline{x_{C_j}}$;) Then $|P| \ge (h-1)+h+3 = 2h+2$.

Case 3.: $\overline{u} = \overline{y_{C_i}}$ and $\overline{v} = \overline{y_{C_j}}$; Then $|P| \ge h+h+2 = 2h+2$.

Case 4.: $\overline{u} = \overline{x_{C_i}}$ and $\overline{v} \notin \{\overline{x_{C_j}}, \overline{y_{C_j}}\}$; (or, by symmetry $\overline{v} = \overline{x_{C_j}}$ and $\overline{u} \notin \{\overline{x_{C_i}}, \overline{y_{C_i}}\}$); Then $|P| \ge (h-1) + (h+1) + 2 = 2h+2$.

Case 5.: all other cases. Then $|P| \ge h + (h+1) + 1 = 2h+2$.

This completes the case analysis and hence the proof of the theorem. □

A graph H is obtained from a graph G by an *elementary identification* if H is obtained from G by identifying two of its vertices u and v and by making the identified vertex adjacent to all neighbors of u as well as to all neighbors of v. It is easy to see that $f_d(G) \ge f_d(H)$ for every d. It also follows that if H is obtained from G by a sequence of elementary identifications then for every d, $f_d(G) \ge f_d(H)$. (Note that this implies that for every $m > n$ and every k and d, $f_d(T(m,k)) \ge f_d(T(n,k))$, as $T(n,k)$ can be obtained from $T(m,k)$ by a sequence of elementary identifications.) This transformation is useful because it makes possible to apply Theorems 3.2, 3.4 to other graphs without translating the proof technique to them. We demonstrate this with the example of the cycle C_n (in fact, for $h = 1$, the next corollary gives another proof for Corollaries 2.2 and 2.4 although with worse constants).

Corollary 3.5 *The values of $f_d(C_n)$ for the n-cycle C_n satisfy the following:*
(i) For every positive integer h and for every n,

$$\lfloor n/(2h-1) \rfloor - 7 \le f_{2h}(C_n) \le \lfloor n/(2h-1) \rfloor.$$

(ii) For every positive integer h and for every n

$$n/(2h-1) - 146 \le f_{2h+1}(C_n) \le \lfloor n/(2h-1) \rfloor.$$

Proof. Let $\{1, \ldots, n\}$ be the vertex set of the cycle where the vertices appear in this order along the cycle. The upper bound comes from adding the diagonals $1i$ for $i = 2h, 4h-1, 6h-2, \ldots$.

To prove the lower bound note that if n is divisible by $2h-1$ then $T(hn/(2h-1), 2h)$ plus one additional edge connecting the two ends of the horizontal path can be obtained from the cycle C_n by a sequence of elementary identifications. To do so pick $2h-1$ consecutive vertices along the cycle, say $1, 2, 3, \ldots, 2h-1$ and identify the pairs $(1, 2h-1)$,$(2, 2h-2)$, $\ldots$, $(h-1, h+1)$. Repeating the same process on each interval of $2h-1$ consecutive vertices along the cycle we obtain the above mentioned graph. Since after deleting one of its edges we get $T(hn/(2h-1), 2h)$ $(= \ T(hn/(2h-1), 2h+1)\ )$ the desired result now follows from Theorems 3.2 and 3.4. If the length of the cycle is not divisible by $(2h-1)$ we first apply the appropriate number of identifications to reduce it to a cycle of length $(2h-1)\lfloor n/(2h-1) \rfloor$. $\square$

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