

STABLE KNESER HYPERGRAPHS AND IDEALS IN \mathbb{N} WITH THE NIKODÝM PROPERTY

NOGA ALON, LECH DREWNOWSKI, AND TOMASZ LUCZAK

ABSTRACT. We use stable Kneser hypergraphs to construct ideals in \mathbb{N} which are not nonatomic yet have the Nikodým property.

1. INTRODUCTION

We say that an ideal \mathcal{I} in $\mathcal{P}(\mathbb{N})$ has the *Positive Summability Property* (PSP) if for any sequence (x_n) of positive reals such that $\sum_{n=1}^{\infty} x_n$ diverges, there exists $I \in \mathcal{I}$ for which $\sum_{n \in I} x_n$ diverges as well. It is not hard to show that the ideal \mathcal{Z} of sets of density zero has (PSP). This ‘folklore’ fact can be traced back (at least) to a short note of Auerbach [2] from 1930, and has later been rediscovered several times (for more information see [3]). However, all known proofs of this and similar results are based on the fact that the ideal in question is nonatomic, in the sense of the definition given in the next section. The aim of this note is to use stable Kneser hypergraphs to provide an example of an ideal which has (PSP), as well as a stronger Nikodým property, but is not nonatomic. This solves in the affirmative Problem 8.1 from [4].

2. IDEALS IN \mathbb{N}

Let us recall that an *ideal* in \mathbb{N} is a family \mathcal{I} of subsets of \mathbb{N} such that if $A, B \in \mathcal{I}$ and $C \subseteq A$, then also $A \cup B \in \mathcal{I}$ and $C \in \mathcal{I}$. We call an ideal \mathcal{I} in \mathbb{N} *nonatomic* if there exists a sequence (\mathcal{P}_n) of finite partitions of \mathbb{N} such that each \mathcal{P}_n is refined by \mathcal{P}_{n+1} , and whenever (A_n) is a decreasing sequence with $A_n \in \mathcal{P}_n$ for each n , and a set $Z \subseteq \mathbb{N}$ is such that $Z \setminus A_n$ is finite for each n , then $Z \in \mathcal{I}$. Observe also that the ideal \mathcal{Z} of sets of density zero is nonatomic (it is enough to consider the sequence of partitions (\mathcal{B}_n) such that \mathcal{B}_n consists of congruence classes modulo 2^n). Thus, Auerbach’s result follows immediately from the following statement.

Theorem 2.1. *Every nonatomic ideal \mathcal{I} in \mathbb{N} has (PSP).*

Proof. Let (\mathcal{P}_n) be a sequence of finite partitions of \mathbb{N} which validates that \mathcal{I} is nonatomic, and let $\sum_{i=1}^{\infty} x_i$ be a divergent sequence of positive numbers. Since each \mathcal{P}_{n+1} is a refinement of \mathcal{P}_n , one can easily find a decreasing sequence of sets (A_n) such that $A_n \in \mathcal{P}_n$ and $\sum_{i \in A_n} x_i = \infty$ for $n = 1, 2, \dots$. Now, proceeding by

2000 *Mathematics Subject Classification.* 05A17, 05C15, 11B05.

Key words and phrases. Kneser graph, divergent series, chromatic number, Nikodým property.

The first author is supported by the Israel Science Foundation, by a USA-Israeli BSF grant, by the Hermann Minkowski Minerva Center for Geometry at Tel Aviv University and by the Von Neumann Fund. The third author is supported by the Foundation for Polish Science.

induction, for every n choose a finite set $B_n \subseteq A_n$ so that $\max B_{n-1} < \min B_n$ and $\sum_{i \in B_n} x_i \geq 1$, and set $Z = \bigcup_{n=1}^{\infty} B_n$. Then, the set $Z \setminus A_n \subseteq \bigcup_{j=1}^{n-1} B_j$ is finite for each n , so $Z \in \mathcal{I}$, but clearly $\sum_{i \in Z} x_i = \infty$. \square

In this note we shall be mainly interested in ideals defined by families of finite subsets of \mathbb{N} in the following way. Let $\mathbf{F} = (F_n)$ be a family of distinct finite subsets of \mathbb{N} and let

$$\mathcal{Z}(\mathbf{F}) = \left\{ A \in \mathbb{N} : \limsup_{n \rightarrow \infty} \frac{|A \cap F_n|}{|F_n|} = 0 \right\}.$$

It is easy to see that $\mathcal{Z}(\mathbf{F})$ is an ideal. In particular, if $\mathbf{H} = ([n])$, where $[n] = \{1, 2, \dots, n\}$, then $\mathcal{Z}(\mathbf{H})$ is the ideal of sets of density zero.

We say that an ideal \mathcal{I} in \mathbb{N} has the *Nikodým property* if a family \mathfrak{M} of bounded finitely additive scalar measures μ defined on \mathcal{I} is uniformly bounded provided that $\sup_{\mu \in \mathfrak{M}} |\mu(A)| < \infty$ for every $A \in \mathcal{I}$. The Nikodým property of ideals (as well as algebras or rings) of sets is important in studying certain types of locally convex spaces; for more information and further references, see, for instance, [4] and [7]. It is easy to see that the Nikodým property implies (PSP). Indeed, for a sequence (x_i) , $x_i \geq 0$, consider a sequence (μ_n) of scalar measures on \mathcal{I} defined as $\mu_n(A) = \sum_{i \in A \cap [n]} x_i$. Clearly, for every $A \in \mathcal{I}$, we have $\sup_n \mu_n(A) = \sum_{i \in A} x_i$. Thus, if \mathcal{I} has the Nikodým property and for every $A \in \mathcal{I}$ we have $\sum_{i \in A} x_i < \infty$, then $\sum_{i=1}^{\infty} x_i < \infty$, and so \mathcal{I} has (PSP). We also mention that, as was shown in [4, Th. 5.4], for ideals of type $\mathcal{Z}(\mathbf{F})$ these two properties are equivalent.

Lemma 2.2. *If \mathbf{F} is a family of finite sets in \mathbb{N} , then $\mathcal{Z}(\mathbf{F})$ has (PSP) if and only if $\mathcal{Z}(\mathbf{F})$ has the Nikodým property.* \square

The main goal of this note is to use stable Kneser hypergraphs to supplement Theorem 2.1 and Lemma 2.2 by the following result.

Theorem 2.3. *There exists a family \mathbf{F} of finite sets in \mathbb{N} such that the ideal $\mathcal{Z}(\mathbf{F})$ is not nonatomic but it has the Nikodým property.*

3. STABLE KNESER HYPERGRAPHS

Let us start with some notation. We say that a set $A \subseteq S$ is k -sparse in $S \subseteq [n]$ if for any two elements $a', a'' \in A$, $a' < a''$, there is a j , $0 \leq j \leq k$, and $2k - 2$ elements $s_1, s_2, \dots, s_{2k-2} \in S \setminus A$, such that

$$s_1 < s_2 < \dots < s_j < a' < s_{j+1} < \dots < s_{j+k-1} < a'' < s_{j+k} < \dots < s_{2k-2},$$

i.e., between any two elements of A there are at least $k - 1$ elements of S in a natural ‘cyclic order’. Given a set $S \subseteq [n]$ the family of all k -sparse m -element subsets of S we denote by $\mathcal{M}(k, m, S)$.

Let us recall that a k -uniform hypergraph H is a pair (V, E) , where V is the set of vertices of H , and the set E of edges of H is a family of k -element subsets of V . The *stable Kneser hypergraph* $H(k, \ell, m)$ is a k -uniform hypergraph defined in the following way. Let $k \geq 2$, ℓ, m , be natural numbers. Set $n = km + \ell$ and consider the set of n ‘points’ $\{1, \dots, n\}$. The set V of vertices of $H(k, \ell, m)$ is the family $\mathcal{M}(k, m, [n])$ of k -sparse m -element subsets of $[n]$. The vertices $A_1, \dots, A_k \in V$ form an edge of $H(k, \ell, m)$ if A_1, \dots, A_k are pairwise disjoint.

The stable Kneser hypergraphs have the following, rather surprising, property.

Lemma 3.1. *For every choice of non-negative weights $x : V \rightarrow [0, \infty)$ of vertices of $H(k, \ell, m)$, and every integer r , where $1 \leq r \leq k$, there exists a subset W of V such that the following two conditions hold:*

- (i) *each edge of $H(k, \ell, m)$ shares with W at most r vertices;*
- (ii)

$$\sum_{w \in W} x(w) \geq \frac{rm}{km + \ell} \sum_{v \in V} x(v).$$

Proof. For every point $i \in [n]$ define the weight of i as

$$\bar{x}(i) = \sum_{v \in V; v \ni i} x(v).$$

Since each vertex contains m points, we have

$$\sum_{i=1}^n \bar{x}(i) = m \sum_{v \in V} x(v).$$

Hence the average weight of a point $i \in [n]$ is

$$\frac{1}{n} \sum_{i=1}^n \bar{x}(i) = \frac{m}{km + \ell} \sum_{v \in V} x(v).$$

Let R denote a set of r consecutive points of $[n]$ (in the cyclic order) of maximum weight, and let W be the set of all vertices of $H(k, \ell, m)$ which intersect R . Then, clearly, each edge of $H(k, \ell, m)$ contains at most $|R| = r$ vertices from W . Moreover, since $r \leq k$ and the elements of R are consecutive, each vertex from W contains at most one point from R . Hence

$$\sum_{w \in W} x(w) = \sum_{i \in R} \bar{x}(i) \geq \frac{rm}{km + \ell} \sum_{v \in V} x(v). \quad \square$$

Let us recall that the *chromatic number* $\chi(H)$ of a k -uniform hypergraph H is the minimum number of sets into which one can partition the set of vertices of H so that no edge of H is contained in any of the sets of the partition. It is easy to find a partition which shows that $\chi(H(k, \ell, m)) \leq \lceil \frac{\ell+k}{k-1} \rceil$, and it is strongly believed that in this estimate the equality holds (see Ziegler [8]). Indeed, as proved in [1], this is the chromatic number of the k -uniform hypergraph whose vertices are *all* m -element subsets of $[km + \ell]$, where A_1, A_2, \dots, A_k form an edge if they are pairwise disjoint.

Conjecture 3.2. *For each k, ℓ , and m ,*

$$\chi(H(k, \ell, m)) = \left\lceil \frac{\ell + k}{k - 1} \right\rceil.$$

Although we cannot settle the above statement we prove the following fact, which, in particular, implies that to check whether Conjecture 3.2 holds it is enough to verify it for prime k 's.

Lemma 3.3. *If Conjecture 3.2 holds for both k_1 and k_2 (and all values of ℓ and m), then it holds also for $k = k_1 k_2$.*

Proof. Let $k = k_1 k_2$, $n = km + \ell$, and $t = \frac{\ell+k}{k-1}$. As we have already mentioned, we only have to show that $\chi(H(k, \ell, m)) \geq \lceil t \rceil$. To this end, let $\mathcal{M} = \mathcal{M}(k, m, [n])$ denote the vertex set of $H(k, \ell, m)$, and consider any coloring $h : \mathcal{M} \rightarrow \llbracket \lceil t \rceil - 1 \rrbracket$ of \mathcal{M} with $\lceil t \rceil - 1$ colors. We need to show that if Conjecture 3.2 holds for both k_1 and k_2 , then there exists a k -tuple of pairwise disjoint m -element sets A_1, A_2, \dots, A_k colored with the same color.

Let $S \subseteq [n]$ be a k_2 -sparse subset of $[n]$ with $k_1 m + \ell_1$ elements, where $\ell_1 = (\lceil t \rceil - 2)(k_1 - 1)$. Observe that any m -element subset $A \subseteq S$ which is k_1 -sparse in S is k -sparse in $[n]$, so it is colored by h with one of $\lceil t \rceil - 1$ colors. Note also that

$$\left\lceil \frac{\ell_1 + k_1}{k_1 - 1} \right\rceil = \left\lceil \lceil t \rceil - \frac{k_1 - 2}{k_1 - 1} \right\rceil = \lceil t \rceil.$$

Consequently, since the coloring h uses only $\lceil t \rceil - 1$ colors and we have assumed that Conjecture 3.2 holds for k_1 , S contains k_1 disjoint k_1 -sparse m -element subsets colored with the same color. We denote this color by $\hat{h}(S)$.

Now let us consider the family $\mathcal{M}_1 = \mathcal{M}(k_2, k_1 m + \ell_1, [n])$ of all k_2 -sparse subsets of $[n]$ with $k_1 m + \ell_1$ elements. By our previous construction there is a coloring $\hat{h} : \mathcal{M}_1 \rightarrow \llbracket \lceil t \rceil - 1 \rrbracket$ of \mathcal{M}_1 with $\lceil t \rceil - 1$ colors. Consider the k_2 -uniform stable Kneser hypergraph $H(k_2, \ell_2, k_1 m + \ell_1)$ which corresponds to this family, where

$$\ell_2 = km + \ell - k_2(k_1 m + \ell_1) = \ell - (\lceil t \rceil - 2)(k_1 - 1)k_2.$$

Note that

$$\frac{\ell_2 + k_2}{k_2 - 1} = t + \frac{(k_1 - 1)k_2}{k_2 - 1}(1 + t - \lceil t \rceil) \geq t,$$

so $\lceil (\ell_2 + k_2)/(k_2 - 1) \rceil \geq \lceil t \rceil$. Hence, since we have assumed that Conjecture 3.2 holds for k_2 , and \hat{h} uses only $\lceil t \rceil - 1$ colors, there exist k_2 disjoint k_2 -sparse sets S_1, \dots, S_{k_2} which are colored with the same color by \hat{h} . This however implies that in the coloring h there are $k = k_1 k_2$ disjoint k -sparse sets which are colored with the same color by h . Hence $\chi(H(k, \ell, m)) \geq \lceil t \rceil$ and the assertion follows. \square

Corollary 3.4. *Conjecture 3.2 holds for $k = 2^i$, for every $i \geq 1$.*

Proof. This follows immediately from Lemma 3.3 and the fact that Conjecture 3.2 holds for $k = 2$ and every m and ℓ , as proved by Schrijver [6] (see also Matoušek [5]). \square

4. PROOF OF THE MAIN RESULT

Proof of Theorem 2.3. Let us recall that our goal is to construct a family of finite sets \mathbf{F} such that the ideal $\mathcal{Z}(\mathbf{F})$ is not nonatomic but has the Nikodým property. For every $k \geq 1$ denote by H_k the stable Kneser 2^k -uniform hypergraph $H_k = H(2^k, 2^{2k}, 2^{3k})$. Note that by Corollary 3.4 the chromatic number of H_k is larger than 2^k . Let $\mathbf{G} = (G_k)$ be a partition of \mathbb{N} into finite sets G_k such that the number of points of G_k is the same as the number of vertices in H_k . Embed in each G_k the hypergraph H_k (i.e., via a bijection replace G_k by the set of vertices of H_k), and define \mathbf{F} to be the collection of all edges of the hypergraphs H_k for $k \geq 1$. We show that \mathbf{F} has both required properties.

Note first that $\mathcal{Z}(\mathbf{F})$ is not nonatomic. Indeed, let us partition \mathbb{N} into ℓ parts, N_1, \dots, N_ℓ . Then, using Corollary 3.4, in each H_k such that $2^k > \ell$ there is an edge which is contained in one of the parts of the partition. Consequently, there is an i , $1 \leq i \leq \ell$, such that N_i contains infinitely many sets from \mathbf{F} . Now, let (\mathcal{P}_n) be a

sequence of finite partitions of \mathbb{N} such that each \mathcal{P}_n is refined by \mathcal{P}_{n+1} . By what we have just observed, one can find a decreasing sequence (A_n) and a sequence (F_n) of pairwise distinct members of \mathbf{F} such that $F_n \subseteq A_n \in \mathcal{P}_n$ for each n . Then, clearly, the union Z of the F_n 's is not in $\mathcal{Z}(\mathbf{F})$ so that the condition required in the definition of a nonatomic ideal is not satisfied. Thus $\mathcal{Z}(\mathbf{F})$ is not nonatomic.

Now let $\sum_{i=1}^{\infty} x_i$ be any divergent series of positive numbers. For each k , set $y_k = \sum_{i \in G_k} x_i$. Since $\sum_k y_k = \infty$, there is a sequence (a_k) of positive numbers such that $a_k \rightarrow 0$, $2^k a_k \geq 1$ for all k , and $\sum_k a_k y_k = \infty$. Select k_0 so that $a_k \leq 1/2$ for $k \geq k_0$. For such k , let $r_k = \lceil 2^k a_k \rceil$, and note that $r_k \leq 2 \cdot 2^k a_k \leq 2^k$. Hence we may apply Lemma 3.1 with $r = r_k$ to find a set W_k of vertices of H_k such that each edge of H_k shares at most r_k vertices with W_k , and yet

$$\sum_{i \in W_k} x_i \geq \frac{r_k 2^{3k}}{2^k 2^{3k} + 2^{2k}} \sum_{i \in G_k} x_i \geq a_k \frac{2^{4k}}{2^{4k} + 2^{2k}} \sum_{i \in G_k} x_i \geq \frac{1}{2} a_k \sum_{i \in G_k} x_i = \frac{1}{2} a_k y_k.$$

Finally, let $W = \bigcup_{k \geq k_0} W_k$. Then $\sum_{i \in W} x_i \geq \frac{1}{2} \sum_{k \geq k_0} a_k y_k = \infty$, while for every set $F \in \mathbf{F}$ such that $F \subseteq G_k$ for some $k \geq k_0$ we have

$$\frac{|W \cap F|}{|F|} = \frac{|W_k \cap F|}{|F|} \leq \frac{r_k}{2^k} \leq 2a_k,$$

so $W \in \mathcal{Z}(\mathbf{F})$. Consequently, $\mathcal{Z}(\mathbf{F})$ has (PSP) property and thus, by Lemma 2.2, also the Nikodým property. \square

REFERENCES

- [1] N. Alon, P. Frankl and L. Lovász, *The chromatic number of Kneser hypergraphs*, Trans. Amer. Math. Soc. 298(1986), 359–370.
- [2] H. Auerbach, *Über die Vorzeichenverteilung in unendlichen Reihen*, Studia Math. **2** (1930), 228–230.
- [3] L. Drewnowski and I. Labuda, *Vector series whose lacunary subseries converge*, Studia Math. **138** (2000), 53–80.
- [4] L. Drewnowski and P.J. Paúl, *The Nikodým property for ideals of sets defined by matrix summability methods*, Rev. R. Acad. Cienc. Exact. Fis. Nat. (Esp) **94** (2000), 485–503.
- [5] J. Matoušek, *Using the Borsuk-Ulam theorem. Lectures on topological methods in combinatorics and geometry*, Springer-Verlag, Berlin, 2003.
- [6] A. Schrijver, *Vertex-critical subgraphs of Kneser graphs*, Nieuw Arch. Wisk. (3) 26 (1978), no. 3, 454–461.
- [7] C.E. Stuart and P. Abraham, *Generalizations of the Nikodým boundedness and Vitali-Hahn-Saks theorems*, J. Math. Anal. Appl. **300** (2004), 351–361.
- [8] G.M. Ziegler, *Generalized Kneser coloring theorems with combinatorial proofs*, Invent. Math. **147** (2002), 671–691. Erratum **163** (2006), 227–228.

SCHOOLS OF MATHEMATICS AND COMPUTER SCIENCE, RAYMOND AND BEVERLY SACKLER FACULTY OF EXACT SCIENCES, TEL AVIV UNIVERSITY, TEL AVIV 69978, ISRAEL AND IAS, PRINCETON, NJ 08540, USA.

E-mail address: `nogaa@tau.ac.il`

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, A. MICKIEWICZ UNIVERSITY, UMULTOWSKA 87, 61–614 POZNAŃ, POLAND

E-mail address: `drewlech@amu.edu.pl`

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, A. MICKIEWICZ UNIVERSITY, UMULTOWSKA 87, 61–614 POZNAŃ, POLAND

E-mail address: `tomasz@amu.edu.pl`