UNIVERSALITY AND TOLERANCE
(Extended Abstract)

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Abstract

For any positive integers r and n, let $\mathcal{H}(r,n)$ denote the family of graphs on n vertices with maximum degree r, and let $\mathcal{H}(r,n,n)$ denote the family of bipartite graphs H on $2n$ vertices with n vertices in each vertex class, and with maximum degree r. On one hand, we note that any $\mathcal{H}(r,n)$-universal graph must have $\Omega(n^{2-2/r})$ edges. On the other hand, for any $n \geq n_0(r)$, we explicitly construct $\mathcal{H}(r,n)$-universal graphs G and Λ on n and $2n$ vertices, and with $O(n^{2-\Omega(\frac{1}{r\log r})})$ and $O(n^{2-\frac{1}{r}}\log^{1/r} n)$ edges, respectively, such that we can efficiently find a copy of any $H \in \mathcal{H}(r,n)$ in G deterministically. We also achieve sparse universal graphs using random constructions. Finally, we show that the bipartite random graph $G = G(n,n,p)$, with $p = cn^{-\frac{1}{2r}}\log^{1/2r} n$ is fault-tolerant; for a large enough constant c, even after deleting any α-fraction of the edges of G, the resulting graph is still $\mathcal{H}(r,a(\alpha)n,a(\alpha)n)$-universal for some $a : [0,1) \to (0,1]$.

1 Introduction

For a family $\mathcal{H}$ of graphs, a graph G is $\mathcal{H}$-*universal* if G contains every member of $\mathcal{H}$ as a subgraph. For example, the complete graph K_n is $\mathcal{H}_n$-universal for the family $\mathcal{H}_n$ of all graphs on at most n vertices. The construction of sparse universal graphs for various families arises in the study of VLSI circuit design, and received a considerable amount of attention, see, e.g., [6], [7], [8] and their references. Since in some applications the cost of a vertex is higher than that of an edge, one is particularly interested in tight $\mathcal{H}$-universal graphs, i.e., graphs whose number of vertices equals $\max_{H\in\mathcal{H}} |V(H)|$.

Most of the previously known constructions of sparse universal graphs for various families are based on the existence of small separators in these families. Since here we consider families of graphs that do not necessarily have small separators, we need some novel techniques. These combine the notion of the strong chromatic number of a graph, introduced in [1], various properties of random graphs, an embedding technique based on matching theorems, developed in [3] and [17], a structure result on sparse regular pairs proved in [12], a sparse version of Szemerédi's regularity lemma, and a hypergraph packing result from [16].

Let $\mathcal{H}(r,n)$ denote the family of all (pairwise nonisomorphic) graphs on n vertices in which every degree is at most r. We claim that the minimum number M of edges in any $\mathcal{H}(r,n)$-universal graph must be at least $\Omega(n^{2-2/r})$ for $r \geq 2$. This lower bound follows from the obvious inequality $\sum_{i\leq rn/2}\binom{M}{i} \geq |\mathcal{H}(r,n)|$ and the well known (see, e.g., [11], Cor. 9.8, page 239) asymptotic formula for the number $L_{r,n}$ of all labelled r-regular graphs on n vertices, nr even:

$$L_{r,n} = (1+o(1))\sqrt{2}e^{-(r^2-1)/4}\left(\frac{r^{r/2}}{e^{r/2}r!}\right)^n n^{rn/2}.$$

Let $M(r,n) = M$ be the minimum number of edges in an $\mathcal{H}(r,n)$-universal graph. The inequalities $\binom{M}{i} \leq \left(\frac{2eM}{i}\right)^i$ for $i \leq rn/2$ and $|\mathcal{H}(r,n)| \geq L_{r,n}/n!$ yield the claimed lower bound

$$M(r,n) = \Omega(n^{2-2/r})$$

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for the number of edges of any $\mathcal{H}(r,n)$-universal graph (on any number of vertices). If rn is odd, simply observe that an $\mathcal{H}(r,n)$-universal graph is also $\mathcal{H}(r,n-1)$-universal.

Having just lower-bounded $M(r,n)$, we now focus on upper-bounding $M(r,n)$, with both explicit and randomized constructions. It follows from the celebrated Blow-up Lemma [14] that $M(r,n) = o(n^2)$. Via explicit constructions we obtain stronger upper-bounds, of the form n^{2-a}, for some strictly positive a that depends only on r, as stated in the following theorem. (Note that $M(1,n) = \lfloor n/2 \rfloor$.)

Theorem 1.1 *There exists an absolute constant $c > 0$ such that for every $r \geq 2$ and $n > n_0(r)$ there is an explicitly described $\mathcal{H}(r,n)$-universal graph G with n vertices and at most $n^{2-c/r \log r}$ edges, such that we can find a copy of any $H \in \mathcal{H}(r,n)$ in G in deterministic polynomial time (in n).*

Thus, in view of the ultimate lower bound, the above construction is not too far from being best possible. We can also describe an even smaller explicit construction (although here the number of vertices is larger.)

Theorem 1.2 *For each r, there is a finite $\phi(r)$ such that for each n there is an explicitly described $\mathcal{H}(r,n)$ universal graph Λ, with at most $\phi(r)n^{2-\frac{1}{r}} \log^{1/r} n$ edges and $2n$ vertices.*

In fact, almost every graph on $2n$ vertices that has as many edges as $\Lambda = \Lambda(r,n)$ has is $\mathcal{H}(r,n)$-universal, as the next theorem implies. We say that a random graph possesses a property asymptotically almost surely, and write *a.a.s.*, if the probability of the event in question tends to 1 as $n \to \infty$.

Theorem 1.3 *For every $\varepsilon > 0$ there exists a positive constant $c = c(\varepsilon)$ such that, for every $r \geq 2$, the random graph $G(\lceil (1+\varepsilon)n \rceil, p)$ with $p = cn^{-1/r}(\log n)^{1/r}$* a.a.s. *is $\mathcal{H}(r,n)$-universal. Consequently, for $n > n_0(r)$ there is an $\mathcal{H}(r,n)$-universal graph G with $\lceil (1+\varepsilon)n \rceil$ vertices and, say, at most $(1+\varepsilon)^2 cn^{2-1/r}(\log n)^{1/r}$ edges.*

A related Ramsey theoretic statement is considered by Kohayakawa, Rödl and Szemerédi [13].

With some more work we are able to get rid of the annoying factor of $1+\varepsilon$, but only in the bipartite case and for a slightly higher edge density. For $r \geq 2$ let $\mathcal{H}(r,n,n)$ be the family of bipartite graphs with n vertices in each vertex class and with maximum degree at most r. Let $G(n,n,p)$ be the random bipartite graph with n vertices in each color class and edge probability p.

Theorem 1.4 *There exists an absolute positive constant c such that, for every $r \geq 2$,* a.a.s. *$G(n,n,p)$ is $\mathcal{H}(r,n,n)$-universal, where $p = c(\log n/n)^{1/2r}$. Consequently, for $n > n_0(r)$ there is an $\mathcal{H}(r,n,n)$-universal, bipartite graph G with n vertices in each color class and at most $2cn^{2-1/2r}(\log n)^{1/2r}$ edges.*

It turns out that the same random graph enjoys a related property. For a real number α, where $0 < \alpha < 1$, we say that a graph G is *α-fault-tolerant* with respect to a family of graphs $\mathcal{H}$, if every subgraph of G with at least a $1-\alpha$ fraction of the edges of G is $\mathcal{H}$-universal. In general, restricting to bipartite graphs is unavoidable here, as for any graph G, there is a bipartite subgraph G' of G with at least half the edges of G.

Theorem 1.5 *For every $r \geq 2$ and $0 < \alpha < 1$ there exist constants $c > 0$ and $C > 0$ such that* a.a.s. *$G(n,n,p)$ is α-fault-tolerant with respect to $\mathcal{H}(r, \lfloor n/C \rfloor, \lfloor n/C \rfloor)$, where $p = c(\log n/n)^{1/2r}$. Consequently, for $n > n_0(r)$ there is a bipartite graph G with n vertices in each color class and at most $2cn^{2-1/2r}(\log n)^{1/2r}$ edges, which is α-fault-tolerant with respect to $\mathcal{H}(r, \lfloor n/C \rfloor, \lfloor n/C \rfloor)$.*

The rest of this extended abstract is organized as follows. In Section 2 we sketch the proof of Theorem 1.1. In Section 3 we describe Λ, and present a proof of a proposition with much of the power of Theorem 1.2. In Section 4 we prove Theorems 1.3 and 1.4. In Section 5, we discuss Theorem 1.5. Finally, Section 6 contains some concluding remarks.

2 The strong chromatic number and universal graphs

The techniques in [8] can be used to construct $\mathcal{H}(r,n)$-universal graphs with $O(n^2/\log^2 n)$ edges. To obtain the better construction needed in the proof of Theorem 1.1 we combine a new technique with the main result of [1]. This construction, besides providing graphs with a rather small number of edges, is simple, and supplies an efficient algorithm for embedding any given member of $\mathcal{H}(r,n)$ in the graph constructed. The construction is in fact so simple that for $r = 3$ and any n which is a power of 16, say, $n = 16^s$, it can be described in one (short) sentence, as follows. The vertices are all vectors of length s over the alphabet $\{1, 2, \ldots, 16\}$, and two are adjacent if and only if they differ in all coordinates.

Let H be a graph with $|V(H)| = n$. If k divides n we say that H is *strongly k-colorable* if for any partition of $V(H)$ into pairwise disjoint sets V_i, each of cardinality k precisely, there is a proper k-vertex coloring of H in which each color class intersects each V_i in exactly one vertex. If k does not divide n, we say that H is strongly k-colorable if the graph obtained from H by adding to it $k\lceil n/k \rceil - n$ isolated vertices is strongly k-colorable. The *strong chromatic number* of H, denoted by $\chi_s(H)$, is the minimum k such that H is strongly k-colorable. It is not difficult to check that if H is strongly k-colorable, then H is also strongly $(k+1)$-colorable.

The notion of strong chromatic number is studied in [1], where the following result is proved. (It is easy to see that if $\Delta(H) \le 1$ then $\chi_s(H) \le 2$.)

Theorem 2.1 ([1]) *For every $r \ge 2$:*
(i) There exists an absolute constant b such that if $\Delta(H) \le r$ then $\chi_s(H) \le br$.
(ii) If $\Delta(H) \le r$ then $\chi_s(H) \le 2^{r+1}$. ■

The constant b is very large, so that for small values of r the assertion of part (ii) is better than that of part (i).

We next show how to apply the above result for the construction of universal graphs. It is easier to describe the construction when the number of vertices is a power of the maximal strong chromatic number of the graphs with maximum degree no greater than r. The general case will be given in the full paper. For two integers k and s, let $G(k,s)$ denote the graph whose $n = k^s$ vertices are all vectors of length s over the alphabet $\{1, 2, \ldots, k\}$, where two are adjacent if and only if they differ in all coordinates. Note that $G(k,s)$ is $(k-1)^s$-regular, and hence its number of edges is

$$\frac{1}{2}n(k-1)^s \le n^2 \left(\frac{k-1}{k}\right)^{\log n/\log k} \le n^{2-1/k\log k}.$$

Proposition 2.2 *For every two positive integers k and s, the graph $G(k,s)$ contains every graph H on $n = k^s$ vertices with $\chi_s(H) \le k$.*

Proof: To prove this proposition, observe that it suffices to show, for each $H = (V, E)$, the existence of an injective mapping $f : V \to \{1, ..., k\}^s$ such that
(a) $f(v)$ and $f(v')$ do not agree on any component if $vv' \in E$.
This is what we do next. Split the vertex set V of H arbitrarily into disjoint subsets of size k each and find a proper coloring of H so that each color class contains precisely one vertex in each of the subsets. Let V_i denote the set of vertices colored i in this coloring. Note that the V_i's partition V, and that each of the k sets V_i is an independent set of cardiality precisely $n/k = k^{s-1}$. Next, split each set V_i arbitrarily into disjoint sets of size k each, and find another proper coloring of H so that each color class contains precisely one vertex in each of the sets. Let V_{ij} denote the set of all vertices colored i in the first coloring and colored j in the second. Note that the V_{ij}'s partition V, each of the k^2 sets V_{ij} is of cardinality k^{s-2}, and if ij and $i'j'$ have a common coordinate, then there is no edge of H connecting a member of V_{ij} with a member of $V_{i'j'}$. Let us continue on with this line of reasoning, for general $l \le s$, to partition V into k^l sets each of cardinality k^{s-l}, and with each such set V_u indexed by a unique $u \in \{1, ..., k\}^l$, such that there is no edge in H between V_u and $V_{u'}$ if u and u' agree on a coordinate. When $l = s$, each such V_u has only one vertex; so for any $v \in V$, put $f(v) = u$ if $\{v\} = V_u$; it follows that f is injective, and satisfies (a). ■

Notice that by Proposition 2.2 and by Theorem 2.1(ii), the graph $G(16, s)$ contains every graph H on at most $n = 16^s$ vertices with maximum degree 3 and has no more than $n^{1.9767}$ edges. Moreover, by the fact that the proof of Theorem 2.1(ii) given in [1] provides a linear time algorithm for the corresponding algorithmic problem, the proof above supplies an $O(n \log n)$ algorithm for finding a copy of any such given H in $G(16, s)$. For larger values of r, Proposition 2.2 and Theorem 2.1(i) supply the assertion of Theorem 1.1 if n is a power of $\lfloor br \rfloor$, and supply an efficient deterministic algorithm that uses the Algorithmic Lovász Local Lemma, given in [5] (see also [2]) for finding the required embeddings as well; with a careful implementation, the running time is $n(\log n)^c$, where $c = O(1)$.

To obtain an $\mathcal{H}(r,n)$-universal graph with precisely n vertices, where n is not a power of $\lfloor br \rfloor$, the construction has to be slightly modified. The details will be given in the full version of this paper.

3 A sparse explicit $\mathcal{H}(r,n)$-universal graph

We now explicitly describe the graph Λ promised by Theorem 1.2. We construct $\Lambda = \Lambda(r,n)$ from Γ, a graph on a set V of $n/\log n$ vertices. Partition V into sets $W_1, ..., W_{r+1}$ each of cardinality $m = n/[(r+1)\log n]$. For each k, associate each $v \in W_k$ with a vector in $\{1, ..., m^{1/r}\}^r$. For any distinct k, k', and any $v \in W_k$, and $v' \in W_{k'}$, put $vv' \in E(\Gamma)$ if and only if v and v' agree on at least one coordinate. Now replace each vertex $v \in \Gamma$ with a set V_v of $2^r r^r \log n$ vertices, and for each edge $uv \in E(\Gamma)$ interconnect each vertex of V_u with each vertex of V_v, and call the resulting graph $\Gamma' = \Gamma'(r,n)$. As we show in the full version of this paper, the graph Γ' is $\mathcal{H}(r,n)$-universal and has $\varphi(r)n^{2-\frac{1}{r}}\log^{1/r} n$ edges, but has $2^r r^r n$ vertices.

To form Λ from Γ', let Ω be a bipartite graph with vertex classes $V(\Gamma')$ and a set Q of $2n$ vertices, such that $|N_\Omega(S)| \ge |S|$ for each $S \subset V(\Gamma')$, and such that the maximum degree of Ω is $\phi'(r)$, for some finite function ϕ'. It is well known that we can explicitly construct such an Ω. We set $V(\Lambda) = Q$. For each two vertices ν, ν' in Q, put the edge $\nu\nu'$ into Λ if there exist two vertices v and v' such that $vv' \in \Gamma'$, and $\nu v, \nu' v' \in \Omega$. The following theorem implies Theorem 1.2.

Theorem 3.1 *The graph $\Lambda(r,n)$ is $\mathcal{H}(r,n)$-universal.*

We will present the proof of Theorem 3.1 in the full version of this paper. It is easier to prove the following proposition that has much of the power of Theorem 3.1. Let

$\Gamma_b = \Gamma_b(r,n)$ be a graph on a set V of dn vertices, where $d = 2r^2+1$. Partition V into sets $V_0, ..., V_d$, each of cardinality n. For each k, associate each $v_k \in V_k$ with a vector in $\{1, ..., n^{1/r}\}^r$. For each $k > 0$, and for each $v_0 \in V_0$, and $v_k \in V_k$, put $v_0 v_k \in E(\Gamma_b)$ if and only if v_0 and v_k agree on at least one component. Now replace each $v \in V$ with a set V_v of $\log n$ vertices, and for each edge $uv \in E(\Gamma_b)$, interconnect each vertex in V_u with each vertex in V_v, and call the resulting graph $\Gamma'_b = \Gamma'_b(r,n)$. Note that $\Gamma'_b(r,n)$ has $(d+1)n\log n$ vertices and $(d+1)rn^{2-\frac{1}{r}}\log^2 n$ edges. We prove the following proposition (using only elementary arguments).

Proposition 3.2 *The graph $\Gamma'_b(r,n)$ is $\mathcal{H}(r,n,n)$-universal for each r, and for n sufficiently large.*

Proof: Fix an arbitrary $H \in \mathcal{H}(r,n,n)$, and let X and Y be the two vertex classes of H. Add edges between X and Y until each vertex in Y has degree exactly r, and each vertex in X has degree at most $2r$, and call the resulting graph H'. We will now show that there exists an $f : X \cup Y \to V$ such that $|f^{-1}(v)| \leq \log n$ for each $v \in V$, and $f(x)f(y)$ is an edge in Γ_b if xy is an edge in H'. Then H' is a subgraph of Γ'_b, and so Γ'_b is $\mathcal{H}(r,n,n)$-universal.

We show the existence of such an f via a probabilistic argument. Let $f : X \to V_0$ be chosen uniformly from V_0^X, where T^S denotes the set of functions from S to T, for any sets S and T. We now extend the domain of f to Y in the following fashion. Each vertex $y \in Y$ shares a neighbor in X with at most $d-1$ other such Y; partition Y into sets $Y_1, ..., Y_d$ such that no two vertices in the same Y_k share neighbors in H'. For any ordered r-set $S = \{x_1, ..., x_r\} \subset X$, let us define $g(S)$ as the vector in $\{1, ..., n^{1/r}\}^r$ whose j-th component, $j = 1, ..., r$, is the j-th component of $f(x_j)$. We now specify $f : Y_k \to V_k$ for each k. For each $y \in Y_k$, order arbitrarily $N_{H'}(y)$, and set $f(y) = g(N_{H'}(y))$. If the edge xy is in H', then $f(x)f(y)$ is an edge in Γ. We make some observations.

Observation 1: If $|f^{-1}(v)| \leq \log n$ for all $v \in V$ with positive probability, then Γ' is $\mathcal{H}(r,n,n)$-universal.

Observation 2: The sets $N_{H'}(y)$ and $N_{H'}(y')$, with $y, y' \in Y_k$, and each Y_k, are mutually disjoint, and each has cardinality r. Therefore, since $f : X \to V_0$ is distributed uniformly on V_0^X, each resulting $f : Y_k \to V_k$ is also distributed uniformly on $V_k^{Y_k}$.

By Observation 2 and elementary probability, for each k, the probability that there exists a $v \in V_k$ such that $|f^{-1}(v)| \geq \log n$ is at most $n/[(\log n)!] < n^{-4}$ for n sufficiently large. Since there are at most $d+1$ such k, and $d+1 < n$, the proposition follows by Observation 1. ■

4 Random graphs as universal graphs

Consider the probability space of all graphs on n labelled vertices in which every pair of vertices forms an edge, randomly and independently, with probability p. We use the notation $G(n,p)$ to denote a graph chosen randomly according to this probability measure; i.e., for any graph G on n labeled vertices and with m edges, $P[G(n,p) = G] = p^m(1-p)^{\binom{n}{2}-m}$. Similarly, we define the bipartite random graph $G(n,n,p)$. For any two disjoint sets of vertices V_1 and V_2, we use the notation $G(V_1, V_2, p)$ to denote a graph chosen according to the probability space in which every pair v_1, v_2 of vertices, such that $v_1 \in V_1$ and $v_2 \in V_2$, forms an edge, randomly and independently, with probablility p.

It has been shown in [3] that, given a particular $H \in \mathcal{H}(r,n)$, the graph H is *a.a.s.* a subgraph of $G(n,p)$, for some $p = cn^{-\frac{1}{r}}\log^{1/r} n$, where c is a sufficiently large constant independent of n. By a simple averaging argument, it follows that $G(n,p)$ *a.a.s.* contains **almost every** $H \in \mathcal{H}(r,n)$ as a subgraph. To show that a random graph $G = G((1+\varepsilon)n, p)$ *a.a.s* contains **every** $H \in \mathcal{H}(r,n)$ as a subgraph for each fixed $\varepsilon > 0$, we show, via Lemma 4.1 that G *a.a.s.* satisfies certain properties, and then prove, via Lemma 4.2 that any graph that satisfies these properties must be $\mathcal{H}(r,n)$-universal; we use Proposition 4.3 in the proof of Lemma 4.2.

The proof of Theorem 1.4 is similar in structure to the proof of Theorem 1.3. We show, via Lemma 4.4 that $G = G(n,n,p)$ *a.a.s.* satifies certain properties, and then prove, via Lemma 4.5, that any graph that satisfies these properties must be $\mathcal{H}(r,n,n)$-universal; we use Proposition 4.6 in the proof of Lemma 4.5.

We now present the proofs of Theorems 1.3 and Theorems 1.4.

Proof of Theorem 1.3

The next two lemmas suffice to prove Theorem 1.3. A *star* (v,S) is a vertex v connected to each member of the set of vertices S. (We write $(v,S) \in G$ (or (v,S) is a star in G) if and only if $vw \in E(G)$ for all $w \in S$.)

Lemma 4.1 *Fix $\varepsilon > 0$ and set $N = (1+\varepsilon)n$. Let $W_1, ...W_d$, where $d = r^2+1$, be fixed subsets that partition the vertex-set V of the random graph $G(N,p)$, such that $|W_k| = N/d$ for each k. If $p = cn^{-1/r}(\log n)^{1/r}$, where $c > (4d(r+1)/\varepsilon)^{1/r}$, then* a.a.s. *$G(N,p)$ satisfies the following properties*
(a) There are at most $(1+\varepsilon)^2cn^{2-\frac{1}{r}}(\log n)^{1/r}$ edges.
(b) For each k, and for every collection $\mathcal{S}$ of $s \leq n/d$ pairwise disjoint, non-empty subsets of $V \setminus W_k$, each of size at most r, and for every set $T \subset W_k$ of $t = N/d - s$ vertices, there is at least one star (v,S), where $v \in T$ and $S \in \mathcal{S}$.

Proof: Part (a) follows by the standard estimates for binomial distributions (or simply by Chebyshev's inequality).

To prove the assertion of part (b), note that the probability that (b) fails to hold is at most

$$d\sum_{s=1}^{n/d}\binom{N}{r}^s\binom{N/d}{s}(1-p^r)^{st} \le d\sum_{s=1}^{n/d}\left[\frac{N^{r+1}}{n^{\varepsilon c^r/d}}\right]^s$$

$$= o(1/n^3). \blacksquare$$

Lemma 4.2 *Let $G=(V,E)$ be a graph on $N \ge n$ vertices, and let $d=r^2+1$. Let $W_1,...,W_d$ be disjoint subsets of V, each of cardinality N/d. If G and $W_1,...,W_d$ satisfy the assertion of part (b) of Lemma 4.1, then G is $\mathcal{H}(r,n)$-universal.*

Proof: We will show that, for each $H \in \mathcal{H}(r,n)$, there exists an $f : V(H) \to V$ such that

(I) f is injective, and

(II) $f(x)f(y) \in E$ if $xy \in E(H)$.

Thus G is $\mathcal{H}(r,n)$-universal. Fix an arbitrary $H \in \mathcal{H}(r,n)$. By the Hajnal-Szemerédi Theorem [10] there are sets $Y_1,...,Y_d$ that partition $V(H)$ so that each Y_i is a two-independent set of H, and $|Y_i| = n/d$ for each i. (We say that a set Y' of vertices is a *2-independent* set in a graph H' if no two vertices in Y' share a neighbor, and if Y' is an independent set in H'.)

We will define f recursively on each Y_i, $i=1,\dots,d$; let f be any injective embedding from Y_1 to W_1. For any $i \in \{1,....,d-1\}$, let us now assume that we have extended f to $Y_1 \cup ... \cup Y_i$, that $f(Y_1 \cup ... \cup Y_i) \subseteq W_1 \cup ... \cup W_i$, and that f is still injective. We now extend f to Y_{i+1}, specifying that $f(Y_{i+1}) \subset W_{i+1}$.

Let us define an auxiliary bipartite graph $A_f^{(i+1)}$ between Y_{i+1} and W_{i+1} where there is an edge between $y \in Y_{i+1}$ and $w \in W_{i+1}$ if and only if $f(N_H(y) \cap [Y_1 \cup ... \cup Y_i]) \subset N_G(w)$. Suppose that $f(y)=w$ for some w such that $yw \in A_f^{(i+1)}$ for each $y \in Y_{i+1}$, and each $i+1$. Since each Y_{i+1} is an independent set in H, that will guarantee that f satisfies (II). Suppose $A_f^{(i+1)}$ has a matching M_{i+1} that saturates Y_{i+1}, for each $i+1$, and that $f(y)=w$, where $yw \in M_{i+1}$, for each $y \in Y_{i+1}$, and each $i+1$. Then f satisfies (I) as well as (II). The next observation follows.

Observation 3: If $A_f^{(i+1)}$ has a matching M_{i+1} that saturates Y_{i+1}, then Lemma 4.2 follows.

We next present Claim 1.

Claim 1: Let f be any injective function that is in $[W_1 \cup ... \cup W_i]^{Y_1 \cup ... \cup Y_i}$. Then $e_{A_f^{(i+1)}}(S,T) > 0$ for each $S \subseteq Y_{i+1}$, and for each $T \subseteq W_{i+1}$ such that $|T| \ge |W_{i+1}| - |S|$.

(Proof of Claim 1: Let S be any nonempty subset of Y_{i+1}, and T be any subset of W_{i+1} such that $|T| = |W_{i+1}|-|S|$. Let us write $S(y) = f(N_{H'}(y) \cap [Y_1 \cup ... \cup Y_i])$ for each $y \in Y_{i+1}$, and $\mathcal{S} = \{S(y) : y \in S\}$. Because f is injective and Y_{i+1} is 2-independent in H', the set $\mathcal{S}$ is a collection of $|S|$ disjoint r-subsets of $W_1 \cup ... \cup W_i$. Therefore, because G and $W_1,...,W_i$ satisfy the assertion of part (b) of Lemma 4.1, $(w, S(y)) \in G$, for some $y \in S$ and $w \in T$. But if $(w,S(y)) \in G$, then $yw \in A_f^{(i+1)}$, and Claim 1 follows.)

Thus, by Claim 1 and Proposition 4.3, each $A_f^{(i+1)}$ has a matching saturating Y_{i+1}. Therefore, in view of Observation 3, Lemma 4.2 follows.

Proposition 4.3 *Let $A=(Y,W,E)$ be a bipartite graph such that $|Y|=m_1$ and $|W|=m_2$, with $m_2 \ge m_1$. If $e_A(S,T) > 0$ for every two sets $S \subset Y$ and $T \subset W$ such that $|T| > m_2 - |S|$, then A has a matching saturating Y.* $\blacksquare$

From Lemmas 4.1 and 4.2, Theorem 1.3 follows immediately. $\blacksquare$

Proof of Theorem 1.4

The next two lemmas suffice to prove Theorem 1.4.

Lemma 4.4 *Let V_1 and V_2 be disjoint sets of vertices each of cardinality n. Fix subsets $W_1,...,W_{r^2}$ that partition V_2 such that $|W_k| = n/r^2 = m$. Then, for all $c \ge 3$ and $r \ge 2$,* a.a.s. *the random bipartite graph $G(V_1,V_2,p)$, where $p = c(\log n/n)^{1/2r}$, satisfies the following properties:*
(a) The minimum and maximum degree are within the range $np \pm \sqrt{n}$.
(b) For each k, every r-element subset S_0 of V_1, the number of stars (w_k, S_0), where $w_k \in W_k$, is at least $mp^r/2$.
(c) For every collection $\mathcal{S}$ of $s = mp^r/2$ pairwise disjoint, r-element subsets of V_1, and for every s-element subset T of V_2, there is at least one star (v,S), where $v \in T$, and $S \in \mathcal{S}$.

Proof: Parts (a-b) follow trivially by Chernoff's bound.

To prove the assertion (c), observe that there are at most $\binom{n}{s}n^{rs} < \exp\{(r+1)s\log n\}$ choices of the set T and collection $\mathcal{S}$. Consider an auxiliary bipartite random graph $G(s,s,p^r)$ between the selected set T and the s r-tuples of $\mathcal{S}$, whose edges are the stars (v,S) of $G(n,n,p)$. The probability of no edge in $G(s,s,p^r)$ is smaller than $(1-p^r)^{s^2} < \exp\{-s^2p^r\} \le \exp\{-\frac{3^{2r}}{2r^2}s\log n\}$.

Lemma 4.5 *Let V_1 and V_2 be disjoint sets of vertices, each of cardinality n, and let G be a bipartite graph on $V_1 \cup V_2$ with vertex classes V_1 and V_2. Let $W_1,...,W_{r^2}$ be subsets of V_2. Suppose that G and $W_1,..,W_{r^2}$ satisfy (a)–(c) of Lemma 4.5. Then G is $\mathcal{H}(r,n,n)$-universal.*

Proof: We will show that for each $H \in \mathcal{H}(r,n,n)$ there exists an $f : V(H) \to V_1 \cup V_2$ such that

(I) f is injective, and

(II) $f(x)f(y) \in E(G)$ if $xy \in E(H)$.

Thus, G is $\mathcal{H}(r,n,n)$-universal. Fix an arbitrary $H \in \mathcal{H}(r,n,n)$ and denote by X and Y its two vertex classes. By the Hajnal-Szemerédi Theorem there is a partition $Y = Y_1 \cup ... \cup Y_{r^2}$, such that each Y_i is two-independent in H and $|Y_i| = |W_i|$ for each i.

We will first choose a suitable bijection $f : X \to V_1$, and then extend f to Y, by simultaneously specifying $f : Y_i \to W_i$ for all i. Given $f : X \to V_1$, define the auxiliary graph $A_f^{(i)} = (Y_i, W_i, E_f^{(i)})$, where there is an edge in $E_f^{(i)}$ between $y \in Y_i$ and $w \in W_i$ if and only if $N_G(w) \supset f(N_H(y))$. Suppose each $A_f^{(i)}$ has a perfect matching M_i. Then for each $y \in Y_i$, and each i, set $f(y) = w$, where $yw \in M_i$. The resulting $f : X \cup Y \to V_1 \cup V_2$ satisfies (I) and (II). The next observation follows.

Observation 4: If there exists some $f : X \to V_1$ such that each resulting $A_f^{(i)}$ has a perfect matching, then Lemma 4.5 follows.

To this end, we will use Proposition 4.6, stated below. Let i be any integer in $\{1, ..., r^2\}$. We first show that the degree of each $y \in Y_i$ in $A_f^{(i)}$ is at least $\delta = mp^r/2$. We then show that $e_{A_f^{(i)}}(S,T) > 0$ for each $S \subset Y_i$ and $T \subset W_i$ such that $|S| = |T| = \delta$. We finally show that the degree of each $w \in W_i$ in $A_f^{(i)}$ is at least δ for a random bijection $f : X \to V_1$.

Claim 2: For any $f \in V_1^X$, the degree of each $y \in A_f^{(i)}$ is at least $\delta = mp^r/2$.

(Proof of Claim 2: Let y_i be any arbitrary vertex in Y_i, and let f be any function in V_1^X. Note that $f(N_{H'}(y)))$ is a set of cardinality no greater than r in V_1. Therefore, because G and $W_1, ..., W_{r^2}$ satisfy the assertion of part (b) of Lemma 4.4, there are at least δ vertices $w_i \in W_i$ such that $(w_i, f(N_H(y_i)))$ is a star in G. But $y_i w_i \in A_f^{(i)}$ if $(w_i, f(N_H(y_i))) \in G$, so Claim 2 follows.)

Claim 3: For any $f \in V_1^X$ that is injective, $e_{A_f^{(i)}}(S,T) > 0$ for any subset $S \subseteq Y_i$ and any subset $T \subseteq W_i$ such that $|S| = |T| = \delta$.

(Proof of Claim 3: Let $S \subset Y_i$, and let $T \subset W_i$ such that $|S| = |T| = \delta$, and let f be any function in V_1^X that is injective. Because f is injective, and Y_i is 2-independent in H', the collection $\{f(N_H(y)); y \in S\}$ is a disjoint collection of δ sets of size at most r in V_1. Therefore, because G satisfies the assertion of part (c) of Lemma 4.4, $(w, f(N_H(y)))$ is a star in G, for some $y \in S$ and $w \in T$. However, $yw \in A_f^{(i)}$ if $(w, f(N_H(y))) \in G$, and Claim 3 follows.)

Thus, to prove Lemma 4.5, all we need to show now is (*) there exists an f such that the degree of each $w \in W_i$ in $A_f^{(i)}$ is at least δ for all $i \in \{1, ..., r^2\}$.

Indeed, Claims 2 and 3, and (*) imply, in view of Proposition 4.6 stated below, that there exists an $f : X \to V_1$ such that each $A_f^{(i)}$ has a perfect matching; in view of Observation 4, this proves Lemma 4.5. To show (*) we will apply the probabilistic method.

For each $w \in W$, let i be such that $w \in W_i$. For each $f \in V_1^X$, let Z_w be the number of vertices y in Y_i such that $N_H(y) \subset f^{-1}(N_G(w))$. Note that Z_w is the degree of w in $A_f^{(i)}$, so (*) follows if there exists an f such that $Z_w \geq mp^r/2$ for all $w \in W$. We now fix an arbitrary $w \in W$ and show that $P(Z_w \geq mp^r/2) = 1 - o(1/n)$, if f is chosen according to the uniform distribution on the set of injective functions of V_1^X. Note that this suffices to show (*). We now compare Z_w to yet another random variable Z'_w which is easier to work with.

Let us write $w \in W_i$. Let X_{q_w} be a random subset of X such that each $x \in X$ belongs to X_{q_w} independently with probability $q_w = |N_G(w)|/n$. Let $Z'_w = |\{y \in Y_i : N_H(y) \subseteq X_{q_w}\}|$. Since $|Y_i| = n/r^2 = m$, and the sets $N_H(y)$, $y \in Y_i$, are mutually disjoint, Z'_w has the binomial distribution $B(m, q_w^r)$, with expectation $\mu = mq_w^r$. However, $q_w^r \geq (cn^{-1/2}\log n)/2$, since G is assumed to satisfy (a) of Lemma 4.4. Therefore, we note that

$$\mu = mq_w^r \geq 2mp^r/3 \gg \sqrt{n}.$$

Hence, by Chernoff's inequality (see e.g. [11], Thm.2.6, page 26), it follows that

$$P(Z'_w \leq mp^2/2) \leq P(Z'_w \leq 3\mu/4)) \leq \exp\{-\mu/48\}$$

$$< \exp\left\{-\sqrt{n}\right\} .$$

However, by Pittel's inequality (see e.g. [11], formula (1.6), page 17),

$$P(Z_w \leq mp^r/2) \leq \sqrt{|N_G(w)|}P(Z'_w \leq mp^r/2)$$

$$= o(1/n).$$

So (*) follows, and so does Lemma 4.5, as discussed above.

Proposition 4.6 *Let $A = (Y, W, E)$ be a bipartite graph such that $|Y| = |W| = n$. If there exists a positive integer δ such that (a) each vertex in $Y \cup W$ has degree at least δ in A, and (b) $e_A(S,T) > 0$ for every two sets $S \subset Y$ and $T \subset W$ such that $|S| = |T| = \delta$, then A has a perfect matching.* ■

Theorem 1.4 follows immediately from Lemmas 4.4 and 4.5. ■

5 Fault-tolerance of random bipartite graphs

The proof of Theorem 1.5 is based on a combination of a sparse version of the regularity lemma with a hypergraph packing result and several additional ideas. This proof is more complicated than the previous proofs in this paper and requires three additional lemmas. We describe these lemmas here and postpone the actual proof to the full version of the paper. The first lemma is a special case of Lemma 20 in [12] (with α, γ, k, q replaced, resp., by $\xi/2, 1/2, r, p$). For a bipartite graph F with vertex classes U and W and for any non-empty subsets $U' \subset U$ and $W' \subset W$, we denote by $e_F(U', W')$ the number of edges between U' and W', and by

$$d_{p,F}(U', W') = \frac{e_F(U', W')}{p|U'||W'|} \tag{1}$$

the p-*density* of the pair (U', W') in F, where $0 < p \leq 1$. In particular, we set $d_F(U', W') = d_{1,F}(U', W')$. We say that the pair (U', W') is $(p; \varepsilon)$-regular in F if

$$|d_{p,F}(U', W') - d_{p,F}(U'', W'')| < \varepsilon \tag{2}$$

for all $U'' \subset U'$ and all $W'' \subset W'$ for which $|U''| \geq \varepsilon|U'|$ and $|W''| \geq \varepsilon|W'|$. A pair which is not $(p; \varepsilon)$-regular is sometimes called $(p; \varepsilon)$-irregular. If $U' = U$ and $W' = W$, we then also say that F is $(p; \varepsilon)$-regular. For a set $R \in [U]^r$ (that is, an r-element subset of U), we define its *joint degree* in F, which we write as $deg_F(R)$, as $|\tilde{N}_G(R)|$; which is the number of vertices that are adjacent in F to every vertex in R. Finally, $G[U, W]$ stands for the subgraph of G induced by $U \cup W$.

Lemma 5.1 ([12]) *For all $r \geq 2$, $\xi > 0$, and $\eta > 0$, there is $\varepsilon > 0$ such that a.a.s. $G = G(n, n, p)$ with $p \geq n^{-3/5r}$ satisfies the following property. Suppose $U \subset V_1$, $W \subset V_2$ are sets of size at least $n^{2/3}$, and F is a $(p; \varepsilon)$-regular subgraph of $G[U, W]$ with $e_F(U, W) \geq \xi e_G(U, W)$. Then*

$$\left|\left\{R \in [U]^r \ : \ \deg_F(R) < \frac{1}{2}(d_F(U, W))^r |W|\right\}\right| \leq \eta \binom{|U|}{r}. \tag{3}$$

■

Our next tool is the following version of Szemerédi's Regularity Lemma (see e.g. [11], page 215, Lemma 8.18). For $0 < p < 1$, $b \geq 1$ and $\beta > 0$, call a graph Γ $(p; b, \beta)$-*bounded* if for every pair of disjoint subsets $U, W \subset V(\Gamma)$ with $|U|, |W| \geq \beta|V(\Gamma)|$ we have $d_{p,\Gamma}(U, W) \leq b$. Furthermore, let $\Pi = (\Pi_1, \ldots, \Pi_k)$ be a partition of $V(\Gamma)$ and suppose k divides $|V(\Gamma)|$. We say that this partition is $(p; \Gamma, \varepsilon, k)$-*regular* if $|\Pi_1| = |\Pi_2| = \ldots = |\Pi_k|$ and for all except at most $\varepsilon\binom{k}{2}$ choices of the indices $1 \leq i < j \leq k$, the pairs (Π_i, Π_j) are $(p; \varepsilon)$-regular in Γ. In the lemma below we begin with a bipartite graph and obtain a partition which subdivides the original bipartition.

Lemma 5.2 *For all $\varepsilon > 0$, $b \geq 1$ and $k_0 > 0$ there exist $\beta > 0$ and K such that the following holds. For every choice of the scaling factor p, and a $(p; b, \beta)$-bounded bipartite graph Γ on a vertex set $V = V_1 \cup V_2$, where $|V_1| = |V_2|$ is divisible by $K!$, there exists a subpartition $\Pi = (\Pi_1^1, \ldots, \Pi_k^1, \Pi_1^2, \ldots, \Pi_k^2)$ of (V_1, V_2) which is $(p; \Gamma, \varepsilon, 2k)$-regular for some $k \leq K$.* ■

We also need a hypergraph packing result. Given an r-uniform hypergraph $\mathcal{B}$ on m vertices and $\eta > 0$, we define a recursive family of j-uniform hypergraphs $\mathcal{B}_j^\eta$, for $j = r, r-1, \ldots, 1$. Set $\mathcal{B}_r^\eta = \mathcal{B}$ and, given $\mathcal{B}_j^\eta$, define $\mathcal{B}_{j-1}^\eta$ as the family of all $(j-1)$-tuples J which are contained in more than ηm j-tuples of $\mathcal{B}_j^\eta$, or in other words, for which $\deg_{\mathcal{B}_j^\eta}(J) > \eta m$.

The hypergraph packing result that we need is a special case of Theorem 2 from [16]; this was proved only for bijections f, which is the tightest case. However, the proof can be literally repeated step by step also in a more relaxed setting when the hypergraph $\mathcal{N}$ has fewer vertices than $\mathcal{B}$. We say that a one-to-one function $f : V(\mathcal{N}) \to V(\mathcal{B})$ is *a packing* of $\mathcal{N}$ and $\mathcal{B}$, if $B \neq f(N)$ for every $B \in \mathcal{B}$ and every $N \in \mathcal{N}$; or shortly, $f(\mathcal{N}) \cap \mathcal{B} = \emptyset$, where $f(\mathcal{N})$ denotes the image of the hypergraph $\mathcal{N}$. We say that f'' is obtained from f' by a *switching* if there exists a pair $\{v, w\}$ such that $f'(v) = f''(w)$, $f'(w) = f''(v)$, and $f'(u) = f''(u)$ for all $u \notin \{v, w\}$.

Lemma 5.3 ([16]) *For all integers $r \geq 2$, and for all $\sigma > 0$, there exist $\eta > 0$ and m_0 such that the following holds. If $\mathcal{N}$ and $\mathcal{B}$ are two r-uniform hypergraphs on, resp., m_1 and m_2 vertices, $m_0 \leq m_1 \leq m_2$, satisfying $\Delta(\mathcal{N}) \leq r$ and $\mathcal{B}_1^{(\eta)} = \emptyset$, then at least half of all $(m_2)_{m_1}$ one-to-one functions $f' : V(\mathcal{N}) \to V(\mathcal{B})$ can be turned into a packing f of $\mathcal{N}$ and $\mathcal{B}$ by a sequence of at most σm_2 switchings.* ■

The derivation of Theorem 1.5 from these lemmas will be given in the full version of the paper.

6 Concluding remarks

It would be interesting to close the gap between the upper and the lower bound for the minimum possible number of edges in an $\mathcal{H}(r, n)$-universal graph on n vertices. Note that the lower bound holds even for the number of edges of any graph that contains a reasonable fraction of all r-regular graphs on n vertices (when rn is even).

The application of the Hajnal-Szemerédi theorem in the proofs of Theorems 1.3–1.5 is not essential, but is convenient. Instead one could use the (algorithmic) packing lemma of Sauer and Spencer [18] (we would pack H and the union of $2r^2 + 2$ cliques of order $n/(2r^2 + 2)$, doubling the number of rounds). Thus, our proofs can be made algorithmic in the sense that, for example, if indeed a given graph G

satisfies the assertions of Lemma 4.1, then one can find an embedding of any member of $\mathcal{H}(r, n)$ in G in polynomial time, using some standard matching algorithms. However, the problem of checking whether a given graph G satisfies these assertions does not seem to be solvable by a polynomial time algorithm.

References

[1] N. Alon, The strong chromatic number of a graph, *Random Structures and Algorithms* 3 (1992), 1–7.

[2] N. Alon, A parallel algorithmic version of the local lemma, *Random Structures and Algorithms* 2 (1991), 367–378. Also: Proc. 32^{nd} Annual FOCS, IEEE (1991), 586–593.

[3] N. Alon and Z. Füredi, Spanning subgraphs of random graphs, *Graphs and Combinatorics* 8 (1992), 91–94.

[4] N. Alon and R. Yuster, Threshold functions for H-factors, *Combinatorics, Probability and Computing* 2 (1993), 137–144.

[5] J. Beck, An algorithmic approach to the Lovász Local Lemma, *Random Structures and Algorithms* 2 (1991), 343–365.

[6] S. N. Bhatt, F. Chung, F. T. Leighton and A. Rosenberg, Universal graphs for bounded-degree trees and planar graphs, *SIAM J. Disc. Math.* 2 (1989), 145–155.

[7] M. R. Capalbo, A small universal graph for bounded-degree planar graphs, Proc. SODA (1999), 150–154.

[8] M. R. Capalbo and S. R. Kosaraju, Small universal graphs, Proc. STOC (1999), 741–749.

[9] P. Erdős and A. Renyi, On random matrices, *Publ. Math. Inst. Hung. Acad. Sci.* 8 (1964), 455–461

[10] A. Hajnal and E. Szemerédi, Proof of a conjecture of Erdős, in: *Combinatorial Theory and its Applications*, Vol. II (P. Erdős, A. Rényi and V. T. Sós eds.), Colloq. Math. Soc. J. Bolyai 4, North Holland, Amsterdam 1970, 601–623.

[11] S. Janson, T. Łuczak and A. Ruciński, *Random Graphs*, Wiley, New York, 2000.

[12] Y. Kohayakawa and V. Rödl, Regular pairs in sparse random graphs I, submitted, 2000.

[13] Y. Kohayakawa, V. Rödl and E. Szemerédi, in preparation.

[14] J. Komlós, G. Sárközy and E. Szemerédi, The Blow-up Lemma, *Combinatorica* 17 (1997) 109–123

[15] V. Rödl, A. Ruciński, Perfect matchings in ε-regular graphs and the Blow-up Lemma, *Combinatorica* 19 (3) (1999) 437–452.

[16] V. Rödl, A. Ruciński and A. Taraz, Hypergraph packing and graph embedding, *Combinatorics, Probability and Computing* 8 (1999) 363–376.

[17] A. Ruciński, Matching and covering the vertices of a random graph by copies of a given graph, *Discrete Math.* 105 (1992), 185–197.

[18] N. Sauer and J. Spencer, Edge disjoint placement of graphs, *J. Combinatorial Theory Ser. B*, 25 (1978), 295–302.